GENERALIZED MATRIX ALGEBRAS

W. P. BROWN

1. Introduction. The algebras considered here arose in the investigation of an algebra connected with the orthogonal group.[1] We consider an algebra $\mathfrak{A}$ of dimension mn over a field K of characteristic zero, and possessing a basis $\{e_{ij}\}$ $(1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant n)$ with the multiplication property

$$e_{ij}e_{pq} = \phi_{jp}e_{iq}, \qquad \phi_{jp} \in K. \tag{1}$$

The field elements ϕ_{ij} form a matrix $\Phi = (\phi_{ij})$ of order $n \times m$. It will be called the multiplication matrix of the algebra relative to the basis $\{e_{ij}\}$.

Such algebras will be called generalized matrix algebras. If $m = n$ and $\phi_{ij} = \delta_{ij}$ (the Kronecker delta) we have a total matrix algebra.

An element b of $\mathfrak{A}$ has an expression in terms of the basis $\{e_{ij}\}$ of the form

$$b = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}\, e_{ij}, \qquad b_{ij} \in K.$$

The correspondence $b \to B = (b_{ij})$ is a one to one correspondence between the elements of $\mathfrak{A}$ and the set of $m \times n$ matrices over K. If $a \in \mathfrak{A}$ and $a \to A = (a_{ij})$, then

$$\begin{aligned} ab &= \sum_{i,j} a_{ij}e_{ij} \sum_{p,q} b_{pq}e_{pq} \\ &= \sum_{i,q} \Big(\sum_{j,p} a_{ij}\phi_{jp}b_{pq} \Big) e_{iq}. \end{aligned}$$

It follows that the product ab corresponds to the matrix $A\Phi B$; $ab \to A\Phi B$. The most general change of basis of $\mathfrak{A}$ may be effected by a transformation of the type

$$f_{ij} = \sum_{\lambda=1}^{m} \sum_{\mu=1}^{n} \sigma_{ij}^{\lambda\mu}\, e_{\lambda\mu}, \qquad \sigma_{ij}^{\lambda\mu} \in K \tag{2}$$

$(1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant n)$. We use double suffix notation with lexicographic ordering to describe the matrix of this transformation. The element of its (ij)th row and $(\lambda\mu)$th column is $\sigma_{ij}^{\lambda\mu}$. The elements f_{ij} of $\mathfrak{A}$ constitute a basis for $\mathfrak{A}$ if and only if this matrix is non-singular. We consider however a special type of transformation, namely:

$$f_{ij} = \sum_{\lambda=1}^{m} s_{i\lambda}e_{\lambda j}, \qquad s_{i\lambda} \in K. \tag{3}$$

This may be written in the form of (2) by setting $\sigma_{ij}^{\lambda\mu} = s_{i\lambda}\delta_{j\mu}$. This is the element of the (ij)th row and $(\lambda\mu)$th column of the Kronecker product $S \times I$ where S is the $m \times m$ matrix (s_{ij}) and I is the $n \times n$ unit matrix. It follows

Received November 2, 1953.

[1]The algebra concerned is $\omega_f{}^n$. For definition see (1) and (2, chap. V, 5).

https://doi.org/10.4153/CJM-1955-023-2 Published online by Cambridge University Press

that the elements f_{ij} in (3) constitute a basis for $\mathfrak{A}$, if and only if $S = (s_{ij})$ is non-singular.

The nature of the multiplication rule is preserved under transformations of the type occurring in (3). Indeed

$$\begin{aligned} f_{ij}f_{pq} &= \sum_{\lambda} s_{i\lambda}e_{\lambda j}\sum_{\mu} s_{p\mu}e_{\mu q} \\ &= \sum_{\lambda,\mu} s_{i\lambda}s_{p\mu}\phi_{j\mu}e_{\lambda q} \\ &= \Big(\sum_{\mu} s_{p\mu}\phi_{j\mu}\Big) f_{iq}, \end{aligned}$$

by (3). Hence $f_{ij}f_{pq} = \psi_{jp}f_{iq}$ where $\psi_{jp} = \sum_{\mu}\phi_{j\mu}s_{p\mu}$. Relative to the new basis $\{f_{ij}\}$, the algebra has therefore the multiplication matrix $\Psi = \Phi S^T$, where S^T denotes the transpose of S.

We make a further change of basis, namely

$$g_{ij} = \sum_{\lambda=1}^{n} f_{i\lambda}r_{\lambda j}, \qquad r_{\lambda j} \in K,$$

$(1 \leqslant i \leqslant m;\ 1 \leqslant j \leqslant n)$. Again the elements g_{ij} of $\mathfrak{A}$ constitute a basis if and only if the $n \times n$ matrix $R = (r_{ij})$ is non-singular. The multiplication rule is again transformed;

$$g_{ij}g_{pq} = \theta_{jp}g_{iq} \text{ where } \theta_{jp} = \sum_{\lambda} r_{\lambda j}\psi_{\lambda p} \in K.$$

Relative to the basis $\{g_{ij}\}$ the algebra has the matrix $\Theta = R^T\Phi S^T$.

If Φ has rank r, non-singular matrices R and S may be chosen so that the $n \times m$ matrix Θ is

$$\Theta = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ unit matrix and all other submatrices of Θ are zero. A basis such as $\{g_{ij}\}$, relative to which the multiplication matrix has this simple form, will be called a special basis. While the types of transformation used preserve the rank of the multiplication matrix, it has not yet been demonstrated that the rank of a multiplication matrix is an invariant for $\mathfrak{A}$. This, however, will be obvious later.

2. The structure of generalized matrix algebras. We now assume that a special basis has been chosen for $\mathfrak{A}$ and that the multiplication matrix Φ has the special form of Θ above and has rank r. Suppose that an element $b \in \mathfrak{A}$ has a matrix B whose partitioned form is

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

the dimensions of the submatrices being $r \times r$ for B_1, $r \times n - r$ for B_2, $m - r \times r$ for B_3 and $m - r \times n - r$ for B_4.

The radical of A consists of all elements that are properly nilpotent, i.e. all elements b of $\mathfrak{A}$ such that for every $a \in \mathfrak{A}$, ab is nilpotent. The matrix cor-

https://doi.org/10.4153/CJM-1955-023-2 Published online by Cambridge University Press

responding to the tth power of b is $(B\Phi)^{t-1}B$. It follows that b is nilpotent if and only if $B\Phi$ is a nilpotent matrix.

Let a and c be elements of $\mathfrak{A}$ whose matrices are A and C respectively. The matrix of the product abc is the product of matrices $A\Phi B\Phi C$. Now suppose $B_1 = 0$. Then

$$\Phi B\Phi = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & B_2 \\ B_3 & B_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

It follows that b is properly nilpotent and has index of nilpotence $\leqslant 3$.

On the other hand, for b to be properly nilpotent it is necessary that the matrix $A\Phi B\Phi$ be nilpotent for all matrices A.

$$A\Phi B\Phi = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_1B_1 & 0 \\ A_3B_1 & 0 \end{pmatrix}$$

The tth power of this matrix has $(A_1B_1)^t$ in the first submatrix position. It follows that b is properly nilpotent only if A_1B_1 is nilpotent for all $r \times r$ matrices A_1. This can only occur if $B_1 = 0$. Hence the radical of $\mathfrak{A}$ consists of all those elements b whose matrices relative to a special basis, have their first submatrix zero.

A Wedderburn decomposition of $\mathfrak{A}$ into a direct sum of a semisimple subalgebra $\mathfrak{B}$ and the radical $\mathfrak{N}$ is now clear. $\mathfrak{B}$ consists of all elements b of $\mathfrak{A}$ whose matrices relative to a special basis, have the submatrices B_2, B_3 and B_4 all zero. For such elements the mapping $b \rightarrow B_1$ is a ring isomorphism of $\mathfrak{B}$ onto the total matrix algebra of degree r over K. Hence $\mathfrak{B}$ is simple and r is an invariant of the algebra.

We see that a generalized matrix algebra is either simple ($m = n =$ rank Φ) or is non semisimple and simple modulo its radical. If the algebra is simple it certainly possesses an identity element. On the other hand let the generalized matrix algebra $\mathfrak{A}$ possess an identity element e so that $ea = ae = a$ for all a in $\mathfrak{A}$. Let E and A be the corresponding matrices. We must have $E\Phi A = A\Phi E = A$ for all matrices A. Hence $E\Phi$ must be the $m \times m$ unit matrix and ΦE must be the $n \times n$ unit matrix. This can happen only if $m = n$ and Φ is non-singular. The algebra is then simple. We restate the above result in a

THEOREM. *A generalized matrix algebra is either*

(i) *simple, or*

(ii) *non-semisimple and simple modulo its radical.*

It is simple if and only if it possesses an identity element.

REFERENCES

1. R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. Math., *38* (1937), 857.
2. H. Weyl, *The classical groups* (Princeton, 1946).

King's College, Aberdeen