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$W^{2,p}_{\omega}$-Solvability of the Cauchy–Dirichlet Problem for Nondivergence Parabolic Equations with BMO Coefficients

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Abstract. In this paper, we establish the regularity of strong solutions to nondivergence parabolic equations with BMO coefficients in nondoubling weighted spaces.

1 Introduction

Barmanti and Cerutti [4] studied the global regularity of the strong solution to the following Dirichlet problem on the second-order parabolic equation in nondivergence form:

$$(1.1)\qquad \begin{cases} Lu = u_t - \sum_{i,j=1}^n a_{ij}(x)u_{x_i'x_j'} = f \quad \text{a.e. in } \Omega_T, \\ u(x) = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x',0) = 0 \text{ in } \Omega, \end{cases}$$

where $x = (x', t) = (x_1', \dots, x_n', t) \in \mathbb{R}^{n+1}$ and $\Omega_T = \Omega \times (0, T)$ (Ω is a bounded domain $C^{1,1}$ of $\mathbb{R}^n$); the coefficients $\{a_{ij}\}_{i,j=1}^n$ of L are symmetric and uniformly elliptic, *i.e.*, for some $\nu \geq 1$ and every $\xi \in \mathbb{R}^n$,

$$(1.2)\qquad a_{ij}(x) = a_{ji}(x) \quad \text{and} \quad \nu^{-1}|\xi|^2 \leq \sum_{i,j=1}^n a_{ji}(x)\xi_i\xi_j \leq \nu|\xi|^2$$

with a.e. $x \in \Omega_T$ and $a_{ij} \in$ VMO (the space VMO, introduced by Sarason in [12], is the space of the functions in the John–Nirenberg space BMO whose BMO norm over a ball vanishes as the radius of the ball tends to zero). A different approach to divergence form parabolic equations with BMO coefficients was developed by Byun [3].

On the other hand, the weighted theory always plays an important role in partial differential equations; see [7, 10, 17]. In this paper, we are interested in global regularity in nondoubling weighted spaces of strong solutions to nondivergent parabolic equations with parameter $\lambda \geq 0$ defined by

$$(1.3)\qquad L_\lambda u = u_t - \sum_{i,j=1}^n a_{ij}(x)u_{x_i'x_j'} + \lambda u = f \quad \textit{a.e. in } \Omega_T.$$

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It is worth pointing out that Krylov [14–16] established the L^p theory of parabolic equations with parameter $\lambda \geq 0$. It also should be pointed out that we obtain non-doubling weight results, which are not only new, but also generalize some well-known results in some sense; see [4, 17].

The paper is organized as follows: Section 2 contains some definitions and lemmas. In Section 3, we obtain weighted interior estimates for the solution to (1.3). The boundary weighted estimates for the solution to (1.3) are obtained in Section 4. By a standard procedure, we can obtain its global weighted estimates in Section 5.

2 Some Definitions and Lemmas

Throughout this paper we will use $x, y, \dots$ to indicate points in $\mathbb{R}^{n+1}$ and $x', y', \dots$ for points in $\mathbb{R}^n$ corresponding to the first n coordinates, *i.e.,* we will write $x = (x', t) = (x'_1, \dots, x'_n, t)$.

The parabolic distance was introduced by Fabes and Riviere [8]:

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + t^2}}{2}}, \quad d(x, y) = \rho(x - y).$$

A ball with respect to the metric d centered at $x_0 = (x'_0, t_0)$ and of radius r is simply an ellipsoid

$$B(x_0, r) = \Big\{ x \in \mathbb{R}^{n+1} : \frac{|x' - x'_0|^2}{r^2} + \frac{(t - t_0)^2}{r^4} < 1 \Big\}.$$

Obviously, the unit sphere with respect to this metric coincides with the sphere in $\mathbb{R}^{n+1}$, a.e. $\partial B(0, 1) = S^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| = (\sum_{i=1}^n x_i'^2 + t^2)^{\frac{1}{2}} = 1\}$.

Let us first recall the definitions and some properties of BMO and VMO spaces. We say that $f \in L^1_{\text{loc}}$ is in the space $\text{BMO}(\mathbb{R}^{n+1})$ if the BMO seminorm

$$\|f\|_* = \sup_B \frac{1}{|B|} \int_B |f(y) - f_B| \, dy < \infty,$$

where B ranges over all balls in $\mathbb{R}^n$ with radius r, and centered at some point x and $f_B = \frac{1}{|B|} \int_B f(y) \, dy$. Then $\|f\|_*$ is a norm in BMO modulo constant functions under which BMO is a Banach space.

For $f \in \text{BMO}$ and $r > 0$ define the VMO modulus of f

$$\eta_f(r) = \sup_{\rho \leq r} \frac{1}{|B_\rho|} \int_{B_\rho} |f(y) - f_{B_\rho}| \, dy,$$

where B_ρ ranges over all balls in $\mathbb{R}^n$ with radius ρ. We say that $f \in \text{BMO}$ is in the space $\text{VMO}(\mathbb{R}^{n+1})$ if $\eta_f(r) \to 0$ as $r \to 0$.

The space $\text{BMO}(\Omega)$ and $\text{VMO}(\Omega)$ can be defined by taking $B \cap \Omega$ and $B_\rho \cap \Omega$ instead of B and B_ρ in the definition of $\|f\|_*$ and $\eta_f(r)$.

Having a function f defined in Ω and belonging to $\text{BMO}(\Omega)$, if Ω is a bounded Lipschitz domain, we may then extend it to $\mathbb{R}^{n+1}$ and the BMO norm of the extension could be estimated by the BMO norm of the original function; see [11]. If, in addition, $f \in \mathrm{VMO}(\Omega)$, then we can extend it preserving its VMO-modulus, as follows by the results of [18].

A weight will always mean a positive function which is locally integrable. We say that a weight ω belongs to the Muckenhoupt class A_p for $1 < p < \infty$ (see [9]), if there is a constant C such that for all balls $B = B(x, r)$

$$\Big(\frac{1}{|B|}\int_B \omega(y)\,dy\Big)\Big(\frac{1}{|B|}\int_B \omega^{-\frac{1}{p-1}}(y)\,dy\Big)^{p-1} \le C.$$

Definition 2.1 A function $k(x)$ is said to be a parabolic Calderón–Zygmund (PCZ) kernel in the space $\mathbb{R}^{n+1}$ if

(i) k is smooth on $\mathbb{R}^{n+1} \setminus \{0\}$;
(ii) $k(rx', r^2 t) = r^{-(n+2)} k(x', t)$ for each $r > 0$;
(iii) $\int_{\rho(x)=r} k(x)\,d\sigma(x) = 0$ for each $r > 0$.

A parabolic Calderón–Zygmund operator T with PCZ kernel k is defined by

$$Tf(x) = \mathrm{pv}\int k(x-y) f(y)\,dy.$$

We define the commutator of T by $[T, a]f = aTf - T(af)$. We have the following well-known result.

Lemma 2.2 ([8]) *Let $a \in \mathrm{BMO}$, $1 < p < \infty$, and $\omega \in A_p$. If T is a parabolic Calderón–Zygmund operator, then there exists a positive constant C such that*

$$\|Tf\|_{p,\omega} \le C\|f\|_{p,\omega} \quad \textit{and} \quad \|[T, a]f\|_{p,\omega} \le C\|a\|_*\|f\|_{p,\omega}.$$

Definition 2.3 A function $k(x, y)$ is said to be a variable parabolic Calderón–Zygmund kernel in the space $\mathbb{R}^{n+1}$ if

(i) $k(x, \cdot)$ is PCZ kernel a.e. $x \in \mathbb{R}^{n+1}$;
(ii) $\sup_{y\in S^{n+1}} |(\frac{\partial}{\partial y})^\beta k(x, y)| \le C_\beta$ independent of x.

A variable parabolic Calderón–Zygmund operator T is defined by

$$Tf(x) = p.v\int k(x, x-y) f(y)\,dy.$$

Similarly, we define the commutator of T by $[T, a]f = aTf - T(af)$. Using the same spherical harmonic expansion as in [4], by Lemma 2.2 we have the following lemma.

Lemma 2.4 *Let $a \in \mathrm{BMO}$, $1 < p < \infty$, and $\omega \in A_p$. If T is a variable parabolic Calderón–Zygmund operator, then there exists a positive constant C such that*

$$\|Tf\|_{p,\omega} \le C\|f\|_{p,\omega} \quad \textit{and} \quad \|[T, a]f\|_{p,\omega} \le C\|a\|_*\|f\|_{p,\omega}.$$

Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1} \cap \{x'_n \geq 0\}$, $\mathbb{R}^{n+1}_- = \mathbb{R}^{n+1} \cap \{x'_n \leq 0\}$,

$$C_t = \{u \in C_0^\infty(\mathcal{A}) : u(x', 0) = 0, \text{ with } \mathcal{A} = \mathbb{R}^{n+1} \cap \{t \geq 0\}\},$$

$$C_{t,x'} = \{u \in C_0^\infty(\mathcal{B}) : u(x', 0) = 0, f \text{ for } t = 0 \vee x'_n = 0, \text{ with } \mathcal{B} = \mathcal{A} \cap \mathbb{R}^{n+1}_+\}.$$

Note that $\mathcal{A}$ and $\mathcal{B}$ being closed, functions in the above spaces do not need to have derivatives vanishing at the boundary.

Let us now turn to equation (1.3). Throughout the paper the coefficient $a_{ij}(x)$ will satisfy (1.2) a.e. in a smooth cylinder $Q_T \subset \mathbb{R}^{n+1}$ and will belong to $L^\infty(Q_T)$.

Now for fixed $x_0 \in Q_T$, consider the constant coefficients operator with parameter $\lambda \geq 0$

$$L^0_\lambda u = u_t - \sum_{i,j=1}^n a_{ij}(x_0) u_{x'_i x'_j} + \lambda u$$

obtained by L_λ freezing the coefficients at x_0. It is easy to see that the fundamental solution of the operator L^0_λ is given by the formula

$$\gamma^0_\lambda(y) = \frac{1}{(4\pi\tau)^{n/2}\sqrt{\det a_0}} \exp\left\{ -\frac{\sum_{i,j=1}^n A_0^{ij} y'_i y'_j}{4\tau} - \lambda\tau \right\}$$

for $\tau > 0$, zero otherwise, where $a_0 = \{a_{ij}(x_0)\}$ is the matrix of the coefficients of L^0_λ and $A_0 = \{A_0^{ij}\} = a_0^{-1}$ is its inverse matrix. Hereafter we denote by $D_i\gamma^0_\lambda$ and $D_{ij}\gamma^0_\lambda$ the derivatives $\partial\gamma^0_\lambda/\partial y_i$ and $\partial^2\gamma^0_\lambda/\partial y_j \partial y_i$. In (1.3), the coefficients of the operator L_λ depend on x. To express this dependence in the fundamental solution we define

$$\gamma^x_\lambda(y) = \frac{1}{(4\pi\tau)^{n/2}\sqrt{\det a(x')}} \exp\left\{ -\frac{\sum_{i,j=1}^n A^{ij}(x) y'_i y'_j}{4\tau} - \lambda\tau \right\}$$

for $\tau > 0$, zero otherwise, where $a(x) = \{a^{ij}(x)\}$ is the matrix of the coefficients of L and $A(x) = \{A^{ij}\} = a^{-1}(x)$ is its inverse matrix.

We remark that $D_{ij}\gamma^0_0(y)$ is a parabolic Calderón–Zygmund kernel.

For the parabolic equation with parameter $\lambda \geq 0$, we have the following result.

Lemma 2.5 *Let $\lambda \geq 0$. Suppose $\gamma^x_\lambda(x - y)$ is the fundamental solution to*

$$\partial_t u - \sum_{i,j=1}^n a_{ij}(x) D_{ij} u + \lambda u = 0$$

in $\mathbb{R}^{n+1}$. Let $u \in C_t$. Then for $x \in \operatorname{supp} u$, the interior representation formula

$$u_{x'_i x'_j}(x) = p.v. \int_{\mathbb{R}^{n+1}} D_{ij}\gamma^x_\lambda(x-y) \Big\{ \sum_{h,k=1}^n [a_{hk}(x) - a_{hk}(y)] u_{y'_i y'_j}(y) + L_\lambda(y) \Big\}\, dy$$
$$+ L_\lambda u \int_{S^{n+1}} D_i \gamma^x_\lambda(y) n_i \, d\sigma(y)$$

holds, and

$$|\gamma^x_\lambda(x-y)| \le \frac{C_N}{(1+|\lambda|^{1/2}\rho(x-y))^N}\frac{1}{\rho(x-y)^n},$$
$$|D_i\gamma^x_\lambda(x-y)| \le \frac{C_N}{(1+|\lambda|^{1/2}\rho(x-y))^N}\frac{1}{\rho(x-y)^{n+1}},$$
$$\left|D_{ij}\gamma^x_\lambda(x-y)\right| \le \frac{C_N}{(1+|\lambda|^{1/2}\rho(x-y))^N}\frac{1}{\rho(x-y)^{n+2}}$$

hold for any $N>0$, where C_N is a constant depending only on n, N, and ν in (1.2).

We also give the following boundary representation formula.

Lemma 2.6 *Let $u \in C_{t,x'}$. Then for x in the support of u, the following holds:*

$$\begin{aligned} u_{x'_i x'_j}(x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^{n+1}_+\cap\{\rho(x-y)>\epsilon\}} & D_{ij}\gamma^x_\lambda(x-y) \\ & \times\Big\{\sum_{h,k=1}^n [a_{hk}(x)-a_{hk}(y)]u_{y'_i y'_j}(y) + L_\lambda(y)\Big\}\,dy \\ + L_\lambda u\int_{S^{n+1}} & D_i\gamma^x_\lambda(y)n_i d\sigma(y) - I^\lambda_{ij}(x),\end{aligned}$$

where

$$\begin{aligned} I^\lambda_{ij}(x) = \int_{\mathbb{R}^{n+1}_+\cap\{\rho(x-y)>\epsilon\}} & D_{ij}\gamma^x_\lambda(T(x)-y) \\ & \times\Big\{\sum_{h,k=1}^n [a_{hk}(x)-a_{hk}(y)]u_{y'_i y'_j}(y) + L_\lambda(y)\Big\}\,dy \quad \textit{for } i,j=1,\cdots,n-1,\end{aligned}$$

$$I^\lambda_{in}(x) = I^\lambda_{ni}(x) = \int_{\mathbb{R}^{n+1}_+}\sum_{l=1}^n B_l(x)D_{il}\gamma^x_\lambda(T(x)-y)\{\cdots\}dy, \quad \textit{for } i,j=1,\dots,n-1,$$

$$I^\lambda_{nn} = I^\lambda_{ni}(x) = \int_{\mathbb{R}^{n+1}_+}\sum_{l,r=1}^n B_l(x)B_r(x)D_{lr}\gamma^x_\lambda(T(x)-y)\{\cdots\}\,dy,$$

where $T(x) = x - 2x'_n\frac{a(x)}{a_{nn}(x)}$ and $a(x)$ is the $(n+1)$-dimensional vector

$$[a_{1,n}(x),\dots,a_{nn}(x),0]$$

constructed with the coefficients of L_λ, where $B_i(x)$ is the i-th component of the vector $B(x) = T(e_n;x)$ and n_i is the i-th component of the outer normal to the surface S^{n+1}. (The expression between braces is always the same).

Adapting the same arguments of [4], we can prove Lemmas 2.4 and 2.5. We omit the details here.

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3 Weighted Interior Estimates

Let $W^{2,p}_\omega(\mathbb{R}^{n+1})$ be the Sobolev space of functions u such that

$$u, u_t, u_{x_i'}, u_{x_i'x_j'} \in L^p_\omega(\mathbb{R}^{n+1}).$$

Obviously, $C_t \subset W^{2,p}_\omega(\mathbb{R}^{n+1})$.

Theorem 3.1 *Let $1 < p < \infty$, $\lambda \geq 0$, and $\omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^\gamma \mu(x)$ with $x_0 \in \mathbb{R}^{n+1}$, $\gamma \in \mathbb{R}$, and $\mu \in A_p$. Then there exist positive constants η and C independent of λ and x_0 such that for all $u \in C_t$,*

$$\sum_{|\nu|\leq 2} \|\lambda^{1-|\nu|/2} D^\nu u\|_{p,\omega} + \|u_t\|_{p,\omega} \leq C\|(\lambda + \partial_t - A)u\|_{p,\omega},$$

provided that $\|a\|_* := \max_{|\beta|=2}\{\|a_\beta\|_*\} \leq \eta$, where $A = \sum_{i,j=1}^n a_{ij}(\,\cdot\,)D_{ij}$.

Remark 1 If $\lambda > 0$ and $\gamma \leq -(n + 2)$, then $\omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^\gamma$ is a nondoubling weight.

To prove Theorem 3.1, we need the following several results.

Lemma 3.2 ([8]) *Let $h \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ be a homogeneous function of degree 0. Set $Tf = \mathcal{F}^{-1}k\mathcal{F}f$ for $f \in L^2(\mathbb{R}^{n+1})$. Then there exists a homogeneous function $k \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ of degree $-n - 2$ with $\int_{S^{n+1}} k(x)\,d\sigma(x) = 0$ such that*

$$Tf(x) = cf(x) + \lim_{\epsilon\to 0} \int_{\rho(y)>\epsilon} k(y)f(x - y)\,dy.$$

Moreover, c and the derivatives of k can be estimated in terms depending on k only.

Lemma 3.3 *Let $u \in C_0^\infty(\mathbb{R}^{n+1})$. If $|\nu| < 2$, then*

$$D^\nu u(x) = \int_{\mathbb{R}^{n+1}} D^\nu \gamma_\lambda^x(x - y)(\lambda + \partial_t - A_x)u(y)\,dy.$$

Moreover, for $|\nu| = 2$,

$$\begin{aligned} D^\nu u(x) = \mathrm{pv}\int_{\mathbb{R}^{n+1}} &k_\nu(x, x - y)(\lambda + \partial_t - A_x)u(y)\,dy + c_\nu(x)(\lambda + \partial_t - A_x)u(x) \\ &- \int_{\mathbb{R}^{n+1}} \lambda\gamma_\lambda^x(x - y)\left(\mathrm{pv}\int_{\mathbb{R}^{n+1}} k_\nu(x, x - y)(\lambda + \partial_t - A_x)u(z)dz\right)dy \\ &- c_\nu(x)\int_{\mathbb{R}^{n+1}} \gamma_\lambda^x(x - y)(\lambda + \partial_t - A_x)u(y)\,dy. \end{aligned}$$

Here $c_\nu \in L^\infty(\mathbb{R}^{n+1})$ and k_ν is a variable parabolic Calderón–Zygmund kernel.

Proof First note that $\mathcal{F}u = (\lambda + i\tau + A_{x_0}(\xi))^{-1}\mathcal{F}(\lambda + \partial_t + A_{x_0})u$ holds for arbitrary $x_0 \in \mathbb{R}^{n+1}$. The first part of the assumption follows easily by setting $x_0 = x$ since, by Lemma 2.5, $D^\nu\gamma_\lambda^{x_0}$ is in $L^1(\mathbb{R}^{n+1})$ for every $|\nu| < 2$.

Hence, we turn to the case where $|\nu| = 2$ and see that

$$\begin{aligned}
\mathcal{F}D^\nu u &= \xi^\nu(\lambda + i\tau - A_{x_0}(\xi))^{-1}\mathcal{F}(\lambda + \partial_t - A_{x_0})u \\
&= \xi^\nu(i\tau - A_{x_0}(\xi))^{-1}\mathcal{F}(\lambda + \partial_t - A_{x_0})u \\
&\qquad + (\xi^\nu(\lambda + i\tau - A_{x_0}(\xi))^{-1} - \xi^\nu(i\tau - A_{x_0}(\xi))^{-1})\mathcal{F}(\lambda + \partial_t - A_{x_0})u \\
&= \xi^\nu(i\tau - A_{x_0}(\xi))^{-1}\mathcal{F}(\lambda + \partial_t - A_{x_0})u \\
&\qquad - \lambda(\lambda + i\tau - A_{x_0}(\xi))^{-1}\xi^\nu(i\tau - A_{x_0}(\xi))^{-1})\mathcal{F}(\lambda + \partial_t - A_{x_0})u.
\end{aligned}$$

Applying $\mathcal{F}^{-1}$ and using Lemma 3.2, we get for all $x \in \mathbb{R}^{n+1}$ that

$$\begin{aligned}
D^\nu u(x) = \operatorname{pv}\int_{\mathbb{R}^{n+1}} & k_\nu(x_0, x - y)(\lambda + \partial_t - A_{x_0})u(y)\,dy + c_\nu(x_0)(\lambda + \partial_t - A_{x_0})u(x) \\
& - \int_{\mathbb{R}^{n+1}} \lambda\gamma_\lambda^{x_0}(x - y)\Big(\operatorname{pv}\int_{\mathbb{R}^{n+1}} k_\nu(x_0, y - z)(\lambda + \partial_t - A_{x_0})u(z)\,dz\Big)\,dy \\
& - \int_{\mathbb{R}^{n+1}} \lambda\gamma_\lambda^{x_0}(x - y)c_\nu(x_0)(\lambda + i\tau - A_{x_0})u(y)\,dy.
\end{aligned}$$

Setting $x_0 = x$ now yields the claim. ∎

For $f \in \mathcal{S}$ we set

$$\begin{aligned}
T_1 f(x) &:= \operatorname{pv}\int_{\mathbb{R}^{n+1}} k_\nu(x, x - y)f(y)\,dy, \\
[T_1, a]f(x) &:= \operatorname{pv}\int_{\mathbb{R}^{n+1}} k_\nu(x, x - y)(a(x) - a(y))f(y)\,dy, \\
T_2 f(x) &:= \int_{\mathbb{R}^{n+1}} \lambda\gamma_\lambda^x(x - y)\operatorname{pv}\int_{\mathbb{R}^{n+1}} k_\nu(x, y - z)f(z)\,dzdy, \\
[T_2, a]f(x) &:= \int_{\mathbb{R}^{n+1}} \lambda\gamma_\lambda^x(x - y)\operatorname{pv}\int_{\mathbb{R}^{n+1}} k_\nu(x, y - z)(a(x) - a(z))f(z)\,dzdy,
\end{aligned}$$

for $|\nu| = 2$, and

$$\begin{aligned}
T_3^\nu f(x) &= \int_{\mathbb{R}^{n+1}} \lambda^{\frac{2-|\nu|}{2}} D^\nu\gamma_\lambda^x(x - y)f(y)\,dy, \\
[T_3^\nu, a]f(x) &= \int_{\mathbb{R}^{n+1}} \lambda^{\frac{2-|\nu|}{2}} D^\nu\gamma_\lambda^x(x - y)(a(x) - a(y))f(y)\,dy,
\end{aligned}$$

for $|\nu| < 2$, where a is assumed to be in BMO.

Lemma 3.4 *Let $|\nu| < 2$. Let $1 < p < \infty$ and $\omega \in A_p$. The operators T_2, T_3^ν, and $[T_2, a]$, $[T_3^\nu, a]$, respectively, are bounded in L^p_ω. Furthermore, the norms of these operators can be estimated by*

$$\|T_2\| + \|T_3^\nu\| \le C \quad \text{and} \quad \|[T_2, a]\| + \|[T_3^\nu, a]\| \le C\|a\|_*,$$

where the constant C can be chosen to be A_p-constant.

Proof We consider first T_3^ν. By Lemma 2.5, we have $|T_3^\nu f(x)| \le CMf(x)$, where M denotes the parabolic Hardy–Littlewood maximal operator.

From this, we obtain that

$$\|T_3^\nu f\|_{p,\omega} \le C\|f\|_{p,\omega}, \tag{3.1}$$

where the constant C is A_p-constant and independent of λ. We now turn our attention to T_2. Using the same spherical harmonic expansion as in [4], we obtain that

$$T_2 f(x) = \sum_{m=1}^{\infty}\sum_{k=1}^{g_m} b_{k,m}(x) \int_{\mathbb{R}^{n+1}} \lambda\gamma_\lambda^x(x-y)\, \mathrm{pv}\int_{\mathbb{R}^{n+1}} \frac{Y_{k,m}(y-z)}{\rho(y-z)^{n+2}} f(z)\, dz dy,$$

where $\frac{Y_{k,m}(y-z)}{\rho(y-z)^{n+2}}$ is a parabolic Calderón–Zygmund kernel.

Hence, by Lemma 2.2, we have

$$\|T_2 f\|_{p,\omega} \le \sum_{m=1}^{\infty}\sum_{k=1}^{g_m} \|b_{k,m}\|_\infty \|f\|_{p,w} \le C\|f\|_{p,\omega}, \tag{3.2}$$

where we used the fact that $\sum_{m=1}^\infty \sum_{k=1}^{g_m} \|b_{k,m}\|_\infty \le C$; see [4].

By (3.1) and (3.2), using [2, Theorem 2], we obtain that

$$\|[T_2, a]\|_{p,\omega} + \|[T_3^\nu, a]\|_{p,\omega} \le C\|a\|_*\|f\|_{p,\omega}. \qquad \blacksquare$$

Proposition 3.5 *Let $1 < p < \infty$ and $\omega \in A_p$. Then there exist positive constants η and C such that for all $u \in W^{2,p}_\omega(\mathbb{R}^{n+1})$ and $\lambda \ge 0$*

$$\sum_{|\nu|\le 2} \|\lambda^{1-|\nu|/2} D^\nu u\|_{p,\omega} + \|u_t\|_{p,\omega} \le C\|(\lambda + \partial_t - A)u\|_{p,\omega},$$

provided that $\|a\|_* \le \eta$. The constants C and η only depend on the A_p-constant of ω.

Proof We recall the representation formulas in Lemma 3.3. Let $|\gamma| < 2$; for $u \in C_0^\infty(\mathbb{R}^{n+1})$ we write

$$\begin{aligned}
\lambda^{1-|\gamma|/2} D^\nu u(x) &= \int_{\mathbb{R}^{n+1}} \lambda^{1-|\gamma|/2} D^\nu \gamma_\lambda^x(x-y)((\lambda + \partial_t - A) + (A - A_x))u(y)\, dy \\
&= \int_{\mathbb{R}^{n+1}} \lambda^{1-|\gamma|/2} D^\nu \gamma_\lambda^x(x-y)(\lambda + \partial_t - A)u(y)\, dy \\
&\quad + \int_{\mathbb{R}^{n+1}} \lambda^{1-|\gamma|/2} D^\nu \gamma_\lambda^x(x-y) \sum_{|\beta|=2} (a_\beta(x) - a_\beta(y)) D^\beta u(y)\, dy \\
&= (T_3^\gamma(\lambda + \partial_t - A))u(x) + \sum_{|\beta|=2} ([T_3^\gamma, a_\beta] D^\beta) u(x).
\end{aligned}$$

By Lemma 3.3, we have

$$\|\lambda^{1-|\gamma|/2} D^\nu u\|_{p,\omega} \le C_\nu \|(\lambda + \partial_t - A)u\|_{p,\omega} + C\|a\|_* \sum_{|\beta|=m} \|D^\beta u\|_{p,\omega}. \tag{3.3}$$

Noting that $c_\gamma \in L^\infty$, for $|\nu| = 2$ we get

$$\begin{aligned} D^\nu u(x) = \Big(& T_1(\lambda + \partial_t - A)u + c_\gamma(\lambda + \partial_t - A)u \\ & - T_2(\lambda + \partial_t - A)u - c_\gamma T_3^0(\lambda + \partial_t - A)\Big)(x) \\ & + \sum_{|\beta|=2} ([T_1, a_\beta]D^\beta u - [T_2, a_\beta]D^\beta u - c_\gamma [T_3, a_\beta]D^\beta u)(x). \end{aligned}$$

By Lemmas 2.4 and 3.3, for $|\nu| = 2$ we have

$$\|D^\nu u\|_{p,\omega} \le C_\gamma \|(\lambda + \partial_t - A)u\|_{p,\omega} + C\|a\|_* \sum_{|\beta|=2} \|D^\beta u\|_{p,\omega}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain that

$$\sum_{|\gamma|\le 2} \|\lambda^{1-|\gamma|/2} D^\nu u\|_{p,\omega} \le C\|(\lambda + \partial_t - A))u\|_{p,\omega},$$

if $C\|a\|_* < 1$. ■

Proposition 3.6 *Let $1 < p < \infty$, $p' = p/(p-1)$, $\lambda \ge 0$, and*

$$\omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^{p\gamma_1} \rho(x - x_0)^{p\gamma_2}$$

with $x_0 \in \mathbb{R}^{n+1}$, $\gamma_1 \in \mathbb{R}$, and $\gamma_2 \in (-(n+2)/p, (n+2)/p'))$. Then there exists positive constants C and η such that for all $u \in C_t$

$$\sum_{|\nu|\le 2} \|\lambda^{1-|\nu|/2} D^\nu u\|_{p,\omega} + \|u_t\|_{p,\omega} \le C\|(\lambda + \partial_t - A)u\|_{p,\omega},$$

provided that $\|a\|_* \le \eta$. The constants C and η only depend on p, γ_1, γ_2 and η.

Proof As in the proof of Proposition 3.5, let $|\nu| < 2$. We write for $u \in C_0^\infty(\mathbb{R}^{n+1})$

$$\begin{aligned} \lambda^{1-|\nu|/2} D^\nu u(x) &= (T_3^\nu(\lambda + \partial_t - A))u(x) + \sum_{|\beta|=2} ([T_3^\nu, a_\beta]D^\beta)u(x) \\ &:= \mathrm{I}_1 + \mathrm{I}_2 . \end{aligned}$$

By Lemma 3.4, we have

$$\|\,\mathrm{I}_1\,\|_{p,\omega} \le C_\gamma \|(\lambda + \partial_t - A)u\|_{p,\omega}, \ \|\,\mathrm{I}_2\,\|_{p,\omega} \le C\|a\|_* \sum_{|\beta|=m} \|D^\beta u\|_{p,\omega}. \tag{3.5}$$

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From Lemma 2.5, we write for $u \in C_t$

$$\begin{aligned} u_{x_i'x_j'}(x) &= \mathrm{pv}\int_{\mathbb{R}^{n+1}} D_{ij}\gamma_\lambda^x(x-y)\Big\{\sum_{h,k=1}^{n}[a_{hk}(x)-a_{hk}(y)]u_{y_i'y_j'}(y)\Big\}\,dy \\ &\quad + \mathrm{pv}\int_{\mathbb{R}^{n+1}} D_{ij}\gamma_\lambda^x(x-y)(\lambda+\partial_t-A)u(y)\,dy \\ &\quad + (\lambda+\partial_t-A)u(x)\int_{S^{n+1}} D_i\gamma_\lambda^x(y)n_i d\sigma(y) \\ &:= \mathrm{II}_1+\mathrm{II}_2+\mathrm{II}_3\,. \end{aligned}$$

Obviously, let ω be as in Proposition 3.6. Then we have

$$\|\,\mathrm{II}_3\,\|_{p,\omega} \le C\|(\lambda+\partial_t-A)u\|_{p,\omega}.$$

Note that $\lambda(\lambda+\partial_t-A_x)^{-1}f(x) = \mathrm{pv}\int_{\mathbb{R}^{n+1}}\lambda\gamma_\lambda^x(x-y)f(y)\,dy$. Let $\omega \in A_p$. Then $\|\lambda(\lambda+\partial_t-A_x)^{-1}f\|_{p,\omega} \le C\|f\|_{p,\omega}$, so

$$\|(\partial_t-A_x)(\lambda+\partial_t-A_x)^{-1}f\|_{p,\omega} \le C\|f\|_{p,\omega}.$$

Hence,

$$\|D_{ij}(\lambda+\partial_t-A_x)^{-1}f\|_{p,\omega} \le C\|(\partial_t-A_x)(\lambda+\partial_t-A_x)^{-1}f\|_{p,\omega} \le C\|f\|_{p,\omega}. \tag{3.6}$$

Here we used the fact that $\|D_{ij}(\partial_t-A_x)^{-1}f\|_{p,\omega} \le C\|f\|_{p,\omega}$. Note that

$$D_{ij}(\lambda+\partial_t-A_x)^{-1}f(x) = \mathrm{pv}\int_{\mathbb{R}^{n+1}} D_{ij}\gamma_\lambda^x(x-y)f(y)\,dy.$$

By (3.6) and using [2, Theorem 2], we obtain

$$\|\,\mathrm{II}_1\,\|_{p,\omega} \le C\|a\|_*\sum_{|\beta|=2}\|D^\beta u\|_{p,\omega},\ \ \|\,\mathrm{II}_2\,\|_{p,\omega} \le C\|(\lambda+\partial_t-A)u\|_{p,\omega}. \tag{3.7}$$

For convenience, we define the operator T_ν by for $|\nu| \le 2$

$$Tf(x) = \int_{\mathbb{R}^{n+1}} \lambda^{1-|\nu|/2}D^\nu\gamma_\lambda^x(x-y)f(y)\,dy.$$

To prove Proposition 3.6, it suffices to prove that

$$\|Tf\|_{p,\omega} \le C\|f\|_{p,w}, \tag{3.8}$$

$$\|[T,a]f\|_{p,\omega} \le C\|a\|_*\|f\|_{p,w}, \tag{3.9}$$

where ω is defined in Proposition 3.6.

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From Lemma 2.5, it is easy to see that

$$|\lambda^{1-|\nu|/2} D^\nu \gamma^x_\lambda(x-y)| \le \frac{C_N}{(1+|\lambda|^{1/2}\rho(x-y))^N}\frac{1}{\rho(x-y)^{n+2}} \tag{3.10}$$

holds for any $N > 0$ and C_N is a constant depending only on n, N.

We first prove (3.8), following [13]. Fix a ball $B = B(x_0, r)$. Write

$$f(y) = \sum_{k=-\infty}^{\infty} f(y)\chi_{A_k}(y) := \sum_{k=-\infty}^{\infty} f_k(y),$$

where $A_k = B(x_0, 2^{k+1}r) \setminus B(x_0, 2^k r)$.

Thus,

$$\begin{aligned}
\|Tf\|^p_{L^p_\omega(\mathbb{R}^{n+1})} &\le C\sum_{k=-\infty}^{\infty}(1+\sqrt{\lambda}2^k)^{p\gamma_1}2^{kp\gamma_2}\|\chi_{A_k}Tf\|^p_{L^p(\mathbb{R}^{n+1})}\\
&\le C\sum_{k=-\infty}^{\infty}(1+\sqrt{\lambda}2^k)^{p\gamma_1}2^{kp\gamma_2}\Big(\Big\|\sum_{j=-\infty}^{k-2}\chi_{A_k}T(f_j)\Big\|^p_{L^p(\mathbb{R}^{n+1})}\Big)\\
&\quad + C\sum_{k=-\infty}^{\infty}(1+\sqrt{\lambda}2^k)^{p\gamma_1}2^{kp\gamma_2}\Big(\sum_{j=k-1}^{k+1}\|\chi_{A_k}T(f_j)\|^p_{L^p(\mathbb{R}^{n+1})}\Big)\\
&\quad + C\sum_{k=-\infty}^{\infty}(1+\sqrt{\lambda}2^k)^{p\gamma_1}2^{kp\gamma_2}\Big(\Big\|\sum_{j=k+2}^{\infty}\chi_{A_k}T(f_j)\Big\|^p_{L^p(\mathbb{R}^{n+1})}\Big)\\
&\equiv \mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3\,.
\end{aligned}$$

For E_2, by the $L^p(\mathbb{R}^{n+1})$ boundedness of T (see (3.6) and (3.7)), we have

$$\mathrm{E}_2 \le C\sum_{k=-\infty}^{\infty}(1+\sqrt{\lambda}2^k)^{p\gamma_1}2^{kp\gamma_2}\Big(\sum_{j=k-1}^{k+1}\|T(f_j)\|^p_{L^p(\mathbb{R}^{n+1})}\Big) \le C\|f\|^p_{L^p_\omega(\mathbb{R}^{n+1})}.$$

For E_1, note that when $x \in A_k$, $j \le k-2$, and $y \in A_j$, then $2\rho(y-x_0) \le \rho(x-x_0)$. Therefore, for $x \in A_k$ and any $N > 0$, by (3.10) we have

$$\begin{aligned}
\big|T(f_j)(x)\big| &\le C_N\int_{\mathbb{R}^{n+1}}\frac{|f_j(y)|\,dy}{(1+\rho(x-y))^N\rho(x-y)^{n+2}}\\
&\le C(1+\sqrt{\lambda}2^k)^{-N}2^{-k(n+2)}\int_{A_j}|f(y)|\,dy\\
&\le C(1+\sqrt{\lambda}2^k)^{-N}2^{-k(n+2)}2^{j(n+2)/p'}\|f\chi_{A_j}\|_{L^p(\mathbb{R}^{n+1})},
\end{aligned}$$

where $1/p' + 1/p = 1$.

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From this, and taking $N \geq |\gamma_1|$, by Hölder's inequality, we get

$$\begin{aligned} E_1 &\leq C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p(\gamma_1-N)} 2^{kp\gamma_2} \\ &\quad \times \Big(\sum_{j=-\infty}^{k-2} 2^{-k(n+2)/p'} 2^{j(n+2)/p'} \| f\chi_{A_j} \|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')} (1+\sqrt{\lambda}2^j)^{(\gamma_1-N)} 2^{j\gamma_2} \| f\chi_{A_j} \|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')} (1+\sqrt{\lambda}2^j)^{\gamma_1} 2^{j\gamma_2} \| f\chi_{A_j} \|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')} \| f\chi_{A_j} \|_{L^p_\omega(\mathbb{R}^{n+1})} \Big)^p \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)p(\gamma_2-(n+2)/p')} \| f\chi_{A_j} \|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big) \\ &\quad \times \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)p'(\gamma_2-(n+2)/p')} \Big)^{p/p'} \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)p(\gamma_2-(n+2)/p')} \| f\chi_{A_j} \|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big) \\ &= C \sum_{j=-\infty}^{\infty} \| f\chi_{A_j} \|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big(\sum_{k=j+2}^{\infty} 2^{-(k-j)p(\gamma_2-(n+2)/p')} \Big) \leq C \| f \|^p_{L^p_\omega(\mathbb{R}^{n+1})}, \end{aligned}$$

since $\gamma_2 < (n+2)/p'$.

For E_3, note that when $x \in A_k$, $j \geq k+2$ and $y \in A_j$, then $2\rho(x-x_0) \leq \rho(y-x_0)$. Therefore, for $x \in A_k$,

$$\begin{aligned} |T(f_j)(x)| &\leq C \int_{\mathbb{R}^{n+1}} \frac{|f_j(y)|\,dy}{(1+\rho(x-y))^N \rho(x-y)^{n+2}} \\ &\leq C(1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)} \int_{A_j} |f(y)|\,dy \\ &\leq C(1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)/p} \| f\chi_{A_j} \|_{L^p(\mathbb{R}^{n+1})}, \end{aligned}$$

where $1/p + 1/p' = 1$. From this, taking $N \geq |\gamma_1|$, we obtain

$$\begin{aligned}
E_3 &\leq C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \\
&\quad \times \Big(\sum_{j=k+2}^{\infty} (1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)/p} \|f\chi_{A_j}\|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\
&\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} 2^{(k-j)(\gamma_2+(n+2)/p)} (1+\sqrt{\lambda}2^j)^{\gamma_1} 2^{-j\gamma_2} \|f\chi_{A_j}\|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\
&\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} 2^{-(k-j)(\gamma_2+(n+2)/p)} \|f\chi_{A_j}\|_{L^p_\omega(\mathbb{R}^{n+1})} \Big)^p \\
&\leq C \sum_{j=-\infty}^{\infty} \|f\chi_{A_j}\|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big(\sum_{j=k+2}^{\infty} 2^{(k-j)p'(\gamma_2+(n+2)/p)} \Big)^{p/p'} \\
&\leq C\|f\|^p_{L^p_\omega(\mathbb{R}^{n+1})},
\end{aligned}$$

since $\gamma_2 > -(n+2)/p$.

Hence,

$$\|Tf\|_{L^p_\omega(\mathbb{R}^{n+1})} \leq C\|f\|_{L^p_\omega(\mathbb{R}^{n+1})}.$$

It remains to prove (3.9), similar to the proof of (3.8). Fix a ball $B = B(x_0, r)$. Write

$$f(y) = \sum_{k=-\infty}^{\infty} f(y)\chi_{A_k}(y) := \sum_{k=-\infty}^{\infty} f_k(y),$$

where $A_k = B(x_0, 2^{k+1}r) \setminus B(x_0, 2^k r)$.

Then we have

$$\begin{aligned}
\|[T,a]f\|^p_{L^p_\omega(\mathbb{R}^{n+1})} &\leq C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \|\chi_{A_k}[T,a]f\|^p_{L^p(\mathbb{R}^{n+1})} \\
&\leq C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\Big\| \sum_{j=-\infty}^{k-2} \chi_{A_k}[T,a](f_j) \Big\|^p_{L^p(\mathbb{R}^{n+1})} \Big) \\
&\quad + C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\sum_{j=k-1}^{k+1} \|\chi_{A_k}[T,a](f_j)\|^p_{L^p(\mathbb{R}^{n+1})} \Big) \\
&\quad + C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\Big\| \sum_{j=k+2}^{\infty} \chi_{A_k}[T,a](f_j) \Big\|^p_{L^p(\mathbb{R}^{n+1})} \Big) \\
&\equiv \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3\,.
\end{aligned}$$

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For E_2, by the $L^p(\mathbb{R}^{n+1})$ boundedness of $[T,a]$(see (3.5) and (3.6)), we have

$$F_2 \le C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\sum_{j=k-1}^{k+1} \|[T,a](f_j)\|^p_{L^p(\mathbb{R}^{n+1})} \Big) \le C\|f\|_{L^p_\omega(\mathbb{R}^{n+1})}.$$

For F_1, note that when $x \in A_k$, $j \le k-2$, and $y \in A_j$, then $2\rho(y-x_0) \le \rho(x-x_0)$. Therefore, for $x \in A_k$ and any $N > 0$, by (3.10) we have

$$\begin{aligned}
\big|[T,a](f_j)(x)\big| &\le C \int_{\mathbb{R}^{n+1}} \frac{|a(x)-a(y)|}{(1+\sqrt{\lambda}\rho(x-y))^N \rho(x-y)^{n+2}} |f_j(y)|\,dy \\
&\le C(1+\sqrt{\lambda}2^k)^{-N} 2^{-k(n+2)} \int_{A_j} |a(x)-a(y)||f(y)|\,dy \\
&\quad + C(1+\sqrt{\lambda}2^k)^{-N} 2^{-k(n+2)} |a(x)-a_{B_j}| \int_{A_j} |f(y)|\,dy \\
&\le C(1+\sqrt{\lambda}2^k)^{-N} 2^{-k(n+2)} 2^{j(n+2)/p'} \\
&\qquad \times (\|a\|_* + |a(x)-a_{B_j}|) \|f\chi_{A_j}\|_{L^p(\mathbb{R}^{n+1})},
\end{aligned}$$

where $a_{B_j} = \frac{1}{|B_j|}\int_{B_j} a(y)\,dy$.

From this, and taking $N \ge |\gamma_1|$, we get

$$\begin{aligned}
F_1 &\le C\|a\|_*^p \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p(\gamma_1-N)} 2^{kp\gamma_2} \\
&\quad \times \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(n+2)/p'} (k-j) \|f\chi_{A_j}\|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\
&\le C\|a\|_*^p \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')} (1+\sqrt{\lambda}2^j)^{\gamma_1} \\
&\qquad\qquad \times 2^{j\gamma_2} (k-j) \|f\chi_{A_j}\|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\
&\le C\|a\|_*^p \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')} (k-j) \|f\chi_{A_j}\|_{L^p_\omega(\mathbb{R}^{n+1})} \Big)^p \\
&\le C\|a\|_*^p \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)p(\gamma_2-(n+2)/p')} \|f\chi_{A_j}\|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big) \\
&\quad \times \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)p'(\gamma_2-(n+2)/p')} (k-j)^{p'} \Big)^{p/p'} \\
&\le C\|a\|_*^p \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)p(\gamma_2-(n+2)/p')} \|f\chi_{A_j}\|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big)
\end{aligned}$$

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$$= C\|a\|_*^p \sum_{j=-\infty}^{\infty} \|f\chi_{A_j}\|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big(\sum_{k=j+2}^{\infty} 2^{-(k-j)p(\gamma_2-(n+2)/p')} \Big)$$
$$\leq C\|a\|_*^p \|f\|^p_{L^p_\omega(\mathbb{R}^{n+1})},$$

since $\gamma_2 < (n+2)/p'$.

For F_3, note that when $x \in A_k$, $j \geq k+2$, and $y \in A_j$, then $2\rho(x-x_0) \leq \rho(y-x_0)$. Therefore, for $x \in A_k$,

$$\big|[T,a](f_j)(x)\big| \leq C_N \int_{\mathbb{R}^{n+1}} \frac{|a(x)-a(y)||f_j(y)|\,dy}{(1+\sqrt{\lambda}\rho(x-y))^N \rho(x-y)^{n+2}}$$
$$\leq C(1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)} \int_{A_j} |a(x)-a(y)||f(y)|\,dy$$
$$+ C(1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)} |a(x)-a_{B_j}| \int_{A_j} |f(y)|\,dy$$
$$\leq C(1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)/p} (\|a\|_* + |a(x)-a_{B_j}|) \|f\chi_{A_j}\|_{L^p(\mathbb{R}^{n+1})}.$$

From this, taking $N \geq |\gamma_1|$, we get

$$F_3 \leq C\|a\|_*^p \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2}$$
$$\times \Big(\sum_{j=k+2}^{\infty} (1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)/p} (j-k) \|f\chi_{A_j}\|_{L^p(\mathbb{R}^{n+1})} \Big)^p$$
$$\leq C\|a\|_*^p \sum_{j=-\infty}^{\infty} \|f\chi_{A_j}\|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big(\sum_{j=k+2}^{\infty} 2^{(k-j)(\gamma_2+(n+2)/p)} (j-k)^{p'} \Big)^{p/p'}$$
$$\leq C\|a\|_*^p \|f\|^p_{L^p_\omega(\mathbb{R}^{n+1})},$$

since $\gamma_2 > -(n+2)/p$.

Hence, $\|[T,a]f\|_{L^p_\omega(\mathbb{R}^{n+1})} \leq C\|a\|_* \|f\|_{L^p_\omega(\mathbb{R}^{n+1})}$. ∎

Proof of Theorem 3.1 Since $\omega_1 \in A_p$, there exists $\alpha > 1$ such that $\omega_1^\alpha \in A_p$. So by Proposition 3.5 we have

$$\text{(3.11)} \qquad \sum_{|\nu|\leq 2} \|\lambda^{1-|\nu|/2} D^\nu u\|_{p,\omega_1^\alpha} + \|u_t\|_{p,\omega_1^\alpha} \leq C\|(\lambda+\partial_t - A)u\|_{p,\omega_1^\alpha}.$$

On the other hand, let $\omega_2(x) = (1+\sqrt{\lambda}\rho(x-x_0))^\gamma$ with $\gamma \in \mathbb{R}$. Using Proposition 3.6 with $\gamma_1 = \gamma$ and $\gamma_2 = 0$, we have

$$\text{(3.12)} \qquad \sum_{|\nu|\leq 2} \|\lambda^{1-|\nu|/2} D^\nu u\|_{p,\omega_2} + \|u_t\|_{p,\omega_2} \leq C\|(\lambda+\partial_t - A)u\|_{p,\omega_2}.$$

Interpolating between (3.11) and (3.12), we obtain the desired result. ∎

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4 Boundary Integral Operators Estimates

In this section L^p_ω norms will always be taken over $\mathbb{R}^{n+1}_+$, where

$$\mathbb{R}^{n+1}_+ = \{x = (x', t) : x' = (x'_1, \dots, x'_n) \in \mathbb{R}^n, x'_n > 0\}.$$

For all $x = (x'_1, \dots, x'_n, t)$, $x'_n > 0$ we define $\bar{x} = (x_1, \dots, x_{n-1}, -x_n, t)$. Then there exist two positive constants C_1 and C_2 depending on ν in (1.1) such that

$$C_1 \rho(\bar{x} - y) \le \rho(T(x) - y) \le C_2 \rho(\bar{x} - y) \tag{4.1}$$

for every $x, y \in \mathbb{R}^{n+1}_+$(see [4]), where $T(x)$ is the transformation introduced in Section 1.

Lemma 4.1 *Let* $1 < p < \infty$, $a \in \mathrm{BMO}$ *and* $\omega \in A_p$. *We define*

$$R f(x) = \int_{\mathbb{R}^{n+1}_+} \frac{|f(y)|}{\rho^{n+2}(\bar{x} - y)}\, dy,$$

$$R_a f(x) = \int_{\mathbb{R}^{n+1}_+} \frac{|a(x) - a(y)||f(y)|}{\rho^{n+2}(\bar{x} - y)}\, dy.$$

Then there exits a positive constant C such that

$$\|Rf\|_{p,\omega} \le C\|f\|_{p,\omega} \quad \text{and} \quad \|R_a f\|_{p,\omega} \le C\|a\|_* \|f\|_{p,\omega}.$$

The proof can be found in [4].

Lemma 4.2 *Let* $1 < p < \infty$, $\lambda \ge 0$, *and* $\omega(x) = (1 + \sqrt{\lambda}|x - x_0|)^{p\gamma_1} |x - x_0|^{p\gamma_2}$ *with* $x_0 \in \mathbb{R}^{n+1}$, $\gamma_1 \in \mathbb{R}$, *and* $\gamma_2 \in (-(n+2)/p, (n+2)/p')$. *We define*

$$R_\lambda f(x) = \int_{\mathbb{R}^{n+1}_+} \frac{|f(y)|}{(1 + \sqrt{\lambda}\rho(\bar{x} - y))^N \rho^{n+2}(\bar{x} - y)} dy.$$

Then there exits a positive constant C such that for any $\lambda \ge 0$ $\|R_\lambda f\|_{p,\omega} \le C\|f\|_{p,\omega}$.

Proof Fix a ball $B = B(x_0, r)$. As usual $B_k = B(x_0, 2^k r)$ for any integer k; $B^k_+ = B_k \cap \{x'_n > 0\}$. Then we can write $f(x)$ as

$$f(y) = \sum_{k=-\infty}^{\infty} f(y)\chi_{A^+_k}(y) := \sum_{k=-\infty}^{\infty} f_k(y),$$

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where $A^+_k = B^+_k \setminus B^+_{k-1}$. Then, we have

$$\begin{aligned}
\|R_\lambda f\|^p_{L^p_\omega(\mathbb{R}^{n+1}_+)} &\le C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \|\chi_{A^+_k} R_\lambda f\|^p_{L^p(\mathbb{R}^{n+1}_+)} \\
&\le C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{\gamma_1} 2^{k\gamma_2} \Big(\Big\| \sum_{j=-\infty}^{k-2} \chi_{A^+_k} R_\lambda(f_j) \Big\|^p_{L^p(\mathbb{R}^{n+1}_+)} \Big) \\
&\quad + C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\sum_{j=k-1}^{k+1} \|\chi_{A^+_k} R_\lambda(f_j)\|^p_{L^p(\mathbb{R}^{n+1}_+)} \Big) \\
&\quad + C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\Big\| \sum_{j=k+2}^{\infty} \chi_{A^+_k} R_\lambda(f_j) \Big\|^p_{L^p(\mathbb{R}^{n+1}_+)} \Big) \\
&\equiv \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 .
\end{aligned}$$

For E_2, by the $L^p(\mathbb{R}^{n+1}_+)$ boundedness of R_λ(see Lemma 4.1), we have

$$\mathrm{E}_2 \le C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\sum_{j=k-1}^{k+1} \|R_\lambda(f_j)\|^p_{L^p(\mathbb{R}^{n+1}_+)} \Big) \le C\|f\|^p_{L^p_\omega(\mathbb{R}^{n+1}_+)}.$$

For E_1, note that when $x \in A^+_k$, $j \le k-2$, and $y \in A^+_j$, then $2\rho(y-x_0) \le \rho(x-x_0)$. Therefore, for $x \in A^+_k$ and any $N > 0$, we have

$$\begin{aligned}
\big|R_\lambda(f_j)(x)\big| &\le C_N \int_{\mathbb{R}^{n+1}_+} \frac{1}{(1+\sqrt{\lambda}\rho(x-y))^N \rho(x-y)^{n+2}} |f_j(y)|\,dy \\
&\le C(1+\sqrt{\lambda}2^k)^{-N} 2^{-k(n+2)} \int_{A^+_j} |f(y)|\,dy \\
&\le C(1+\sqrt{\lambda}2^k)^{-N} 2^{-k(n+2)} 2^{j(n+2)/p'} \|f\chi_{A^+_j}\|_{L^p(\mathbb{R}^{n+1})}.
\end{aligned}$$

From this, and taking $N \ge |\gamma_1|$, we get

$$\begin{aligned}
\mathrm{E}_1 &\le C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p(\gamma_1-N)} 2^{kp\gamma_2} \\
&\quad \times \Big(\sum_{j=-\infty}^{k-2} 2^{-k(n+2)/p'} 2^{j(n+2)/p'} \|f\chi_{A^+_j}\|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\
&\le C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')} (1+\sqrt{\lambda}2^j)^{(\gamma_1-N)} 2^{j\gamma_2} \|f\chi_{A^+_j}\|_{L^p(\mathbb{R}^{n+1})} \Big)^p
\end{aligned}$$

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$$\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')} (1+\sqrt{\lambda}2^j)^{\gamma_1} 2^{j\gamma_2} \| f\chi_{A_j^+} \|_{L^p(\mathbb{R}^{n+1})} \Big)^p$$

$$\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')} \| f\chi_{A_j^+} \|_{L^p_\omega(\mathbb{R}^{n+1})} \Big)^p \leq C\|f\|^p_{L^p_\omega(\mathbb{R}^{n+1}_+)},$$

since $\gamma_2 < (n+2)/p'$.

For E_3, note that when $x \in A_k^+$, $j \geq k+2$, and $y \in A_j^+$, then $2\rho(x-x_0) \leq \rho(y-x_0)$. Therefore, for $x \in A_k^+$,

$$\begin{aligned} |R_\lambda(f_j)(x)| &\leq C \int_{\mathbb{R}^{n+1}_+} \frac{|f_j(y)|\, dy}{(1+\sqrt{\lambda}\rho(x-y))^N \rho(x-y)^{n+2}} \\ &\leq C(1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)} \int_{A_j^+} |f(y)|\, dy \\ &\leq C(1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)/p} \| f\chi_{A_j^+} \|_{L^p(\mathbb{R}^{n+1})}. \end{aligned}$$

From this and taking $N \geq |\gamma_1|$, we obtain

$$\begin{aligned} E_3 &\leq C \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \\ &\quad \times \Big(\sum_{j=k+2}^{\infty} (1+\sqrt{\lambda}2^j)^{-N} 2^{-j(n+2)/p} \| f\chi_{A_j^+} \|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} 2^{(k-j)(\gamma_2+(n+2)/p)} (1+\sqrt{\lambda}2^j)^{\gamma_1} 2^{-j\gamma_2} \| f\chi_{A_j^+} \|_{L^p(\mathbb{R}^{n+1})} \Big)^p \\ &\leq C \sum_{k=-\infty}^{\infty} \Big(\sum_{j=k+2}^{\infty} 2^{-(k-j)(\gamma_2+(n+2)/p)} \| f\chi_{A_j^+} \|_{L^p_\omega(\mathbb{R}^{n+1})} \Big)^p \\ &\leq C \sum_{j=-\infty}^{\infty} \| f\chi_{A_j^+} \|^p_{L^p_\omega(\mathbb{R}^{n+1})} \Big(\sum_{j=k+2}^{\infty} 2^{(k-j)p'(\gamma_2+(n+2)/p)} \Big)^{p/p'} \leq C\|f\|^p_{L^p_\omega(\mathbb{R}^{n+1}_+)}, \end{aligned}$$

since $\gamma_2 > -(n+2)/p$.

Hence, $\|R_\lambda f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \leq C\|f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)}$. Thus, the lemma is proved. ∎

For the linear commutator on $\mathbb{R}^{n+1}_+$, we have the following lemma.

Lemma 4.3 *Let $1 < p < \infty$, $a \in BMO$, $\lambda \geq 0$, and*

$$\omega(x) = (1+\sqrt{\lambda}\rho(x-x_0))^{p\gamma_1} \rho(x-x_0)^{p\gamma_2}$$

with $x_0 \in \mathbb{R}^{n+1}$, $\gamma_1 \in \mathbb{R}$, and $\gamma_2 \in (-(n+2)/p, (n+2)/p')$. Then

$$\|[a, R_\lambda]f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \leq C\|a\|_* \|f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)},$$

where the constant C is independent of x_0, f and λ.

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Proof Note that for any $N > 0$

$$[a, R_\lambda]|f|(x) \le T_a|f|(x) := \int_{\mathbb{R}^{n+1}_+} \frac{|a(x) - a(y)|}{(1 + \sqrt{\lambda}\rho(\bar{x} - y))^N \rho^{n+2}(\bar{x} - y)} |f(y)|\, dy.$$

Fix a ball $B = B(x_0, r)$. As usual $B_k = B(x_0, 2^k r)$ for any integer k, $B_+ = B \cap \{x_n > 0\}$. Then we can write $f(x)$ as

$$f(y) = \sum_{k=-\infty}^{\infty} f(y)\chi_{A_k^+}(y) := \sum_{k=-\infty}^{\infty} f_k(y),$$

where $A_k^+ = B_k^+ \setminus B_{k-1}^+$.

Then we have

$$\begin{aligned}
\|T_a f\|^p_{L^p_\omega(\mathbb{R}^{n+1}_+)} &\le C \sum_{k=-\infty}^{\infty} (1 + \sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \|\chi_{A_k^+} T_a f\|^p_{L^p(\mathbb{R}^{n+1}_+)} \\
&\le C \sum_{k=-\infty}^{\infty} (1 + \sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\Big\| \sum_{j=-\infty}^{k-2} \chi_{A_k^+} T_a(f_j) \Big\|^p_{L^p(\mathbb{R}^{n+1}_+)} \Big) \\
&\quad + C \sum_{k=-\infty}^{\infty} (1 + \sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\sum_{j=k-1}^{k+1} \|\chi_{A_k^+} T_a(f_j)\|^p_{L^p(\mathbb{R}^{n+1}_+)} \Big) \\
&\quad + C \sum_{k=-\infty}^{\infty} (1 + \sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\Big\| \sum_{j=-\infty}^{k-2} \chi_{A_k^+} T_a(f_j) \Big\|^p_{L^p(\mathbb{R}^{n+1}_+)} \Big) \\
&\equiv \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3\,.
\end{aligned}$$

For F_2, by the $L^p(\mathbb{R}^{n+1}_+)$ boundedness of T_a (see Lemma 4.1), we have

$$\mathrm{F}_2 \le C \sum_{k=-\infty}^{\infty} (1 + \sqrt{\lambda}2^k)^{p\gamma_1} 2^{kp\gamma_2} \Big(\sum_{j=k-1}^{k+1} \|T_a(f_j)\|^p_{L^p(\mathbb{R}^{n+1}_+)} \Big) \le C\|f\|^p_{L^p_\omega(\mathbb{R}^{n+1}_+)}.$$

For F_1, note that when $x \in A_k$, $j \le k - 2$, and $y \in A_j$, then $2\rho(y - x_0) \le \rho(x - x_0)$. Therefore, for $x \in A_k$ and any $N > 0$, we have

$$\begin{aligned}
\big|T_a(f_j)(x)\big| &\le C \int_{\mathbb{R}^{n+1}_+} \frac{|a(x) - a(y)|}{(1 + \sqrt{\lambda}\rho(x - y))^N \rho(x - y)^{n+2}} |f_j(y)|\, dy \\
&\le C(1 + \sqrt{\lambda}2^k)^{-N} 2^{-k(n+2)} \int_{A_j^+} |a(x) - a(y)||f(y)|\, dy \\
&\quad + C(1 + \sqrt{\lambda}2^k)^{-N} 2^{-k(n+2)} |a(x) - a_{B_j}| \int_{A_j^+} |f(y)|\, dy \\
&\le C(1 + \sqrt{\lambda}2^k)^{-N} 2^{-k(n+2)} 2^{j(n+2)/p'} \\
&\qquad \times (\|a\|_* + |a(x) - a_{B_j}|) \|f\chi_{A_j^+}\|_{L^p(\mathbb{R}^{n+1})},
\end{aligned}$$

where $a_{B_j} = \frac{1}{|B_j|}\int_{B_j} a(y)\,dy$. From this, and taking $N \geq |\gamma_1|$, we get

$$\begin{aligned}
\mathrm{F}_1 &\leq C\|a\|_*^p \sum_{k=-\infty}^{\infty} (1+\sqrt{\lambda}2^k)^{p(\gamma_1-N)} 2^{kp\gamma_2} \\
&\qquad\qquad \times \Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(n+2)/p'}(k-j)\|f\chi_{A_j^+}\|_{L^p(\mathbb{R}^{n+1})}\Big)^p \\
&\leq C\|a\|_*^p \\
&\quad \times \sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')}(1+\sqrt{\lambda}2^j)^{\gamma_1}2^{j\gamma_2}(k-j)\|f\chi_{A_j^+}\|_{L^p(\mathbb{R}^{n+1})}\Big)^p \\
&\leq C\|a\|_*^p \sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2} 2^{-(k-j)(\gamma_2-(n+2)/p')}(k-j)\|f\chi_{A_j^+}\|_{L^p_\omega(\mathbb{R}^{n+1})}\Big)^p \\
&\leq C\|a\|_*^p\|f\|^p_{L^p_\omega(\mathbb{R}^{n+1})},
\end{aligned}$$

since $\gamma_2 < (n+2)/p'$.

For F_3, note that when $x \in A_k$, $j \geq k+2$, and $y \in A_j$, then $2\rho(x-x_0) \leq \rho(y-x_0)$. Therefore, for $x \in A_k$,

$$\begin{aligned}
\big|T_a(f_j)(x)\big| &\leq C_N \int_{\mathbb{R}^{n+1}_+} \frac{|a(x)-a(y)||f_j(y)|\,dy}{(1+\sqrt{\lambda}\rho(x-y))^N\rho(x-y)^{n+2}} \\
&\leq C(1+\sqrt{\lambda}2^j)^{-N}2^{-j(n+2)}\int_{A_j^+}|a(x)-a(y)||f(y)|\,dy \\
&\qquad + C(1+\sqrt{\lambda}2^j)^{-N}2^{-j(n+2)}|a(x)-a_{B_j}|\int_{A_j^+}|f(y)|\,dy \\
&\leq C(1+\sqrt{\lambda}2^j)^{-N}2^{-j(n+2)/p}(\|a\|_* + |a(x)-a_{B_j}|)\|f\chi_{A_j^+}\|_{L^p(\mathbb{R}^{n+1})}.
\end{aligned}$$

From this and taking $N \geq |\gamma_1|$, we get

$$\begin{aligned}
F_3 &\leq C\|a\|_*^p \sum_{k=-\infty}^{\infty}(1+\sqrt{\lambda}2^k)^{p\gamma_1}2^{kp\gamma_2} \\
&\quad \times \Big(\sum_{j=k+2}^{\infty}(1+\sqrt{\lambda}2^j)^{-N}2^{-j(n+2)/p}(j-k)\|f\chi_{A_j^+}\|_{L^p(\mathbb{R}^{n+1})}\Big)^p \\
&\leq C\|a\|_*^p \sum_{j=-\infty}^{\infty}\|f\chi_{A_j^+}\|^p_{L^p_\omega(\mathbb{R}^{n+1})}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)(\gamma_2+(n+2)/p)}(j-k)^{p'}\Big)^{p/p'} \\
&\leq C\|a\|_*^p\|f\|^p_{L^p_\omega(\mathbb{R}^{n+1}_+)},
\end{aligned}$$

since $\gamma_2 > -(n+2)/p$.

Hence, $\|T_a f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \le C\|a\|_* \|f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)}$. Thus, the lemma is proved. ■

By Lemmas 4.1 and 4.2, we can obtain the following result.

Proposition 4.4 *Let* $1 < p < \infty$, $a \in \mathrm{BMO}$, $\lambda \ge 0$, *and*

$$\omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^\gamma \mu(x)$$

with $x_0 \in \mathbb{R}^{n+1}$, $\gamma \in \mathbb{R}$, *and* $\mu(x) \in A_p$. *Then*

$$\|R_\lambda f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \le C\|f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)}$$

$$\|[a, R_\lambda]f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \le C\|a\|_* \|f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)},$$

where the constant C *is independent of* f, x_0 *and* λ.

By (4.1) and Proposition 4.4 and using the same spherical harmonic expansion as in [4], we deduce the following result.

Theorem 4.5 *Let* $1 < p < \infty$, $a \in \mathrm{BMO}$, $\lambda \ge 0$, *and*

$$\omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^\gamma \mu(x)$$

with $x_0 \in \mathbb{R}^{n+1}$, $\gamma \in \mathbb{R}$ *and* $\mu(x) \in A_p$. *If* k *is a variable PCZ kernel, define*

$$\bar{T} f(x) = \int_{\mathbb{R}^{n+1}_+} k_\lambda(x, T(x) - y) f(y)\, dy.$$

Then

$$\|\bar{T} f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \le C\|f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \quad \text{and} \quad \|[a, \bar{T}]f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \le C\|a\|_* \|f\|_{L^p_\omega(\mathbb{R}^{n+1}_+)},$$

where the constant C *is independent of* f, x_0, *and* λ.

Finally, we give the boundary estimate in weighted spaces.

Theorem 4.6 *Let* $1 < p < \infty$, $\lambda \ge 0$, *and* $\omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^\gamma \mu(x)$ *with* $x_0 \in \mathbb{R}^{n+1}$, $\gamma \in \mathbb{R}$ *and* $\mu \in A_p$. *Then there exist positive constants* η *and* C *independent of* λ *and* x_0 *such that for all* $u \in C_{t,x'}$

$$\sum_{|\nu| \le 2} \|\lambda^{1-|\nu|/2} D^\nu u\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} + \|u_t\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \le C\|(\lambda + \partial_t - A)u\|_{L^p_\omega(\mathbb{R}^{n+1}_+)},$$

provided that $\|a\|_* := \max_{|\beta|=2}\{\|a_\beta\|_*\} \le \eta$, *where* $A = \sum_{i,j=1}^n a_{ij}(\,\cdot\,)D_{ij}$.

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Proof From Theorem 4.5, we obtain

$$\|D^2u\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \leq C\|(\lambda + \partial_t - A)u\|_{L^p_\omega(\mathbb{R}^{n+1}_+)}, \tag{4.2}$$

provided that $\|a\|_* := \max_{|\beta|=2}\{\|a_\beta\|_*\} \leq \eta$. Note that the boundary representation of u is as follows

$$u(x) = \int_{\mathbb{R}^{n+1}_+} G^x_\lambda(x-y)L_\lambda u(y)\,dy,$$

where

$$G^x_\lambda(x-y) = \gamma^x_\lambda(x-y) - \gamma^x_\lambda(T(x)-y) \text{ and } L_\lambda u = (\lambda + \partial_t - A)u.$$

Obviously,

$$\lambda|G^x_\lambda(x-y)| \leq \frac{C_N\lambda}{(1+\sqrt{\lambda}\rho(x-y))^N\rho^n(x-y)}, \tag{4.3}$$

$$\sqrt{\lambda}|D_xG^x_\lambda(x-y)| \leq \frac{C_N\sqrt{\lambda}}{(1+\sqrt{\lambda}\rho(x-y))^N\rho^{n+1}(x-y)}. \tag{4.4}$$

By (4.3) and (4.4), for $\mu(x) \in A_p$, we then have

$$\sum_{|\nu|\leq 1}\|\lambda^{1-|\nu|/2}D^\nu u\|_{L^p_\mu(\mathbb{R}^{n+1}_+)} \leq C\|M(L_\lambda u)\|_{L^p_\mu(\mathbb{R}^{n+1}_+)} \leq C\|L_\lambda u\|_{L^p_\mu(\mathbb{R}^{n+1}_+)}. \tag{4.5}$$

On the other hand, it is easy to see that for any $x, y \in \mathbb{R}^{n+1}_+$

$$\lambda|G^x_\lambda(x-y)| + \sqrt{\lambda}|D_xG^x_\lambda(x-y)| \leq \frac{C_N}{(1+\sqrt{\lambda}\rho(x-y))^N\rho^{n+2}(x-y)}. \tag{4.6}$$

Adapting the same arguments in the proof of Lemma 4.2, by (4.6), we have

$$\sum_{|\nu|\leq 1}\|\lambda^{1-|\nu|/2}D^\nu u\|_{L^p_{\omega_1}(\mathbb{R}^{n+1}_+)} \leq C\|L_\lambda u\|_{L^p_{\omega_1}(\mathbb{R}^{n+1}_+)}, \tag{4.7}$$

$\omega_1(x) = (1+\sqrt{\lambda}\rho(x-x_0))^\gamma$ with $x_0 \in \mathbb{R}^{n+1}$ and $\gamma \in \mathbb{R}$. Interpolating (4.5) and (4.7), we obtain

$$\sum_{|\nu|\leq 1}\|\lambda^{1-|\nu|/2}D^\nu u\|_{L^p_\omega(\mathbb{R}^{n+1}_+)} \leq C\|L_\lambda u\|_{L^p_\omega(\mathbb{R}^{n+1}_+)}, \tag{4.8}$$

where $\omega(x) = (1+\sqrt{\lambda}\rho(x-x_0))^\gamma\mu(x)$ with $x_0 \in \mathbb{R}^{n+1}$, $\gamma \in \mathbb{R}$ and $\mu \in A_p$. Finally, we observe that $u_t = L_\lambda u + a_{ij}D_{ij}u + \lambda u$, combining with (4.2), (4.5), and (4.8), we have $\|u_t\|_{L^p_\mu(\mathbb{R}^{n+1}_+)} \leq C\|(\lambda + \partial_t - A)u\|_{L^p_\mu(\mathbb{R}^{n+1}_+)}$. ■

5 The Dirichlet Problem

Let $W^{2,p}_{0,\omega}(\Omega_T)$ be the closure in the $W^{2,p}_\omega$ norm of the space

$$\mathcal{C} = \{\phi \in C^\infty(\Omega_T) : \phi = 0 \text{ for } t = 0 \text{ or } x \in \partial\Omega\}.$$

In this section, we are interested in the Cauchy–Dirichlet problem

$$\begin{cases} L_\lambda u = f \text{ in } \Omega_T, \\ u \in W^{2,p}_{0,\omega}(\Omega_T), \end{cases} \tag{5.1}$$

with $f \in L^p_\omega(Q_T)$, a_{ij}'s satisfying (1.2), $a_{ij} \in BMO(\Omega_T)$ and $\lambda \geq 0$.

We first give the following result concerning interior estimates in weighted spaces.

Theorem 5.1 *Let $1 < p < \infty, 0 \leq \lambda < \infty$, $\omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^\gamma \mu(x)$ with $x_0 \in \mathbb{R}^{n+1}$, $\gamma \in \mathbb{R}$, and $\mu(x) \in A_p$. There exist C and η such that $B_{2r} \Subset \Omega \times \mathbb{R}$, and $u \in W^{2,p}_{0,\omega}(\Omega_T)$. We have*

$$\lambda\|u\|_{L^p_\omega(B_r^+)} + \sqrt{\lambda}\|Du\|_{L^p_\omega(B_r^+)} + \|u_t\|_{L^p_\omega(B_r^+)} + \|D_{ij}u\|_{L^p_\omega(B_r^+)} \\ \leq C\big(\|L_\lambda u\|_{L^p_\omega(B_{2r}^+)} + r^{-1}\|Du\|_{L^p_\omega(B_{2r}^+)} + r^{-2}\|u\|_{L^p_\omega(B_{2r}^+)}\big),$$

provided that $\|a\|_* \leq \eta$, where $B_r^+ = B_r \cap \{t \geq 0\}$ and the constants C and η are independent of x_0, λ and u.

We then give the following result concerning boundary estimates in weighted spaces.

Theorem 5.2 *Let $1 < p < \infty, 0 \leq \lambda < \infty$, $\omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^\gamma \mu(x)$ with $x_0 \in \mathbb{R}^{n+1}$, $\gamma \in \mathbb{R}$, and $\mu(x) \in A_p$. There exist C and η such that for* supp $u \subset B_r$, *and $u \in W^{2,p}_{0,\omega}(\Omega_T)$. We have*

$$\lambda\|u\|_{L^p_\omega(\widetilde{B}_r^+)} + \sqrt{\lambda}\|Du\|_{L^p_\omega(\widetilde{B}_r^+)} + \|u_t\|_{L^p_\omega(\widetilde{B}_r^+)} + \|D_{ij}u\|_{L^p_\omega(\widetilde{B}_r^+)} \\ \leq C\big(\|L_\lambda u\|_{L^p_\omega(\widetilde{B}_{2r}^+)} + r^{-1}\|Du\|_{L^p_\omega(\widetilde{B}_{2r}^+)} + r^{-2}\|u\|_{L^p_\omega(\widetilde{B}_{2r}^+)}\big),$$

provided that $\|a\|_* \leq \eta$, where $\widetilde{B}_r^+ = B_r \cap \{x'_n \geq 0, t \geq 0\}$ and the constants C and η are independent of x_0, λ and u.

Adapting the same arguments in [4], by Theorems 3.1 and 4.6, we can obtain Theorems 5.1 and 5.2, respectively. We omit the details here.

Using Theorems 5.1 and 5.2, we can establish well-posedness of the Dirichlet problem (5.1) in weighted spaces.

Theorem 5.3 *Let* $1 < p < \infty, 0 \le \lambda < \infty, \omega(x) = (1 + \sqrt{\lambda}\rho(x - x_0))^\gamma \mu(x)$ *with* $x_0 \in \mathbb{R}^{n+1}$, $\gamma \in \mathbb{R}$, *and* $\mu(x) \in A_p$. *Then there exists a positive constant* η *and* λ_0 *such that* $\|a\|_* \le \eta$ *and* $\lambda \ge \lambda_0 = \lambda_0(n, p, \gamma, \beta, \mu, \eta, |\Omega|, T)$. *The Cauchy–Dirichlet problem* (5.1) *has a unique solution* $u \in W^{2,p}_\omega(Q_T)$ *for every* $f \in L^p_\omega(\Omega_T)$. *Moreover, there exists a constant* $C = C(n, p, \gamma, \mu, \eta, |\Omega|, T)$ *such that*

$$\lambda\|u\|_{L^p_\omega(\Omega_T)} + \sqrt{\lambda}\|Du\|_{L^p_\omega(\Omega_T)} + \|u_t\|_{L^p_\omega(\Omega_T)} + \|D_{ij}u\|_{L^p_\omega(\Omega_T)} \le C\|f\|_{L^p_\omega(\Omega_T)}.$$

We remark that our proofs could be modified in order to replace the smallness of the BMO norm assumption by the VMO norm; see [4].

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