SYMMETRISABLE OPERATORS

PART II

OPERATORS IN A HILBERT SPACE $\mathfrak{H}$

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Introduction

In the first paper of this series [4] I gave a brief summary of the properties of symmetrisable operators in Hilbert Space. A detailed discussion of these properties will be given now, but the properties of operators symmetrisable by bounded operators will be dealt with further in Part III.

6. Definitions and preliminary discussion

It was mentioned in the first paper that we intended to use the term symmetrisable for an operator A provided HA was self-adjoint and H was a non-negative definite self-adjoint operator. However, it is clearly of some interest to see what happens when H, say, is merely essentially self-adjoint or HA closed symmetric. Some light will be thrown on the more general case. As was mentioned before, the domain of A will be assumed such that $\mathfrak{D}_{HA} = \mathfrak{D}_A$. The conditions governing the null-space of H were given as

(2.1) $$(Hx, x) \geqq 0 \qquad \text{all} \qquad x \in \mathfrak{D}_H$$

(2.2) $$\overline{\mathfrak{N}}_A \supset \mathfrak{N}_H.$$

Remark 6.1. In condition (2.2) it was necessary to use $\overline{\mathfrak{N}}_A$ in place of $\mathfrak{N}_A$, since $\mathfrak{N}_H$ is necessarily closed whereas $\mathfrak{N}_A$ could be otherwise. However, this condition is not altogether satisfactory since it admits some highly pathological cases. It follows from the fact that HA is closed and that $\mathfrak{N}_{HA} \supseteqq \mathfrak{N}_A$ that A must be defined on $\overline{\mathfrak{N}}_A$. Hence if $x \in \overline{\mathfrak{N}}_A$ and $Ax = y \neq 0$ then $y \in \mathfrak{N}_H$. Such an A cannot have a closed, single-valued extension. Since $\mathfrak{N}_A \cap \mathfrak{N}_H$ need not be closed it is therefore possible to have $Ay = \lambda y$ for any complex λ without affecting symmetrisability. It is natural, therefore, that whenever the spectrum of A is being discussed condition (2.2) will be replaced by

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$$\mathfrak{R}_A \supset \mathfrak{R}_H. \tag{2.3}$$

Definition 6.1. It will be convenient to use von Neumann's notation [2] $\tilde{A}$ for the "closure" of A, i.e. $\tilde{A}$ is the closed linear extension of A whose graph is the closure of the graph of A.

Note 6.1. We always require that linear operators be single valued. Von Neumann [2] does admit more general operators so that some of the results stated by him would not be true in our convention, this applies most particularly to adjoints.

It was seen in Part I that it is advantageous to use a symmetrising operator whose null-space is as small as possible. The best we can achieve is given by

LEMMA 6.1. *Let H be a non-negative essentially self-adjoint operator which symmetrises A. Then a symmetrising operator can always be found which is self-adjoint and has as null-space the intersection of the closure of the range of A with the closure of the null-space of A.*

PROOF. Let $H_1 = \tilde{H}+P$ where P is the orthogonal projector onto $\mathfrak{R}_A^\perp$, the orthogonal complement of $\mathfrak{R}_A$.

Clearly H_1 is self-adjoint and $H_1A = HA$ since $\tilde{H}A = HA$ because $\mathfrak{D}_H \supset \mathfrak{R}_A$. Also for all $f \in \mathfrak{D}_{\tilde{H}}$ $(= \mathfrak{D}_{H_1})$

$$\begin{aligned}(H_1 f, f) &= (\tilde{H}f, f)+(Pf, f)\\ &= (\tilde{H}f, f)+\|Pf\|^2\\ &\geqq 0\end{aligned}$$

so that H_1 is non-negative and satisfies the conditions of the lemma.

It will be assumed in the future that the symmetrising operators H satisfy lemma 6.1.

As foreshadowed in section 4 of our first paper [4] we shall have occasion to embed $\mathfrak{H}$, the domain and range space of our operators in a larger space $\mathfrak{H}+\mathfrak{H}'$ where $\mathfrak{H}'$ is an exact replica of $\mathfrak{H}$ but the elements of $\mathfrak{H}'$ are orthogonal to the elements of $\mathfrak{H}$. $\mathfrak{H}+\mathfrak{H}'$ is isometrically isomorphic with $\mathfrak{H}\times\mathfrak{H}$, i.e. the points $[x, y]$ of $\mathfrak{H}\times\mathfrak{H}$ are mapped onto $x+y$ where $x \in \mathfrak{H}$, $y \in \mathfrak{H}'$ and the isometry is established by defining the inner product in $\mathfrak{H}\times\mathfrak{H}$ by $([x_1, y_1], [x_2, y_2]) = (x_1, x_2)+(y_1, y_2)$ which is clearly equal to (x_1+y_1, x_2+y_2) since $(x_i, y_j) = 0$ $(i, j = 1, 2)$.

It is convenient to add the definition of two terms used in reference [1] and again extensively here.

Definition 6.2. A subspace $\mathfrak{M}$ of $\mathfrak{H}$ is a Julia manifold (J-manifold for short) if there exists a closed vector subspace $\mathfrak{V}$ in $\mathfrak{H}+\mathfrak{H}'$ such that $\mathfrak{M} = P_{\mathfrak{H}}\mathfrak{V}$.

Definition 6.3. Let $\mathfrak{V}$ be a closed subspace of $\mathfrak{H}\times\mathfrak{H}'$ and $\mathfrak{V}'$ its orthogonal

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complement. Then $\mathfrak{M} = P_{\mathfrak{H}}\mathfrak{B}$ and $\mathfrak{M}' = P_{\mathfrak{H}}\mathfrak{B}'$ are complementary J-manifolds. It can be shown that there exist two closed mutually orthogonal vector subspaces $\mathfrak{F}$ and $\mathfrak{F}'$ and two J-manifolds $\mathfrak{N}$ and $\mathfrak{N}'$ such that $\overline{\mathfrak{N}} = \mathfrak{F}$, $\overline{\mathfrak{N}'} = \mathfrak{F}'$ and $\mathfrak{M} = \mathfrak{F}+\mathfrak{N}'$, $\mathfrak{M}' = \mathfrak{F}'+\mathfrak{N}$ (Cf. [1] Prop. 2.5).

Remark 6.2. Let A be an operator with dense domain then A^* the adjoint of A can be defined by means of the orthogonal complement of the graph of A (cf. [2] p. 62). It follows immediately that when A is a one-one operator (i.e. $\mathfrak{D}_A$ and $\mathfrak{R}_A$ dense in $\mathfrak{H}$) then $\mathfrak{D}_A$ and $\mathfrak{R}_{A^*}$ are complementary J-manifolds.

7. Some general properties

The excellent paper by Dixmier [1] referred to previously throws a lot of light on the nature of products of operators. It is well known that closed operators do not form a group under multiplication. On the other hand operators whose graphs are Julia-manifolds in $\mathfrak{H}\times\mathfrak{H}$ do form a group. Such operators are called J-operators. Closed operators are, of course, J-operators and Dixmier is able to prove a number of interesting conditions which ensure that the product of two operators is closed. We can use these to prove properties of symmetrisable operators.

We commence by proving that for J-operators (2.2) and (2.3) are equivalent.

THEOREM 7.1. *If A is a J-operator and HA closed then $\mathfrak{R}_A$ is closed.*

PROOF. By Dixmier (Prop. 3.2) HA closed implies

$$z_n \to 0,\ HAz_n \to 0 \Rightarrow Az_n \to 0.$$

As was mentioned in Remark 6.1., any $x \in \overline{\mathfrak{N}}_A$ is in the domain of A. Now suppose $\mathfrak{N}_A$ not closed, then for some $x \in \overline{\mathfrak{N}}_A$ $Ax = y \neq 0$, but $Hy = 0$ since HA closed. Now there exists a sequence (x_n), $x_n \in \mathfrak{N}_A$, such that $x_n \to x$. Hence $(x-x_n) = (z_n)$ is a sequence such that $z_n \to 0$, $HAz_n \to 0$ but $Az_n = y \neq 0$ for all n, which is not possible.

Another easy result is

PROPOSITION 7.1. *If A is unbounded and closed it cannot be symmetrised by a compact H.*

PROOF. By Dixmier [1] proposition 3.4. the product HA is not closed under the hypotheses.

It will be shown later that the theory of operators symmetrisable by operators with bounded inverses is much simpler than the general theory. It is therefore interesting to consider

THEOREM 7.2. *If $\widetilde{A^{-1}}$ is compact and if H symmetrises A then H must have a bounded inverse.*

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PROOF. Since $\mathfrak{D}_A$ and $\mathfrak{R}_A$ are dense, $\mathfrak{N}_A = [0]$ and the ranges of H and HA are dense; thus HA is a one-one operator. Since $\widetilde{A^{-1}}$ is closed and compact, $\mathfrak{R}_{A^{-1}} = \mathfrak{D}_A$ contains no closed, infinite dimensional vector subspace.

By Remark 6.2. $\mathfrak{D}_A$ and $\mathfrak{R}_{HA}$ are complementary J-manifolds since HA is self-adjoint and one-one. By Dixmier's [1] proposition 2.5. this implies that $\mathfrak{D}_A$ and $\mathfrak{R}_{HA}$ contain closed vector subspaces that are orthogonal complements in $\mathfrak{H}$. Hence $\mathfrak{R}_{HA}$ can have at most finite deficiency in $\mathfrak{H}$ (i.e. the quotient space $\mathfrak{H}/\mathfrak{R}_{HA}$ is finite dimensional) and is therefore closed. But as we have observed $\mathfrak{R}_{HA}$ is dense and hence $\mathfrak{R}_{HA} = \mathfrak{H}$. Also $\mathfrak{R}_H \supset \mathfrak{R}_{HA}$ so that $\mathfrak{R}_H = \mathfrak{H}$ and since H^{-1} is closed this implies H^{-1} bounded.

8. On adjoints and on closure of A

For most of the subsequent work it is necessary to assume some relationship between $\mathfrak{D}_H$ and $\mathfrak{D}_A$. This is obvious if we observe that for spectral theory we shall be dealing with the operator $A-\lambda I$. Without further conditions we can state

LEMMA 8.1. *If $x \in \mathfrak{D}_H$ and $Hx \in \mathfrak{D}_{A^*}$ then $x \in \mathfrak{D}_A$ and $A^*Hx = HAx$.*

PROOF. For any $y \in \mathfrak{D}_A$, $x \in \mathfrak{D}_H$

$$(HAy, x) = (Ay, Hx) = (y, A^*Hx)$$

provided $Hx \in \mathfrak{D}_{A^*}$

To obtain a more satisfactory result we have to impose the condition

$$\mathfrak{D}_H \supset \mathfrak{D}_A. \tag{8.1}$$

It can, of course, be re-interpreted as meaning that we concern ourselves with the restrictions of operators A to domain contained in $\mathfrak{D}_H$. However, there is no a priori reason to suppose that in general $\mathfrak{D}_H \cap \mathfrak{D}_A$ is dense and this matter will not be pursued here.

LEMMA 8.2. *If (8.1) is satisfied $x \in \mathfrak{D}_A$ implies $Hx \in \mathfrak{D}_{A^*}$ and $A^*Hx = HAx$.* For all x, $y \in \mathfrak{D}_A$ $(HAy, x) = (Ay, Hx) = (y, HAx)$. Hence $A^*(Hx) = HAx$ as required.

The above lemma defines a restriction of A^* which we shall call A^+, thus

Definition 8.1. A^+ is the linear operator defined on $H(\mathfrak{D}_A)$ such that $A^+Hx = HAx$ for all $x \in \mathfrak{D}_A$. A^+ is a specialisation of A^*.

If $H(\mathfrak{D}_A)$ is dense in $\mathfrak{H}$ — which incidentally implies that the null-space of H, $\mathfrak{N}_H$, is $[0]$ — then A^+ is defined with a dense domain. Hence A^{+*} is a closed linear extension of A, $\hat{A}$ say. (These remarks are true even if HA is merely symmetric).

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It is evidently of interest to know under what conditions $H(\mathfrak{D}_A)$ might be dense. A sufficient condition is given in

LEMMA 8.3. *If H is positive ($\mathfrak{N}_H = [0]$) and (8.1) satisfied (in particular if H bounded) and $\mathfrak{D}_A$ is everywhere dense, then $H(\mathfrak{D}_A)$ is everywhere dense.*

PROOF. We suppose $H(\mathfrak{D}_A)$ not dense. Then for some $y \neq 0$ in $\mathfrak{H}$ and all $x \in \mathfrak{D}_A$

$$(Hx, y) = 0.$$

Since $\mathfrak{D}_A$ is dense we can find a sequence (y_n) such that $y_n \to y$, $y_n \in \mathfrak{D}_A$. By the continuity of the linear functional

$$\lim_{n\to\infty} (Hx, y_n) = \lim_{n\to\infty} (Hx, (y-y_n)) = 0.$$

Also $(Hx, y_n) = (x, Hy_n)$ and in particular putting $x = y_n$ the above gives

$$(y_n, Hy_n) \to 0$$

i.e. $Hy_n \to 0$. But since H is closed and $y_n \to y$, $Hy_n \to 0$ implies $Hy = 0$ contrary to assumption.

COROLLARY. *If the hypotheses of the lemma are satisfied, A is closed or has a closed linear extension A^{+*}.*

Remark 6.1. suggested that when $\mathfrak{N}_H \neq [0]$ certain pathological cases could arise which would make it impossible to find closed extensions for A. To avoid the elaboration of this case we shall, for the remainder of section 8 assume that $\mathfrak{N}_H = [0]$ or — what amounts to the same thing — that all our operators are specialised to the space $\mathfrak{N}_H^{\perp}$.

Let B be a closed linear extension of A — which we have seen always exists when $H(\mathfrak{D}_A)$ is dense. If A is symmetrisable in the strict sense then HA is maximal and HB cannot be symmetric if it is a proper extension of HA. (It would be possible for B to be symmetrisable not by H but by some other operator H_1, say.)

On the other hand if A is only essentially symmetrisable, or HA merely symmetric, then it is sensible to enquire whether HB is symmetric. It is found that if H has a sufficiently large domain HB is certainly symmetric. We have in fact

THEOREM 8.1. *If A is an operator with domain $\mathfrak{D}_A$ and H is positive self-adjoint and such that $H(\mathfrak{D}_A)$ is dense in $\mathfrak{H}$ and HA is symmetric then A has closed linear extensions $\tilde{A}$, $\hat{A}$. If the domain and range of $\tilde{A}$ is in the domain of H then $H\tilde{A}$ is also symmetric. ($\tilde{A}$ is defined in Definition* 6.1.).

The existence of the closed extension $\hat{A}$ was established earlier, when it was defined as A^{+*}. The extension $\tilde{A}$ must therefore exist (Stone [5] Theorem 2.10). Let x be an element of $\mathfrak{D}_{\tilde{A}}$ not belonging to $\mathfrak{D}_A$. (If there

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is no such element there is nothing to prove.) Let (x_n) be a sequence of elements of $\mathfrak{D}_A$ such that $x_n \to x$ and let $\tilde{A}x = y$. Then for all $z \in \mathfrak{D}_A$

$$(HAx_n, z) = (x_n, HAz) = (x_n, A^*Hz) = (Ax_n, Hz)$$

so that

$$(x_n, HAz) = (Ax_n, Hz).$$

Letting $n \to \infty$

$$(x, HAz) = (\tilde{A}x, Hz)$$
$$= (H\tilde{A}x, z)$$

provided $\tilde{A}x$ is in the domain of H. If further $x \in \mathfrak{D}_H$

$$(HAx_n, x) = (x_n, H\tilde{A}x) = (Ax_n, Hx)$$

and letting $n \to \infty$ in the last two expressions

$$(x, H\tilde{A}x) = (\tilde{A}x, Hx)$$
$$= (H\tilde{A}x, x).$$

Hence $H\tilde{A}$ is symmetric on the subspace $\mathfrak{D}_A + \{x\}$ if both x and $\tilde{A}x$ belong to $\mathfrak{D}_H$. The above argument can now be repeated for an element x' of $\mathfrak{D}_{\tilde{A}}$ not belonging to $\mathfrak{D}_A + \{x\}$, if it exists. The process can clearly be continued until $\mathfrak{D}_{\tilde{A}}$ is exhausted.

For symmetrisable operators we have

THEOREM 8.2. *If A is symmetrised by a strictly positive definite operator H which is such that $H(\mathfrak{D}_A)$ is dense and $\mathfrak{D}_H \supset \mathfrak{R}_{\tilde{A}}$ then A is closed; $A^{+*} (\equiv \hat{A}) = A$ if $\mathfrak{D}_H \supset \mathfrak{R}_{\tilde{A}}$,* i.e. $A^{+*} = A^{**} = A$.

PROOF. Let $x_n \to x$ and $Ax_n = y_n \to y$. Then since $\tilde{A}$ is closed $\tilde{A}x = y$. Also for all $z \in \mathfrak{D}_A$

$$(HAx_n, z) = (x_n, HAz) \text{ is equivalent to } (x_n, HAz) = (x_n, A^+Hz).$$

Letting $n \to \infty$ in the latter

$$(x, HAz) = (x, A^+Hz) = (\tilde{A}x, Hz) = (H\tilde{A}x, z).$$

Hence, if HA self-adjoint $x \in \mathfrak{D}_A$ and A is closed. In the second part of the proof let u be any element of $\mathfrak{D}_{\tilde{A}}$ and z any element of $\mathfrak{D}_{HA}$ then

$$(u, HAz) = (u, A^+Hz) = (\hat{A}u, Hz) = (H\hat{A}u, z)$$

and again $u \in \mathfrak{D}_A$.

Remark 8.1. The condition $A^{+*} = A^{**}$ implies that the graph of A^* is the closure of the graph of A^+. Using Definition (6.1) $\widetilde{A^+} = A^*$. We have proved that if A has a closed extension then $A^{**} = \tilde{A}$ in the strict sense of note (6.1), not merely in the sense of von Neumann [2] theorem 13.13.

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The last question we pose in this section is whether the symmetrisability of A implies the symmetrisability of A^*, as it did for operators in unitary spaces. A partial answer is provided by

THEOREM 8.3. *Let A be symmetrised by H then, if $H(\mathfrak{D}_A)$ is dense, $H^{-1}A^+$ is symmetric. If H is bounded and $H^{-1}A^+$ is essentially self-adjoint then $H^{-1}A^*$ is self-adjoint. If in particular $H^{-1}A^+ = H^{-1}A^*$ then $H^{-1}A^+$ is self-adjoint.*

PROOF. By definition A^+ has domain $H(\mathfrak{D}_A)$ and since $HA = A^+H$ $\mathfrak{R}_{A^+} \subset \mathfrak{R}_H$. Hence $H^{-1}A^+$ is defined on $H(\mathfrak{D}_A)$ with range $\mathfrak{R}_A$. Hence for any $x = Hu$, $y = Hv$ where $u, v \in \mathfrak{D}_A$ we have

$$\begin{aligned}(H^{-1}A^+x, y) &= (H^{-1}A^+Hu, y) = (Au, y) = (Au, Hv) = (u, HAv) \\ &= (x, H^{-1}A^+Hv) = (x, H^{-1}A^+y).\end{aligned}$$

The question of a self-adjoint extension of $H^{-1}A^+$ is very difficult when H is unbounded and hence bounded H only are considered. Now the graph of $H^{-1}A^+$ is $[z, H^{-1}A^+z]$ for all $z = Hx$, $x \in \mathfrak{D}_A$. The orthogonal complement of this in $\mathfrak{H} \times \mathfrak{H}$ is $[-H^{-1}A^+z, z] + [-w, u]$ where we suppose the latter subspace distinct from the former, i.e. assume $H^{-1}A^+$ not maximal. Then for the latter subspace and all $z \in H(\mathfrak{D}_A)$

$$(w, z) = (u, H^{-1}A^+z)$$

or

$$\begin{aligned}(w, Hx) &= (u, H^{-1}A^+Hx) \\ &= (u, H^{-1}HAx)\end{aligned}$$

and since $w \in \mathfrak{D}_H = \mathfrak{H}$

$$(Hw, x) = (u, H^{-1}HAx)$$

and the condition $u = Hy$ would imply $y \in \mathfrak{D}_A$ and $HAy = Hw$ or $H^{-1}A^+u = w$ and $[-w, u]$ would not be distinct. Hence $u \notin \mathfrak{R}_H$ but

$$(Hw, x) = (u, Ax)$$

which implies $A^*u = Hw$. Also $A^*u \in \mathfrak{R}_H$ and $[-w, u] = [-H^{-1}A^*u, u]$. $H^{-1}A^*$ is closed since A^* is closed and H bounded (Dixmier [1] prop. 3.3). Also $H^{-1}A^* \supset H^{-1}A^+$ and hence since by the above $(H^{-1}A^+)^* = H^{-1}A^*$

$$H^{-1}A^* = (H^{-1}A^+)^* \supset (H^{-1}A^+)^{**} = (H^{-1}A^*)^* \supset H^{-1}A^+,$$

which is all that is required.

COROLLARY. *If $A = BH$ where B and H are self-adjoint and H is positive and bounded, then $H^{-1}A^+ = H^{-1}A^* = B$ is self adjoint.*

Clearly $A^+ = HB$ and $H^{-1}A^+ = B$ which is self-adjoint; the theorem then proves $H^{-1}A^+ = H^{-1}A^*$.

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Remark 8.2. The statement "$H^{-1}A^*$ is self-adjoint" does not imply A^* symmetrisable since in general $\mathfrak{D}_{H^{-1}} \not\supset \mathfrak{R}_{A^*}$ as would be required by our definition of symmetrisability.

9. Operators symmetrisable by operators H with positive lower bound

Before dealing with the general spectral theory we shall deal with a special case which exemplifies all the properties one would like to find in the general case.

We shall deal with strictly symmetrisable operators A with symmetrising operator H such that $\mathfrak{D}_H \supset \mathfrak{D}_A$ and for simplicity $\mathfrak{N}_H = [0]$. Since

$$(HAx, y) = (x, HAy)$$

for all x, y in $\mathfrak{D}_{HA} = \mathfrak{D}_{(HA)^*}$ it is clear that if we introduce a new inner product

$$(x, y)_1 = (Hx, y)$$

we should have A symmetric in the linear subspace $\mathfrak{D}_H$ of $\mathfrak{H}$. However, the metric induced by this new inner product may make $\mathfrak{D}_H$ an incomplete space $\mathfrak{H}_1$, and for our present purposes that would make it useless. We require that

$$||x_m - x_n||_1 \to 0 \text{ i.e. } ||\sqrt{H}(x_m - x_n)|| \to 0 \text{ i.e. } \sqrt{H}x_m \to g$$

implies the existence of an $x \in \mathfrak{D}_H$ such that $\sqrt{H}x = g$. Hence we require $\mathfrak{R}_{\sqrt{H}}$ to be closed and since H is positive this means $\mathfrak{R}_{\sqrt{H}} = \mathfrak{H}$. We thus require: $\sqrt{H}$ and hence H have bounded inverse, and so

$$||x_m - x_n||_1 \to 0 \Rightarrow ||x_m - x_n|| \to 0$$

and $x_m \to x$.

We therefore obtain the following

THEOREM 9.1. *If A is symmetrisable by an operator H with bounded inverse then we can define a Hilbert space $\mathfrak{H}_1$, consisting of all elements of $\mathfrak{D}_{\sqrt{H}}$ (and no others) and an inner product defined by*

$$(x, y)_1 = (\sqrt{H}x, \sqrt{H}y)$$

then A is a self-adjoint operator in $\mathfrak{H}_1$, its eigenvalues are real, its continuous spectrum is real and its residual spectrum empty. A has resolution of the identity $E(\lambda)$, say,

$$(Ax, y)_1 = \int_{-\infty}^{\infty} \lambda d(E(\lambda)x, y)_1.$$

The case when $\mathfrak{H}_1$ introduced above is incomplete will be dealt with at the end of Part III.

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10. The spectrum of symmetrisable operators

The point spectrum of symmetrisable operator is real under very general conditions as can be seen from

THEOREM 10.1. *The eigenvalues of A (if any) are real if HA is symmetric, $\mathfrak{D}_H \supset \mathfrak{R}_A$ and $\mathfrak{N}_A \supset \mathfrak{N}_H$.*

The proof of this is the same as the proof of Theorem 3.1. statement (i) because all eigenvectors are both in the domain and range of A.

Another easy result is

THEOREM 10.2. *If B and H are symmetric and non-negative (positive) definite and $A = BH$ then the eigenvalues of A are non-negative (positive).*

PROOF. Let x_i be an eigenvector of A. Then $x_i \in \mathfrak{D}_H$, $Hx_i \in \mathfrak{D}_B$, $Ax_i \in \mathfrak{D}_H$ and

$$(HAx_i, x_i) = \lambda_i(Hx_i, x_i).$$

Also

$$(HAx_i, x_i) = (HBHx_i, x_i)$$
$$= (BHx_i, Hx_i).$$

Then unless $BHx_i = 0$ (which cannot happen under the strictly positive assumptions)

$$\lambda_i^{-1} = \frac{(Hx_i, x_i)}{(BHx_i, x_i)}$$

which is clearly positive. The case $BHx_i = 0$ satisfies the lemma trivially.

THEOREM 10.3. *Eigenvectors of A belonging to different eigenvalues are H-orthogonal. All elements y such that $Ay \in \mathfrak{N}_H$ are H-orthogonal to eigenvectors with non-zero eigenvalues.* (We have $A^2y = 0$ if $\mathfrak{N}_A \supset \mathfrak{N}_H$).

Let x_i, x_j be any eigenvectors with eigenvalues λ_i, λ_j then since $x_i, x_j \in \mathfrak{D}_H$

$$HAx_i = \lambda_i Hx_i, \quad HAx_j = \lambda_j Hx_j$$
$$(HAx_i, x_j) = \lambda_i(Hx_i, x_j)$$
$$(x_i, HAx_j) = \lambda_j(x_i, Hx_j) = \lambda_j(Hx_i, x_j).$$

By the symmetry of HA one obtains on subtracting

$$(\lambda_i - \lambda_j)(Hx_i, x_j) = 0$$

and since $\lambda_i \neq \lambda_j$, $(Hx_i, x_j) = 0$.

Further

$$(HAx_i, y) = \lambda_i(Hx_i, y) = (x_i, HAy) = 0$$

and since $\lambda_i \neq 0$, $(Hx_i, y) = 0$.

To make further progress we reintroduce the condition

(8.1) $$\mathfrak{D}_H \supset \mathfrak{D}_A.$$

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LEMMA 10.1. *If A is symmetrisable then $\{H(A-\lambda I)\}^* = H(A-\lambda I)$ for any real λ. If A^p has dense domain then HA^p is symmetric and $\{H(A-\lambda I)^p\}^* \supset H(A-\lambda I)^p$ for $p = 2, 3, \cdots$.*

PROOF. The proof of Lemma 3.1. stands except that the elements x, y used in the proof must now belong to $\mathfrak{D}_{A^p}$ for the particular p under discussion.

We can now generalise the remainder of Theorem 3.1.

THEOREM 10.4. (i) *All eigenvalues $\lambda \neq 0$ of A are simple, i.e.*

$$(A-\lambda I)^p y = 0 \Rightarrow (A-\lambda I)y = 0 \text{ for } \lambda \neq 0,\ p > 1.$$

(ii) *If 0 is an eigenvalue of A it is of multiplicity 2 at most in the sense that*

$$A^p y = 0 \Rightarrow HAy = 0 \Rightarrow A^2 y = 0 \text{ for } p = 3, 4, \cdots.$$

PROOF. The proof of theorem 3.1. (Parts (ii) (iii)) holds by allowing y to be any element of $\mathfrak{D}_A$ since $\mathfrak{R}_{A^p} \subset \mathfrak{R}_A \subset \mathfrak{D}_H$ for $p = 1, 2, \cdots$. The proof can also be slightly modified to avoid the use of $\sqrt{H}$ because

$$Hx_i = 0 \Rightarrow (Hx_i, x_i) = 0$$

since H is non-negative (Cf. [2] p. 71).

COROLLARY. *If H is positive definite all eigenvalues are simple.* (In this case $HAy = 0 \Rightarrow Ay = 0$.)

From the discussion in section 8 it will be seen that the relationship between A and A^* is not as convenient as it was for operators in $\mathfrak{U}_n$. However, we still have

THEOREM 10.5. *If λ is an eigenvalue of A and x the corresponding eigenvector then λ is also an eigenvalue of A^* and Hx is a corresponding eigenvector except when $x \in \mathfrak{N}_H = \mathfrak{N}_A \cap \mathfrak{R}_A$. Then $x = Ay$ and Hy is in the null-manifold of A^*.*

PROOF. By assumption $Ax = \lambda x$ so that $HAx = \lambda Hx$ and hence

$$A^+ Hx = \lambda Hx.$$

Hence Hx is eigenvector of A^+ and hence of A^* unless it is the null vector. In the latter case we can choose y such that $Ay = x$. For all $f \in \mathfrak{D}_A$

$$(A^* Hy, f) = (y, HAf) = (HAy, f) = 0$$

and since $\mathfrak{D}_A$ is dense and $Hy \neq 0$ the theorem is proved.

To illustrate the difficulty of proving results about the continuous and residual spectrum we start with a most discouraging result.

THEOREM 10.6. *The continuous spectrum of a symmetrisable operator A need not be restricted to the real axis.* (For a particular type of symmetrisable A the continuous spectrum can be shown to be real).

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We prove this by constructing a symmetrisable A with a complex continuous spectrum. In order to do this we first investigate a special class of operator which can be used to construct examples of this sort. We consider an A which is such that for some sequence of projectors P_n with n-dimensional range and such that $P_n(\mathfrak{H}) \supset P_{n-1}(\mathfrak{H})$ and $\lim_{n\to\infty} P_n = I$ the operator $A_n = P_n A P_n$ is symmetrisable for all n and $A_n \to A$. (It is evident that not all symmetrisable A are of this type.) Let $x_i^{(n)}$ be the eigenvectors of A_n corresponding to eigenvalues $\mu_i^{(n)}$. Let $x_i^{(n)} = T_n e_i$ where (e_i) is a complete orthonormal system. We first prove

Lemma 10.2. *If T_n is as defined above and if T_n and T_n^{-1} are uniformly bounded with bounds $|T_n| = \alpha$, $|T_n^{-1}| = \beta$, say, a non-real λ cannot belong to the continuous spectrum.*

Proof of Lemma. Let $x_i^{*(n)}$ denote the eigenvectors of A_n^* which can be regarded as suitably normalised so that $x_i^{*(n)} = T_n^{*-1} e_i$. If λ belongs to the continuous spectrum there exists for are $\varepsilon > 0$ an x with $||x|| = 1$ such that

$$||(A-\lambda I)x|| < \varepsilon.$$

If we take n large enough $x_n = P_n x$ will be such that $||x_n|| \geqq \frac{1}{2}$ and

$$||(A_n-\lambda I)x_n|| < 2\varepsilon.$$

But

$$\begin{aligned}(A_n-\lambda I)x_n &= \sum_i(\mu_i^{(n)}-\lambda)(x_n, x_i^{*(n)})x_i^{(n)}\\ &= T_n \sum(\mu_i^{(n)}-\lambda)(x_n, T_n^{*-1}e_i)e_i\\ &= T_n \sum(\mu_i^{(n)}-\lambda)(T_n^{-1}x_n, e_i)e_i\\ &= y.\end{aligned}$$

Now

$$\begin{aligned}||T_n^{-1}y||^2 &= \sum|(\mu_i^n-\lambda)|^2\,|(T_n^{-1}x_n, e_i)|^2\\ &\geqq \mathscr{I}(\lambda)^2||T_n^{-1}x_n||^2 \geqq \mathscr{I}(\lambda)^2\frac{||x_n||^2}{|T_n|^2}.\end{aligned}$$

Inserting the bounds for T_n and T_n^{-1} we have

$$||y|| \geqq \frac{1}{2\alpha\beta}\,|\mathscr{I}(\lambda)|$$

which is bounded below if $\mathscr{I}(\lambda) \neq 0$ and thus the Lemma is proved.

Now we return to the main theorem and observe that the boundedness of T_n and T_n^{-1} implies the following (dropping the upper bracketted index for convenience):

$$y = \sum(y, x_i^*)x_i, \qquad z = \sum(z, x_i)x_i^*$$
$$T_n^{-1}y = \sum(y, x_i^*)e_i, \qquad T_n z = \sum(z, x_i)e_i$$

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so that

$$\beta^2 \geqq \frac{\sum |(y, x_i^*)|^2}{\|\sum (y, x_i^*) x_i\|^2} \qquad \alpha^2 \geqq \frac{\sum |(z, x_i)|^2}{\|\sum (z, x_i) x_i^*\|}.$$

Therefore for all sequences α_i, β_i

$$\|\sum \alpha_i x_i\|^2 \geqq \frac{1}{\beta^2} \sum |\alpha_i|^2$$

$$\|\sum \beta_i x_i^*\| \geqq \frac{1}{\alpha^2} \sum |\beta_i|^2$$

which is a relative measure of their linear independence. Hence unboundedness of T_n and T_n^{-1} implies a loss of linear independence (asymptotically) of (x_i).

Since $T_n^{*-1} T_n^{-1}$ is a symmetrising operator it can be put equal to H_n and in fact without loss of generality the following relations can be assumed: $T_n^{-1} = T_n^{*-1} = \sqrt{H_n}$. We further take $\sqrt{H_n}$ to be bounded by 1, say, by multiplying the x_i by $|T|_{-1}^{n} = \beta$. (The danger of this type of definition is the possibility that H may not be self-adjoint in the limit. However, Dixmier has shown that for operators so definied, which in his notation are written $\sqrt{H^{-1}} = \mathscr{T}_2(e_i, x_i)$, a necessary condition for $(\sqrt{H})^{-1} = (\sqrt{H})^{-1*} = \mathscr{T}_2(x_i^*, e_i)$ is that $(\sqrt{H})^{-1}$ or $\sqrt{H}$ be bounded. We must also have (x_i) as basis and (x_i^*) as dual basis). Under these conditions the only way in which the limiting operator A can have a non-real continuous spectrum is for $|T_n| = \alpha_n$ to be unbounded, and we now show how this can be arranged to construct our example.

Let $\mathfrak{U}_n$ be a set of n-dimensional unitary spaces, $\mathfrak{H}$ the Hilbert sum of the $\mathfrak{U}_n$, i.e. $\mathfrak{U}_n$ mutually orthogonal subspaces of $\mathfrak{H}$ such that $\mathfrak{H} = \mathfrak{U}_1 + \mathfrak{U}_2 + \mathfrak{U}_3 + \cdots$. Let $A_n = P_{\mathfrak{U}_n} A P_{\mathfrak{U}_n}$ be an operator in $\mathfrak{U}_n$ defined by its matrix with respect to a suitable orthonormal system, viz.

$$A_n = \begin{bmatrix} \mu_1 & 0 & 0 & \cdot & 0 & 0 \\ \lambda - \mu_2 & \mu_2 & 0 & \cdot & 0 & 0 \\ \dfrac{\lambda-\mu_3}{2} & \dfrac{\lambda-\mu_3}{2} & \mu_3 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{\lambda-\mu_{n-1}}{n-2} & \dfrac{\lambda-\mu_{n+1}}{n-2} & \dfrac{\lambda-\mu_{n+1}}{n-2} & \cdot & \mu_{n-1} & 0 \\ \dfrac{\lambda-\mu_n}{n-1} & \dfrac{\lambda-\mu_n}{n-1} & \dfrac{\lambda-\mu_n}{n-1} & \cdot & \dfrac{\lambda-\mu_n}{n-1} & \mu_n \end{bmatrix}$$

where the μ are real and unequal and λ is complex. By theorem **3.3** A_n

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is symmetrisable. The symmetrising operator H_n is given by a matrix whose elements h_{pq} are given by the recurrence relation:

$$h_{pq}(\mu_q-\mu_p) = h_{pq+1}\left(\mu_{q+1}-\mu_p+\frac{\mu_{q+1}-\lambda}{q}\right)+\frac{\bar{\lambda}-\mu_{p+1}}{p}(h_{p+1,q}-h_{p+1,q+1})$$
$$+\frac{\bar{\lambda}-\mu_{p+2}}{p+1}(h_{p+2,q}-h_{p+2,q+1})+\cdots+\frac{\bar{\lambda}-\mu_r}{n-1}(h_{nq}-h_{n,q+1})$$

for $p > q$; also $h_{pq} = \bar{h}_{qp}$.

Also the h_{pp} are real and positive but arbitrary except for the fact that they have to decrease rapidly enough to ensure that H_n is positive definite (e.g. $h_{nn} < h_{n-1,n-1}(\mu_{n-1}-\mu_n)$). It is easily verified that the vector $x_n = \{1/\sqrt{n}, 1/\sqrt{n}, \cdots, 1/\sqrt{n}\}$ is such that

$$A_n x_n - \lambda x_n = y_n = \left\{\frac{\mu_1-\lambda}{\sqrt{n}}, 0, 0, \cdots, 0\right\} \text{ so that } \|A_n x_n - \lambda x_n\| = \frac{|\mu_1-\lambda|}{\sqrt{n}}\|x_n\|.$$

Now let A_n be extended to the rest of $\mathfrak{H}$ by putting $A_n = 0$ on $\mathfrak{U}_n^\perp$. Then $A = \lim_{N\to\infty}\sum_{n=1}^N A_n$ which will be bounded and closed if the μ are bounded. By suitable choice of the h_{pp} also H_n will be uniformly bounded and the operator H defined by the same procedure as A will also be bounded. It then follows that A is a symmetrisable operator for which there exists a sequence x_n with $\|x_n\| = 1$ such that $(A-\lambda I)x_n \to 0$. Hence $(A-\lambda I)^{-1}$ is unbounded. Further by theorem 10.8 to be proved presently, or by inspection, $\bar{\lambda}$ is not in the point spectrum of A^* so that λ belongs to the continuous spectrum of A.

This completes the construction.

Before leaving the discussion of the continuous spectrum we shall show how Stone's [5] proof for symmetric operators generalises to symmetrisable operators, provided we introduce severe restrictions about the symmetrising operator.

THEOREM 10.7. *The operator* $(A-\lambda I)^{-1}$ *is bounded if* $|\mathscr{I}(\lambda)| > 0$ *provided* $\sqrt{H}$ and $(\sqrt{H})^{-1}$ *is bounded.* This theorem is actually contained in theorem 9.1.).

The proof is as follows: If x belongs to the range of $A_\lambda = A-\lambda I$ it belongs to the domain of A_λ^{-1} and H (a priori if we assume $\mathfrak{D}_H \supset \mathfrak{D}_A$) then

$$(HAA_\lambda^{-1}xA_\lambda^{-1}x) = (HA_\lambda^{-1}x, AA_\lambda^{-1}x)$$
$$(Hx+\lambda HA_\lambda^{-1}x, A_\lambda^{-1}x) = (HA_\lambda^{-1}x, x+\lambda A_\lambda^{-1}x)$$
$$(\lambda-\bar{\lambda})(HA_\lambda^{-1}x, Ax) = -(Hx, A_\lambda^{-1}x)+(HA_\lambda^{-1}x, x)$$
$$2\mathscr{I}(\lambda)(HA_\lambda^{-1}x, A_\lambda^{-1}x) \leqq 2|(HA_\lambda^{-1}x, x)|$$
$$\leqq 2\|\sqrt{H}\,A_\lambda^{-1}x\|\,\|\sqrt{H}x\|$$
$$\|\sqrt{H}\,A_\lambda^{-1}x\| \leqq \frac{\|\sqrt{H}x\|}{|\mathscr{I}(\lambda)|}.$$

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It is clear that this only proves the stated result if $\sqrt{H}$ has strictly positive upper and lower bounds.

We now turn to the residual spectrum. First a lemma.

LEMMA 10.3. *If y is in the range of H then it is not orthogonal to the range of A_λ* if $\mathscr{I}(\lambda) \neq 0$.

PROOF. If for all $x \in \mathfrak{D}_A$ and some $y = Hz$

$$(A_\lambda x, Hz) = 0$$

then

$$(HAx, z) = \lambda(Hx, z).$$

If $z \in \mathfrak{D}_A$ then we put $x = z$ and

$$(HAz, z) = \lambda(Hz, z)$$

which is impossible unless λ real since H and HA are symmetric. If $z \notin \mathfrak{D}_A$ then

$$(HAx, z) = (x, \bar{\lambda}Hz)$$

so that $(HA)^* \neq HA$, i.e. HA not self-adjoint.

Next we prove

THEOREM 10.8. *If A is symmetrised by a bounded positive H, or in any case if $\tilde{A}^+ = A^*$ and A is symmetrisable the eigenvalues of A^* are real. A complex λ cannot belong to the residual spectrum of A.*

PROOF. We suppose the theorem false. Let λ be an eigenvalue of A^* with $\mathscr{I}(\lambda) \neq 0$ and y be the corresponding characteristic element with $||y|| = 1$, say. By Lemma 10.3 y does not belong to $\mathfrak{R}_H$. However, by Theorem 8.2. and Remark 8.1., or by assumption, $\tilde{A}^+ = A^*$ and there exists a sequence (x_n) such that each $x_n \in \mathfrak{D}_A$ and

$$Hx_n \to y, \ A^+Hx_n \to \lambda y.$$

Now for all $z \in \mathfrak{H}$

$$(A^*y - \lambda y, z) = 0$$

so that by the continuity of the linear functional

$$((A^* - \lambda I)Hx_n, z) = (H(A - \lambda I)x_n, z)$$

tends to 0 as n tends to infinity for all $z \in \mathfrak{H}$ and in particular for all $z \in \mathfrak{D}_H$. Hence for all $z \in \mathfrak{D}_H$

$$\lim_{n\to\infty} ((A - \lambda I)x_n, Hz) = 0 \tag{10.1}$$

and since $H(\mathfrak{D}_H)$ is dense in $\mathfrak{H}$ this means $(A - \lambda I)x_n$ tends weakly to 0.

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It follows that $||(A-\lambda I)x_n|| \leqq \alpha$ for all n and some positive α (cf. S. Banach [6]). Further

$$\begin{aligned}|((A-\lambda I)x_n, Hx_n)| &\leqq |((A-\lambda I)x_n, H(x_n-x_m))|+|((A-\lambda I)x_r, Hx_m)| \\ &\leqq \alpha||Hx_n-Hx_m||+|((A-\lambda I)x_n, Hx_m)|.\end{aligned}$$

Since $Hx_n \to y$ there exists for every $\varepsilon > 0$ an n_0 such that $\alpha||Hx_n-Hx_m|| < \frac{1}{2}\varepsilon$ provided only m, $n \geqq n_0$. Also by (10.1) we see that once an $m \geqq n_0$ has been chosen the term $|((A-\lambda I)x_n, Hx_m)| < \frac{1}{2}\varepsilon$ for all $n \geqq n_1$. Thus for $n \geqq \sup (n_1, n_0)$

$$|((A-\lambda I)x_n, Hx_n)| \leqq \varepsilon.$$

Taking the imaginary part of the inner product on the left hand side and using the self-adjointness of HA we obtain

$$|\mathscr{I}(\lambda)|(Hx_n, x_n) \leqq \varepsilon.$$

Since ε is arbitrary this implies $\lim_{n\to\infty} \sqrt{H}x_n = 0$. But this is impossible since $\sqrt{H}(\sqrt{H}\, x_n) \to y$ and $\sqrt{H}$ is closed single valued. We conclude that $\mathscr{I}(\lambda) = 0$.

The last statement in the theorem is an immediate consequence of the preceding. For, as is well known (Cf. Stone [4] Theorem 4.15), if λ belongs to the residual spectrum of A then $\bar{\lambda}$ belongs to the point spectrum of A^*; hence $\mathscr{I}(\bar{\lambda}) = -\mathscr{I}(\lambda) = 0$.

By making specific assumptions we can prove a stronger result

THEOREM 10.9. *If A is symmetrisable by H and A^* is such that there exists a non-negative definite self-adjoint K such that $\mathfrak{D}_K \supset \mathfrak{R}_{A^*}$, $\mathfrak{R}_{A^*} \supset \mathfrak{R}_K$ and KA^* is symmetric, then the residual spectrum is confined to $\lambda = 0$ at most. If K is positive then the residual spectrum is empty and if λ is an unrepeated eigenvalue then the corresponding eigenvectors of A^* belong to $\mathfrak{R}_{H^*}$.*

PROOF. By theorem 10.1. since KA^* is symmetric, the eigenvalues of A^* are real and hence the residual spectrum of A, if it exists, must be real. If λ is real, "λ in the residual spectrum of A" implies "λ in the point spectrum of A^*". Since KA^* is symmetric we have for any eigenvector x_1 of A^*.

$$KA^*x_1 = \lambda Kx_1$$

and

$$KA^*x_1 = A^{**}Kx_1$$

because $x_1 \in \mathfrak{R}_{A^*} \subset \mathfrak{D}_K$.

(Suppose $KA^* \neq A^{**}K$ for some $y \in \mathfrak{D}_{A^*} \cap \mathfrak{D}_K$. For all $x \in \mathfrak{D}_{A^*}$, $(KA^*x, y) = (x, KA^*y)$ and $(A^*x, Ky) = (xA^{**}Ky)$, so that clearly $KA^* = A^{**}K$ on $\mathfrak{D}_{A^*} \cap \mathfrak{D}_K$.)

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Hence of $A^{**} = A$ (i.e. if $\mathfrak{R}_{\bar{A}} \subset \mathfrak{D}_H$) Kx_1 is an eigenvector of A and λ belongs to the point spectrum of A unless $Kx_1 = 0$.

If λ is an unrepeated eigenvalue and K positive then $x_1 \in \mathfrak{R}_H$. For HKx_1 is an eigenvector of A^* and $KHKx_1$ another characteristic element of A. By assumption $KHKx_1 = \alpha Kx_1$ and since K is positive $HKx_1 = \alpha x_1 \neq 0$ by theorem 10.2.

Finally we have a very simple result, which unfortunately requires very strong hypotheses.

THEOREM 10.10. *If A is symmetrisable by H and if $A^+ = A^*$, i.e. if $\mathfrak{D}_{A^*} \subset \mathfrak{R}_H$ then the residual spectrum is empty.*

Suppose λ belongs to the residual spectrum of A, then

$$(A^* - \bar{\lambda} I)x = 0$$

which implies

$$(A^* - \bar{\lambda} I)Hy = 0$$

for some y. Hence

$$H(A - \bar{\lambda} I)y = 0$$

which implies

$$(A - \bar{\lambda} I)y = 0.$$

Hence by theorem 10.1 $\bar{\lambda}$ is real. Hence $\lambda = \bar{\lambda}$ and λ belongs to the point spectrum.

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