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Cokernels of Homomorphisms from Burnside Rings to Inverse Limits

Masaharu Morimoto

Abstract. Let G be a finite group and let $A(G)$ denote the Burnside ring of G. Then an inverse limit $L(G)$ of the groups $A(H)$ for proper subgroups H of G and a homomorphism res from $A(G)$ to $L(G)$ are obtained in a natural way. Let $Q(G)$ denote the cokernel of res. For a prime p, let $N(p)$ be the minimal normal subgroup of G such that the order of $G/N(p)$ is a power of p, possibly 1. In this paper we prove that $Q(G)$ is isomorphic to the cartesian product of the groups $Q(G/N(p))$, where p ranges over the primes dividing the order of G.

1 Introduction

Throughout this paper, let G be a finite group, let $\mathcal{S}(G)$ denote the set of all subgroups, and let $\mathcal{F}$ be a conjugation-invariant lower-closed subset of $\mathcal{S}(G)$. Let $P(G,\mathcal{F})$ denote the cartesian product of the Burnside rings $A(H)$ of H (*cf.* [3,4]), where H runs over $\mathcal{F}$, *i.e.*, $P(G,\mathcal{F}) = \prod_{H\in\mathcal{F}} A(H)$. For the sake of convenience, if $\mathcal{F}$ is the empty set, then by $P(G,\mathcal{F})$ we mean the trivial group. Let $\mathrm{res}^G_{\mathcal{F}}$ denote the restriction homomorphism $: A(G) \to P(G,\mathcal{F})$; $\mathrm{res}^G_{\mathcal{F}}(x) = (\mathrm{res}^G_H x)_{H\in\mathcal{F}}$ for $x \in A(G)$. Let $B(G,\mathcal{F})$ denote the ring with unit obtained as the image of $\mathrm{res}^G_{\mathcal{F}}\colon A(G) \to P(G,\mathcal{F})$. As free $\mathbb{Z}$-modules, $A(G)$ and $B(G,\mathcal{F})$ are of rank $c_{\mathcal{S}(G)}$ and $c_{\mathcal{F}}$, respectively, where $c_{\mathcal{S}(G)}$ and $c_{\mathcal{F}}$ are the numbers of G-conjugacy classes of subgroups contained in $\mathcal{S}(G)$ and $\mathcal{F}$, respectively. Let V be a real G-module containing a G-submodule isomorphic to $\mathbb{R}[G] \oplus \mathbb{R}[G]$. Then there is a canonical one-to-one correspondence from the set of all G-homotopy classes of G-maps $: S(V) \to S(V)$ to the Burnside ring $A(G)$ of G (*cf.* [3, p. 157], [8, §2]). For a set $\boldsymbol{f} = (f_H)_{H\in\mathcal{F}}$ consisting of H-maps $f_H\colon S(V) \to S(V)$, we wonder if there exists a G-map $f_G\colon S(V) \to S(V)$ such that $\mathrm{res}^G_H f_G$ is H-homotopic to f_H for all $H \in \mathcal{F}$. An obstruction group $O(G,\mathcal{F})$ of the existence problem is $P(G,\mathcal{F})/B(G,\mathcal{F})$. Let $L(G,\mathcal{F})$ denote the subgroup

$$\{x \in P(G,\mathcal{F}) \mid mx \in B(G,\mathcal{F}) \text{ for some positive integer } m\}.$$

By definition, $L(G,\mathcal{F})$ is a ring with unit. By Corollary 2.2, we can describe $L(G,\mathcal{F})$ as an inverse limit of $\{A(H) \mid H \in \mathcal{F}\}$. Clearly, $P(G,\mathcal{F})/L(G,\mathcal{F})$ is a free $\mathbb{Z}$-module and $Q(G,\mathcal{F}) = L(G,\mathcal{F})/B(G,\mathcal{F})$ is a finite module. Note that the exact sequence

$$0 \longrightarrow Q(G,\mathcal{F}) \longrightarrow O(G,\mathcal{F}) \longrightarrow P(G,\mathcal{F})/L(G,\mathcal{F}) \longrightarrow 0$$

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splits, because $P(G,\mathcal{F})/L(G,\mathcal{F})$ is $\mathbb{Z}$-torsion free. We remark that $B(G,\mathcal{F})$, $L(G,\mathcal{F})$, $P(G,\mathcal{F})$, and $Q(G,\mathcal{F})$ are modules over $A(G)$. Let $\mathcal{F}(G)$ denote the set of all proper subgroups of G, *i.e.*, $\mathcal{F}(G) = \mathcal{S}(G) \smallsetminus \{G\}$, and set $P(G) = P(G,\mathcal{F}(G))$, $B(G) = B(G,\mathcal{F}(G))$, $L(G) = L(G,\mathcal{F}(G))$, and $Q(G) = Q(G,\mathcal{F}(G))$. Y. Hara and the author found that for a nontrivial nilpotent group G, $Q(G)$ is trivial if and only if G is a cyclic group of which the order is a prime or a product of distinct primes (*cf.* [5, Theorem 1.4]). M. Sugimura showed that $Q(A_5)$ is trivial, where A_5 is the alternating group on five letters. Furthermore, we can show that $Q(G)$ is trivial for any nontrivial perfect group G (*cf.* [9, Corollary 1.5]).

For a prime p, let $G^{\{p\}}$ denote the smallest normal subgroup of G with p-power index (*cf.* [7]). Let G^{nil} denote the intersection of the subgroups $G^{\{p\}}$, where p ranges the primes dividing $|G|$. Let k_G denote the product of the primes p such that $G^{\{p\}} \neq G$. We can show that $k_G L(G) \subset B(G)$, *i.e.*, $k_G Q(G) = 0$ (*cf.* [9, Corollary 1.5]). For a $\mathbb{Z}$-module M and a prime p, let $M_{(p)}$ denote the localization of M at p, *i.e.*, $M_{(p)} = S^{-1}M$ for $S = \{m \in \mathbb{N} \mid (m,p) = 1\}$. It is remarkable that $Q(G)$ is isomorphic to $\prod_p Q(G)_{(p)}$, where p ranges over the primes dividing k_G, and $Q(G)_{(p)}$ is an elementary abelian p-group, possibly the trivial group (*cf.* Corollary 3.7). In addition, we note that the canonical map $Q(G/G^{\{p\}}) \to Q(G/G^{\{p\}})_{(p)}$ is an isomorphism.

The next feature of $Q(G)$ is interesting.

Theorem 1.1 *Let G be a finite group. For an arbitrary prime p, the finite module $Q(G)_{(p)}$ is canonically isomorphic to $Q(G/G^{\{p\}})_{(p)}$. Therefore, the equalities*

$$\begin{aligned} Q(G) &= \prod_p Q(G)_{(p)} = \prod_p Q(G/G^{\{p\}})_{(p)} = \prod_p Q((G/G^{\text{nil}})/(G/G^{\text{nil}})^{\{p\}})_{(p)} \\ &= \prod_p Q(G/G^{\text{nil}})_{(p)} = Q(G/G^{\text{nil}}) \end{aligned}$$

hold up to isomorphisms, where p ranges over the primes dividing k_G.

This theorem follows from Lemmas 4.1 and 4.2. Combining the theorem with [5, Theorem 1.4], we immediately obtain the next corollary.

Corollary 1.2 *Let G be a finite group. The group $Q(G)$ is trivial if and only if G/G^{nil} is a cyclic group of which the order is a prime or a product of distinct primes.*

2 Preliminary

For a category $\mathfrak{C}$, let $\text{Obj}(\mathfrak{C})$ denote the totality of all objects in $\mathfrak{C}$; for objects x, y in $\mathfrak{C}$, let $\text{Mor}_{\mathfrak{C}}(x,y)$ denote the set of all morphisms in $\mathfrak{C}$ from x to y, and let $\text{Mor}(\mathfrak{C})$ denote the totality of all morphisms in $\mathfrak{C}$, *i.e.*,

$$\text{Mor}(\mathfrak{C}) = \coprod_{x,y \in \text{Obj}(\mathfrak{C})} \text{Mor}_{\mathfrak{C}}(x,y).$$

Let $\mathfrak{S}(G)$ denote the category in which the objects are all elements in $\mathcal{S}(G)$, the morphisms from objects H to K are all triples (H,a,K) such that $a \in G$ and $aHa^{-1} \subset$

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K, and the compositions of morphisms are given by

$$(K,b,L)\circ(H,a,K)=(H,ba,L)\quad \text{for } (H,a,K),(K,b,L)\in \operatorname{Mor}(\mathfrak{S}(G))$$

(*cf.* [2]). For $(H,a,K)\in \operatorname{Mor}(\mathfrak{S}(G))$, we have an associated homomorphism

$$\iota_{(H,a,K)}\colon H\longrightarrow K;\quad \iota_{(H,a,K)}(x)=axa^{-1}\qquad \text{for } x\in H.$$

Let $\mathcal{F}$ be a conjugation-invariant, lower-closed subset of $\mathcal{S}(G)$; *i.e.*, if $H\in\mathcal{F}$, then $(H)\subset\mathcal{F}$, where $(H)=\{gHg^{-1}\mid g\in G\}$, and if $H\in\mathcal{F}$, then $\mathcal{S}(H)\subset\mathcal{F}$. Let $\mathfrak{F}$ denote the full subcategory of $\mathfrak{S}(G)$ such that $\operatorname{Obj}(\mathfrak{F})=\mathcal{F}$. By definition, $\operatorname{Mor}(\mathfrak{F})$ consists of all triples (H,a,K) such that $H,K\in\mathcal{F}$, $a\in G$ satisfying $aHa^{-1}\subset K$. Let $\mathfrak{Ab}$ denote the category of which the objects are all abelian groups and the morphisms are all group homomorphisms between objects.

Let $M\colon\mathfrak{S}(G)\to\mathfrak{Ab}$ be a contravariant functor such that

$$M((H,a,H))=\mathrm{id}_{M(H)}\qquad \text{for all } H\in\mathcal{S}(G) \text{ and } a\in H.$$

In the sequel, we should read the notation $(H,a,K)^*$ as $M((H,a,H))$ and the expression $x=(x_H)_{H\in\mathcal{F}}$ for $x\in\prod_{H\in\mathcal{F}}M(H)$ as one satisfying $x_H\in M(H)$. Let $\varprojlim_{\mathfrak{F}}M(\star)$ denote the inverse limit defined in [1, p. 243], *i.e.*, $\varprojlim_{\mathfrak{F}}M(\star)$ consists of all elements $(x_H)\in\prod_{H\in\mathcal{F}}M(H)$ such that $x_H=f^*x_K$ for all H, $K\in\mathcal{F}$, and $f\in\operatorname{Mor}_{\mathfrak{F}}(H,K)$. There is a canonical restriction homomorphism

$$\operatorname{res}^G_{\mathcal{F}}\colon M(G)\longrightarrow \varprojlim_{\mathfrak{F}}M(\star);\quad x\longmapsto(\operatorname{res}^G_H x)_{H\in\mathcal{F}},$$

where res^G_H stands for $(H,e,G)^*$. For $K\in\mathcal{F}$, we have the restriction homomorphism

$$\operatorname{res}^{\mathcal{F}}_K\colon\varprojlim_{\mathfrak{F}}M(\star)\longrightarrow M(K);\quad x=(x_H)_{H\in\mathcal{F}}\longmapsto x_K.$$

Let $A(G,\mathcal{F})$ denote the submodule of $A(G)$ generated by $\{[G/H]\mid (H)\subset\mathcal{F}\}$. Then $A(G,\mathcal{F})$ is a direct summand of $A(G)$ of rank $c_{\mathcal{F}}$. By definition, the inclusions

$$\operatorname{res}^G_{\mathcal{F}}\big(A(G,\mathcal{F})\big)\subset B(G,\mathcal{F})\subset L(G,\mathcal{F})\subset\varprojlim_{\mathfrak{F}}A(\star)\subset P(G,\mathcal{F})$$

hold, where $B(G,\mathcal{F})=\operatorname{res}^G_{\mathcal{F}}(A(G))$ and $P(G,\mathcal{F})=\prod_{H\in\mathcal{F}}A(H)$. For a finite CW complex C, let $\chi(C)$ denote the Euler characteristic of C. For $K\in\mathcal{F}$ and $x=(x_H)_{H\in\mathcal{F}}\in\varprojlim_{\mathfrak{F}}A(\star)$, we define $\chi_K(x)$ by $\chi_K(x)=\chi(X^K)$, where X is a finite K-CW complex representing x_K. We regard χ_K as a homomorphism from $\varprojlim_{\mathfrak{F}}A(\star)$ to $\mathbb{Z}$.

Lemma 2.1 *For an arbitrary conjugation-invariant lower-closed set $\mathcal{F}$ of subgroups of G, the homomorphism $\operatorname{res}^G_{\mathcal{F}}\colon A(G,\mathcal{F})\to\varprojlim_{\mathfrak{F}}A(\star)$ is injective and the equalities*

$$\operatorname{rank}\operatorname{res}^G_{\mathcal{F}}(A(G,\mathcal{F}))=\operatorname{rank}B(G,\mathcal{F})=\operatorname{rank}\varprojlim_{\mathfrak{F}}A(\star)=c_{\mathcal{F}}$$

hold.

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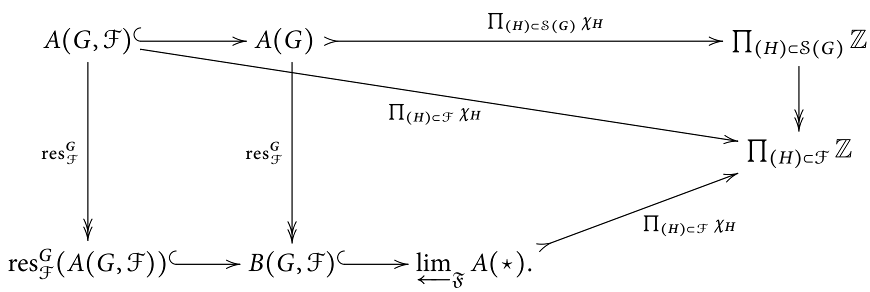
Proof We have the commutative diagram



By the Burnside congruence formula (*cf.* [3, IV, Theorem 5.7]), we readily see

$$|G|\prod_{(H)\subset\mathcal{F}}\mathbb{Z} \subset \big(\prod_{(H)\subset\mathcal{F}}\chi_H\big)(A(G,\mathcal{F})).$$

Therefore the rank of $(\prod_{(H)\subset\mathcal{F}}\chi_H)(A(G,\mathcal{F}))$ is equal to $c_{\mathcal{F}}$. Since $A(G,\mathcal{F})$ is a free $\mathbb{Z}$-module of rank $c_{\mathcal{F}}$, the homomorphism $\prod_{(H)\subset\mathcal{F}}\chi_H\colon A(G,\mathcal{F}) \to \prod_{(H)\subset\mathcal{F}}\mathbb{Z}$ is injective. By the commutative diagram above, we obtain the lemma. ∎

Corollary 2.2 *The module $L(G,\mathcal{F})$ coincides with $\varprojlim_{\mathfrak{F}} A(\star)$.*

Proof The conclusion follows from the fact that $L(G,\mathcal{F}) \subset \varprojlim_{\mathfrak{F}} A(\star)$, $L(G,\mathcal{F})$ is a direct summand of $P(G,\mathcal{F})$, $\varprojlim_{\mathfrak{F}} A(\star)$ is a submodule of $P(G,\mathcal{F})$, and the two modules $L(G,\mathcal{F})$ and $\varprojlim_{\mathfrak{F}} A(\star)$ have same rank, because $L(G,\mathcal{F})/B(G,\mathcal{F})$ is torsion. ∎

Now let $M\colon \mathfrak{S}(G) \to \mathfrak{Ab}$ be a covariant functor such that

$$(H,a,H)_* = \mathrm{id}_{M(H)} \qquad \text{for all } H \in \mathcal{S}(G) \text{ and } a \in H,$$

where $(H,a,K)_*$ stands for $M((H,a,K))$. Let $\varinjlim_{\mathfrak{F}} M(\star)$ denote the colimit defined in [1, p. 243]. In order to understand the colimit, let $\mathcal{C}$ be the family of pairs $(V, (h_H)_{H\in\mathcal{F}})$, where each V is an abelian group and each h_H is a homomorphism $M(H) \to V$, satisfying the following two conditions.

(C1) The set $\{h_H(x) \mid H \in \mathcal{F}, x \in M(H)\}$ generates V.

(C2) If $(H,a,K)_* x = y$ for $(H,a,K) \in \mathrm{Mor}(\mathfrak{F})$, and $x \in M(H)$, $y \in M(K)$, then $h_H(x) = h_K(y)$.

Let $(V_0, (h_{0,H})_{H\in\mathcal{F}})$ be a universal object in the family $\mathcal{C}$; *i.e.*, for $(V,(h_H)_{H\in\mathcal{F}}) \in \mathcal{C}$, there exists a homomorphism $\varphi\colon V_0 \to V$ such that $h_H = \varphi \circ h_{0,H}$ for all $H \in \mathcal{F}$. Since we have a canonical epimorphism $k\colon \prod_{H\in\mathcal{F}} M(H) \to V_0$,

$$k(x) = \sum_{H\in\mathcal{F}} h_{0,H}(x_H),$$

where $x = (x_H)_{H\in\mathcal{F}} \in \prod_{H\in\mathcal{F}} M(H)$ with $x_H \in M(H)$, we can identify V_0 with a module consisting of equivalence classes of elements of $\prod_{H\in\mathcal{F}} M(H)$, which is the colimit $\varinjlim_{\mathfrak{F}} M(\star)$ defined in [1, p. 243]. Thus, we get a universal object in $\mathcal{C}$ of the form $(\varinjlim_{\mathfrak{F}} M(\star), (\mathrm{ind}_H^{\mathcal{F}})_{H\in\mathcal{F}})$.

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There is a canonical homomorphism

$$\mathrm{ind}_{\mathcal{F}}^{G} \colon \varinjlim_{\mathfrak{F}} A(\star) \longrightarrow A(G); \quad \sum_{H \in \mathcal{F}} \mathrm{ind}_{H}^{\mathcal{F}} x_H \longmapsto \sum_{H \in \mathcal{F}} \mathrm{ind}_{H}^{G} x_H,$$

where each x_H is an element of $A(H)$ and ind_H^G stands for $(H, e, G)_*$. The image of this homomorphism is $A(G, \mathcal{F})$.

Proposition 2.3 *For an arbitrary conjugation-invariant lower-closed set $\mathcal{F}$ of subgroups of G, the homomorphism $\mathrm{ind}_{\mathcal{F}}^{G} \colon \varinjlim_{\mathfrak{F}} A(\star) \to A(G)$ is injective.*

Proof It is readily seen that $\varinjlim_{\mathfrak{F}} A(\star)$ is a module generated by $c_{\mathcal{F}}$ elements $\mathrm{ind}_H^{\mathcal{F}}[H/H]$ with $(H) \subset \mathcal{F}$, where

$$\mathrm{ind}_H^{\mathcal{F}} \colon A(H) \longrightarrow \varinjlim_{\mathfrak{F}} A(\star)$$

and $c_{\mathcal{F}}$ is the number of the G-conjugacy classes of subgroups belonging to $\mathcal{F}$. Since $A(G, \mathcal{F})$ is a free $\mathbb{Z}$-module of rank $c_{\mathcal{F}}$, the homomorphism $\mathrm{ind}_{\mathcal{F}}^{G}$ is injective. ■

By the homomorphism $\mathrm{ind}_{\mathcal{F}}^{G}$ above, we can identify $\varinjlim_{\mathfrak{F}} A(\star)$ with the submodule $A(G, \mathcal{F})$ of $A(G)$.

Let N be a normal subgroup of G. We have the homomorphism $\mathrm{fix}_{G,N} \colon A(G) \to A(G/N)$ that maps $[X]$ to $[X^N]$ for finite G-sets X. Let $\mathrm{fix}_{\mathcal{F}(G),N} \colon L(G) \to L(G/N)$ be the homomorphism for which the diagram

$$\begin{array}{ccc} L(G) & \xrightarrow{\ \mathrm{fix}_{\mathcal{F}(G),N}\ } & L(G/N) \\ {\scriptstyle \mathrm{res}}\downarrow & & \downarrow \\ \prod_{H \in \mathcal{G}} A(H) & \xrightarrow[\ \mathrm{fix}\ ]{} & \prod_{H \in \mathcal{G}} A(H/N) \end{array}$$

commutes, where $\mathcal{G} = \{H \in \mathcal{S}(G) \mid N \subset H \neq G\}$. It is a ring homomorphism and induces a homomorphism $\overline{\mathrm{fix}}_{\mathcal{F}(G),N} \colon Q(G) \to Q(G/N)$.

3 Operation of $A(G, \mathcal{F})$ on $L(G, \mathcal{F})$

Recall that $L(G, \mathcal{F})$ is a module over $A(G)$:

$$A(G) \times L(G, \mathcal{F}) \longrightarrow L(G, \mathcal{F}); \quad (\alpha, x) \longmapsto \big((\mathrm{res}_H^G \alpha) x_H \big)_{H \in \mathcal{F}},$$

where $\alpha \in A(G)$ and $x = (x_H)_{H \in \mathcal{F}} \in L(G, \mathcal{F})$.

Let α be an element of $A(G, \mathcal{F})$ (resp. $A(G, \mathcal{F})_{(p)}$ for a prime p) with

$$\alpha = \sum_{(H) \subset \mathcal{F}} a_H [G/H],$$

where $a_H \in \mathbb{Z}$ (resp. $\mathbb{Z}_{(p)}$) and $x = (x_H)_{H \in \mathcal{F}} \in L(G, \mathcal{F})$ (resp. $L(G, \mathcal{F})_{(p)}$). Then we define an element $\alpha \circ x$ of $A(G, \mathcal{F})$ (resp. $A(G, \mathcal{F})_{(p)}$) by

$$\alpha \circ x = \sum_{(H) \subset \mathcal{F}} a_H \, \mathrm{ind}_H^G x_H.$$

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Lemma 3.1 *For $\alpha \in A(G,\mathcal{F})$ (resp. $A(G,\mathcal{F})_{(p)}$) and $x = (x_H)_{H\in\mathcal{F}} \in L(G,\mathcal{F})$ (resp. $L(G,\mathcal{F})_{(p)}$), the equality $\mathrm{res}^G_{\mathcal{F}}(\alpha \circ x) = \alpha x$ (= $(\mathrm{res}^G_{\mathcal{F}}\,\alpha)x$) holds, and therefore αx belongs to $B(G,\mathcal{F})$ (resp. $B(G,\mathcal{F})_{(p)}$).*

Proof Let $K \in \mathcal{F}(G)$. Then we have the equalities

$$\begin{aligned}\mathrm{res}^G_K(\alpha \circ x) &= \sum_{(H)\subset\mathcal{F}} a_H\,\mathrm{res}^G_K(\mathrm{ind}^G_H\, x_H)\\ &= \sum_{(H)\subset\mathcal{F}} a_H\Big(\sum_{KgH\in K\backslash G/H} \mathrm{ind}^K_{K\cap gHg^{-1}}(c_g)_*(\mathrm{res}^H_{H\cap g^{-1}Kg}\, x_H)\Big)\\ &= \sum_{(H)\subset\mathcal{F}} a_H\Big(\sum_{KgH\in K\backslash G/H} \mathrm{ind}^K_{K\cap gHg^{-1}}\, x_{K\cap gHg^{-1}}\Big)\end{aligned}$$

and

$$\begin{aligned}(\mathrm{res}^G_K\,\alpha)(\mathrm{res}^{\mathcal{F}}_K\, x) &= \Big(\sum_{(H)\subset\mathcal{F}} a_H\,\mathrm{res}^G_K[G/H]\Big)x_K\\ &= \sum_{(H)\subset\mathcal{F}} a_H\Big(\mathrm{res}^G_K[G/H]\Big)x_K\\ &= \sum_{(H)\subset\mathcal{F}} a_H\Big(\sum_{KgH\in K\backslash G/H} \mathrm{ind}^K_{K\cap gHg^{-1}}(c_g)_*\,\mathrm{res}^H_{H\cap g^{-1}Kg}[H/H]\Big)x_K\\ &= \sum_{(H)\subset\mathcal{F}} a_H\Big(\sum_{KgH\in K\backslash G/H} [K/K\cap gHg^{-1}]\Big)x_K\\ &= \sum_{(H)\subset\mathcal{F}} a_H\Big(\sum_{KgH\in K\backslash G/H} \mathrm{ind}^K_{K\cap gHg^{-1}}\, x_{K\cap gHg^{-1}}\Big),\end{aligned}$$

where $(c_g)_*$ stands for $(H\cap g^{-1}Kg, g, K\cap gHg^{-1})_*$. Hence we obtain the lemma. ∎

The next fact can be obtained implicitly from R. Oliver [10, Lemma 8] and explicitly from C. Kratzer and J. Thévenaz [6, Proposition 3.2].

Lemma 3.2 ([9, Lemma 1.3], [8, Proposition 2.1]) *For an arbitrary finite group G, there exists a unique element $\gamma_G \in A(G)$ such that $\chi_G(\gamma_G) = k_G$ and $\chi_H(\gamma_G) = 0$ for all $H \in \mathcal{F}(G)$.*

This gives the following corollaries.

Corollary 3.3 *For an arbitrary finite group G, there exists a unique element $\tau_G \in A(G)$ such that $\chi_G(\tau_G) = 0$ and $\chi_H(\tau_G) = k_G$ for all $H \in \mathcal{F}(G)$.*

Corollary 3.4 ([9, Corollary 1.5]) *For an arbitrary finite group G, $k_G\,L(G)$ is contained in $B(G)$, and hence $k_G\,Q(G) = 0$.*

Corollary 3.5 *For an arbitrary finite group G and an arbitrary prime p, there exists $\gamma_{G,p} \in A(G)_{(p)}$ such that $\chi_G(\gamma_{G,p}) = p$ and $\chi_H(\gamma_{G,p}) = 0$ for all $H \in \mathcal{F}(G)$.*

Corollary 3.6 *For an arbitrary finite group G and an arbitrary prime p, there exists $\tau_{G,p} \in A(G)_{(p)}$ such that $\chi_G(\tau_{G,p}) = 0$ and $\chi_H(\tau_{G,p}) = p$ for all $H \in \mathcal{F}(G)$.*

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Corollary 3.7 *For an arbitrary finite group G and an arbitrary prime p, $p\,L(G)_{(p)}$ is contained in $B(G)_{(p)}$, and hence $p\,Q(G)_{(p)} = 0$.*

For a prime p, let $\mathcal{L}_p(G)$ denote the set of all subgroups of G containing $G^{\{p\}}$ and set $\mathcal{M}_p(G) = \mathcal{S}(G) \smallsetminus \mathcal{L}_p(G)$. Let $\mathcal{L}(G)$ (resp. $\mathcal{M}(G)$) be the union of $\mathcal{L}_p(G)$ (resp. the intersection of $\mathcal{M}_p(G)$) for all primes p dividing $|G|$.

Lemma 3.8 ([7, Theorem 1.3]) *For an arbitrary finite group G, there exists an element β_G of $A(G)$ such that $\chi_G(\beta_G) = 1$ and $\chi_H(\beta_G) = 0$ for all $H \in \mathcal{M}(G)$.*

Corollary 3.9 *For an arbitrary finite group G and an arbitrary prime p, there exists an element $\beta_{G,p}$ of $A(G)_{(p)}$ such that $\chi_G(\beta_{G,p}) = 1$ and $\chi_H(\beta_{G,p}) = 0$ for all $H \in \mathcal{M}_p(G)$.*

Proof Let $Q = G/G^{\mathrm{nil}}$. Note that Q is isomorphic to the cartesian product of Sylow subgroups of Q. Let Q_p be the Sylow p-subgroup of Q and let $q\colon Q \to \overline{Q} = Q/Q_p$ denote the quotient homomorphism. There exists an element $u \in A(\overline{Q})_{(p)}$ such that $\chi_{\overline{Q}}(u) = 1$ and $\chi_T(u) = 0$ for all $T < \overline{Q}$. Set $\beta_{Q,p} = q^* u \in A(Q)_{(p)}$. Then $\chi_T(\beta_{Q,p}) = 1$ for $T \in \mathcal{L}_p(Q)$ and $\chi_T(\beta_{Q,p}) = 0$ for $T \in \mathcal{M}_p(Q)$. Let $f\colon G \to Q$ be the quotient homomorphism. Then the element $\beta_{G,p} = \beta_G \cdot f^*\beta_{Q,p}$ possesses the required properties. ∎

Let p be a prime. The element $\alpha = [G/G] - \beta_{G,p} \in A(G)_{(p)}$ has the form

$$\alpha = \sum_{(H)\subset\mathcal{F}(G)} a_H [G/H] \qquad (a_H \in \mathbb{Z}_{(p)})$$

and belongs to $A(G, \mathcal{F}(G))_{(p)}$.

4 Comparison of $Q(G)_{(p)}$ and $Q(G/G^{\{p\}})_{(p)}$

Throughout this section, let N stand for $G^{\{p\}}$. Let p be a prime, $\beta_{G.p}$ the element given in Corollary 3.9, and set $\alpha = [G/G] - \beta_{G,p}$.

Let $x = (x_H)_{H\in\mathcal{F}(G)}$ be an element of $L(G)_{(p)}$. Then we have $x = x - \alpha x + \alpha x$ and the last term $\alpha x = \mathrm{res}^G_{\mathcal{F}(G)}(\alpha \circ x)$ belongs to $B(G)_{(p)} = \mathrm{res}^G_{\mathcal{F}(G)}(A(G)_{(p)})$ by Lemma 3.1. In addition, we have $\mathrm{res}^{\mathcal{F}(G)}_{\mathcal{M}_p(G)}(x - \alpha x) = 0$. Recall the commutative diagram

$$\begin{array}{ccccc}
B(G)_{(p)} & \longrightarrow & L(G)_{(p)} & \twoheadrightarrow & Q(G)_{(p)} \\
\big\downarrow & & \big\downarrow {\scriptstyle \mathrm{fix}_{\mathcal{F}(G),N}} & & \big\downarrow {\scriptstyle \overline{\mathrm{fix}}_{\mathcal{F}(G),N}} \\
B(G/N)_{(p)} & \longrightarrow & L(G/N)_{(p)} & \twoheadrightarrow & Q(G/N)_{(p)}.
\end{array}$$

Lemma 4.1 *The homomorphism $\overline{\mathrm{fix}}_{\mathcal{F}(G),N}\colon Q(G)_{(p)} \to Q(G/N)_{(p)}$ is injective.*

Proof Let $x \in L(G)_{(p)}$ such that $[\mathrm{fix}_{\mathcal{F}(G),N}(x)] = 0$ in $Q(G/N)_{(p)}$. Then the element $\mathrm{fix}_{\mathcal{F}(G),N}(x)$ belongs to $B(G/N)_{(p)}$. Therefore,

$$\mathrm{fix}_{\mathcal{F}(G),N}(x) = \mathrm{fix}_{\mathcal{F}(G),N}(\mathrm{res}^G_{\mathcal{F}(G)}(z))$$

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holds for some $z \in A(G)_{(p)}$. It means that $v = x - \mathrm{res}^G_{\mathcal{F}(G)}(z)$ belongs to the kernel of $\mathrm{fix}_{\mathcal{F}(G),N}$. Set $w = v - \alpha v$. Since $\mathrm{fix}_{\mathcal{F}(G),N}(w) = 0$ and $\mathrm{res}^{\mathcal{F}(G)}_{\mathcal{M}_p(G)}(w) = 0$, we get $w = 0$ in $L(G)_{(p)}$. Clearly we have $[x] = [v] = [w]$ in $Q(G)_{(p)}$. Therefore, we conclude that $[x] = 0$ in $Q(G)_{(p)}$, which shows the injectivity of $\overline{\mathrm{fix}}_{\mathcal{F}(G),N}$. ■

Lemma 4.2 *The homomorphism* $\overline{\mathrm{fix}}_{\mathcal{F}(G),N}\colon Q(G)_{(p)} \to Q(G/N)_{(p)}$ *is surjective.*

Proof Let $x = (x_K)_{K \in \mathcal{F}(G/N)}$ be an arbitrary element of $L(G/N)_{(p)}$. Define an element $y = (y_H)_{H \in \mathcal{F}(G)}$ of $P(G,\mathcal{F}(G))_{(p)}$, where $P(G,\mathcal{F}(G))_{(p)} = \prod_{H \in \mathcal{F}(G)} A(H)_{(p)}$, by

$$y_H = \begin{cases} f|_H{}^* x_{f(H)} & \text{if } H \supset N, \\ 0 & \text{otherwise} \end{cases} \qquad (H \in \mathcal{F}(G)),$$

where $f\colon G \to G/N$ is the quotient map. Then the element $z = y - \alpha y$ belongs to $L(G)_{(p)}$, and the equalities

$$\begin{aligned} \big[\mathrm{fix}_{\mathcal{F}(G),N}(z)\big] &= \big[(\mathrm{fix}_{H,N}(z_H))_H\big] \\ &= \big[(\mathrm{fix}_{H,N}(y_H - (\mathrm{res}^G_H \alpha) y_H))_H\big] \\ &= \big[(x_{f(H)} - (\mathrm{fix}_{H,N}(\mathrm{res}^G_H \alpha)) x_{f(H)})_H\big] \\ &= \big[x - \mathrm{fix}_{G,N}(\alpha)x\big] = [x] \end{aligned}$$

hold in $Q(G/N)_{(p)}$, where H ranges over $\mathcal{L}_p(G) \cap \mathcal{F}(G)$. This shows the surjectivity of $\overline{\mathrm{fix}}_{\mathcal{F}(G),N}$. ■

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Graduate School of Natural Science and Technology, Okayama University, Tsushimanaka 3-1-1, Kitaku, Okayama, 700-8530 Japan
e-mail: morimoto@ems.okayama-u.ac.jp

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