ON THE ZEROS OF FUNCTIONS WITH DERIVATIVES IN H_1 AND H_∞

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1. Introduction. Let $\{z_k\}$, $0 < |z_k| < 1$, be a given sequence of points in the open unit disc $D = \{z: |z| < 1\}$ and let E be its set of limit points on the unit circle T. In this note we consider the problem of finding conditions on the sequence $\{z_k\}$ which will ensure the existence of a function f analytic in D satisfying

$$\text{(A)} \qquad f(0) = 1, \qquad f(z_k) = 0, \qquad z_k = r_k e^{i\theta_k}$$

and whose derivative f' belongs to the Hardy class H_1 or, alternatively, whose derivatives of all orders are bounded in D. We shall prove the following two theorems.

THEOREM 1. *If*

$$\text{(1)} \qquad \sum_{k=1}^{\infty} (1 - |z_k|) < \infty,$$

$$\text{(2)} \qquad E \textit{ is a Carleson set,}$$

and

$$\text{(3)} \qquad \sum_{k=1}^{\infty} \operatorname{dist}(\theta_k, E)^\alpha < \infty \quad \textit{for some } \alpha > 1,$$

then there is a function f analytic in D which satisfies (A) *and its derivative f' belongs to H_1.*

THEOREM 2. *If conditions* (1) *and* (2) *hold and for some $\alpha \geqq 1$ and constant M we have*

$$\text{(4)} \qquad \operatorname{dist}(z_k, E)^\alpha < M(1 - |z_k|) \quad \text{for } k = 1, 2, \ldots,$$

then there is a function f analytic in D which satisfies (A) *and whose derivatives of all orders are bounded in D.*

The special case in which only a finite number of derivatives is required to be bounded is due to Caughran [**4**].

Condition (3) allows z_k to approach E in a "very tangential" manner while condition (4) may be described by saying $\{z_k\}$ has *finite degree of contact α at E*. For example, the sequence $z_k = (1 - 1/k!) \exp(i/k)$ satisfies (3) for $\alpha = 2$, $E = \{1\}$, and (1) holds but not (4) for any $\alpha \geqq 1$. Clearly (1) and (4) with

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$\alpha \geqq 1$ imply (3). However, the necessity of some restriction such as (3) in Theorem 1 is pointed out by taking

$$z_k = [1 - (k(\log k)^2)^{-1}] \exp(i/\log k).$$

This sequence satisfies (1) and $E = \{1\}$, and hence (2) holds, but $\{z_k\}$ is not the zero set of any non-zero analytic function with derivative in H^1. This example, due to Carleson, is discussed in detail in [**5**]. Also in this connection and for a related study of the zero sets of functions with finite Dirichlet integral see the papers of Carleson [**2**] and Shapiro and Shields [**11**].

A Carleson set is a closed subset of the unit circle T of measure zero whose complement is the union of open arcs whose lengths ϵ_k satisfy

$$\sum \epsilon_k \log(1/\epsilon_k) < \infty.$$

H_p $(0 < p < \infty)$ is the space of functions f analytic in D for which

$$\sup_{0<r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta < \infty;$$

H_∞ is the space of functions analytic and bounded in D. We shall write $d(z, E)$ for $\operatorname{dist}(z, E)$ and $d(\theta, E)$ for $\operatorname{dist}(\theta, E) = \operatorname{dist}(e^{i\theta}, E)$.

Our proofs rest on certain estimates which are concerned with the order of growth of a Blaschke product near its singularities on T.

2. Derivatives of Blaschke products. The following lemmas have as their motivation the fact that if B is a Blaschke product whose zeros lie on the segment $(0, 1)$, then $|B'(z)| = O(|z - 1|^{-2})$ (see [**10**, p. 311, problem 23]).

The Blaschke product associated with a sequence $z_k = r_k e^{i\theta_k}$, $0 < r_k < 1$, satisfying the convergence condition (1) is

$$B(z) = \prod_{k=1}^{\infty} \frac{\bar{z}_k}{|z_k|} \frac{z_k - z}{1 - \bar{z}_k z}. \tag{5}$$

Convergence in this product is uniform on any closed subset of the plane which is disjoint from E and the points $1/\bar{z}_k$ [**8**, p. 68]. Its derivative

$$B'(z) = B(z) \sum_{k=1}^{\infty} \frac{|z_k|^2 - 1}{(z_k - z)(1 - \bar{z}_k z)} \tag{6}$$

becomes

$$B'(e^{i\theta}) = e^{-i\theta} B(e^{i\theta}) \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2} \tag{7}$$

at points $e^{i\theta} \notin E$. Since $|B(e^{i\theta})| = 1$ at such points,

$$|B'(e^{i\theta})| = \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2} \tag{8}$$

when $e^{i\theta} \notin E$.

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We make repeated use of the inequalities

(9a) $$|1 - z|^2 \leqq (1 - r)^2 + |\theta|^2,$$

(9b) $$|1 - z|^2 \geqq (1 - r)^2 + 4^{-1}r|\theta|^2,$$

where $z = re^{i\theta}$, $0 \leqq r \leqq 1$, and $-\pi \leqq \theta \leqq \pi$.

LEMMA 1. *If the sequence* $\{z_k\}$ *satisfies* (1) *and* (2) *and* B *is the associated Blaschke product* (5), *then the series* (3) *converges for* $\alpha > 1$ *if and only if*

(10) $$\int_{-\pi}^{\pi} d(\theta, E)^{\alpha}|B'(e^{i\theta})|\, d\theta < \infty.$$

Proof. Since E has measure zero, $B'(e^{i\theta})$ exists for almost all θ and at every point of $T - E$ the function $|B'(e^{i\theta})|$ is given by the series (8). An application of the monotone convergence theorem shows that the existence of the integral (10) is equivalent to

(11) $$\sum_{k=1}^{\infty} (1 - r_k^2) \int_{-\pi}^{\pi} \frac{d(\theta, E)^{\alpha}}{|e^{i\theta} - z_k|^2}\, d\theta < \infty.$$

First we show that convergence of this series implies convergence of the series in (3). Assume that $-1 \in E$ so that $F = \{\theta \in [-\pi, \pi]: e^{i\theta} \in E\}$ is the union of disjoint open intervals (a_n, b_n). If $\theta_k \in F$, the corresponding term in (3) vanishes; hence suppose that $\theta_k \in (a_n, b_n)$ and, to make a choice, assume that $\theta_k - a_n \leqq b_n - \theta_k$ (inequalities similar to the following hold in case $\theta_k - a_n > b_n - \theta_k$). Put $\Delta = 2^{-1}d(\theta_k, E)$ and let I_k denote the interval $(\theta_k - \Delta, \theta_k + \Delta)$. There exists a constant $c > 0$, depending only on E, such that $d(\theta, E) \geqq c|\theta - a_n|$ for $\theta \in I_k$. If S_k denotes the integral in the kth term of (11), this last inequality and (9a) imply that

$$\begin{aligned} S_k &> c^{\alpha} \int_{I_k} \frac{|\theta - a_n|^{\alpha}}{(1 - r_k)^2 + (\theta - \theta_k)^2}\, d\theta \\ &= c^{\alpha} \int_{-\Delta}^{\Delta} \frac{|\theta + (\theta_k - a_n)|^{\alpha}}{(1 - r_k)^2 + \theta^2}\, d\theta \\ &\geqq c^{\alpha} 2^{1-\alpha} d(\theta_k, E)^{\alpha} \int_0^{\Delta} \frac{d\theta}{(1 - r_k)^2 + \theta^2} \\ &= c^{\alpha} 2^{1-\alpha} d(\theta_k, E)^{\alpha} (1 - r_k)^{-1} \tan^{-1}\left[\frac{d(\theta_k, E)}{2(1 - r_k)}\right]. \end{aligned}$$

From this we infer that convergence in (11) implies that

$$\sum_{k=1}^{\infty} d(\theta_k, E)^{\alpha} \tan^{-1}\left[\frac{d(\theta_k, E)}{2(1 - r_k)}\right] < \infty,$$

and if J denotes the integers k for which $d(\theta_k, E) > 2(1 - r_k)$, then, by (1),

$$\sum_{k=1}^{\infty} d(\theta_k, E)^{\alpha} < 4\pi^{-1} \sum_{k \in J} d(\theta_k, E)^{\alpha} \tan^{-1}\left[\frac{d(\theta_k, E)}{2(1 - r_k)}\right] + 2^{\alpha} \sum_{k \in J} (1 - r_k) < \infty,$$

as required.

Now suppose that the series (3) converges. Write

$$u_k(\theta) = |e^{i\theta} - z_k|^{-2} d(\theta, E)^\alpha$$

and, as above, denote the integral of u_k by S_k. By (1) there is a $\delta > 0$ such that $|z_k| \geqq \delta$. When $\theta_k \in F$ we have

$$d(\theta, E)^\alpha \leqq |\theta - \theta_k|^\alpha \text{ and } u_k(\theta) \leqq 4\delta^{-1}|\theta - \theta_k|^{\alpha-2};$$

thus, since $\alpha > 1$, S_k is bounded by a constant independent of k. Otherwise, suppose that $\theta_k \in (a_n, b_n)$ and let A_1 and A_2 denote the integral of u_k over I_k and $[\theta_k - \pi, \theta_k + \pi] - I_k$, respectively. It follows from (9b) and the inequality

$$d(\theta, E)^\alpha \leqq 2^{\alpha-1}[|\theta - \theta_k|^\alpha + d(\theta_k, E)^\alpha]$$

that

$$u_k(\theta) \leqq \delta^{-1}2^{\alpha+1}[|\theta - \theta_k|^{\alpha-2} + |\theta - \theta_k|^{-2} d(\theta_k, E)^\alpha],$$

and since $\theta \notin I_k$ implies that $2^\alpha|\theta - \theta_k|^\alpha > d(\theta_k, E)^\alpha$, one has

$$u_k(\theta) \leqq \delta^{-1}2^{\alpha+1}[|\theta - \theta_k|^{\alpha-2} + 2^\alpha|\theta - \theta_k|^{\alpha-2}]$$

when $\theta \notin I_k$. Therefore A_2 is bounded by a constant independent of k.

To obtain a bound on A_1 write

$$\begin{aligned} A_1 = \int_{I_k} u_k(\theta)\, d\theta &\leqq \text{const } d(\theta_k, E)^\alpha \int_{I_k} \frac{d\theta}{|e^{i\theta} - z_k|^2} \\ &\leqq \text{const } d(\theta_k, E)^\alpha \int_0^\Delta \frac{d\theta}{(1 - r_k)^2 + 4^{-1}\delta\theta^2} \\ &\leqq \text{const } d(\theta_k, E)^\alpha (1 - r_k)^{-1}. \end{aligned}$$

From these various estimates we conclude that

$$\sum_{k=1}^\infty (1 - r_k^2) \int_{-\pi}^{\pi} \frac{d(\theta, E)^\alpha}{|e^{i\theta} - z_k|^2} d\theta \leqq \text{const} \sum_{k=1}^\infty (1 - r_k^2) + \text{const} \sum_{k=1}^\infty d(\theta_k, E)^\alpha < \infty$$

by (1) and (3); hence the integral in (10) exists and the proof is complete.

The requirement $\alpha > 1$ in Lemma 1 is essential since convergence of the series (3) for $\alpha = 1$ does not imply convergence in (10). For example, take $1 - r_k = \theta_k = \epsilon_k = [k(\log k)^2]^{-1}$; then (1) and (2) hold and (3) converges with $\alpha = 1$ but the integral in (10) exceeds

$$\text{const} \sum \epsilon_k + \text{const} \sum \epsilon_k \log(1/\epsilon_k) = +\infty.$$

We omit the routine proofs for the remaining lemmas.

LEMMA 2. *If the sequence* $\{z_k\}$ *satisfies* (4) *and* $|z_k| \geqq \delta > 0$, *then there exists a constant* L, *depending only on* α, δ, *and* M *such that*

$$\sup_{|z|<1} \frac{d(z, E)^\alpha}{|1 - \bar{z}_k z|} \leqq L \quad \textit{for } k = 1, 2, \ldots . \tag{12}$$

LEMMA 3. *If the sequence* $\{z_k\}$ *satisfies* (1) *and* (4) *and* B *is the associated Blaschke product, then there is a sequence of constants* M_p *such that*

$$|C^{(p)}(z)|d(z, E)^{2\alpha p} \leqq M_p \text{ in } D \tag{13}$$

for $p = 0, 1, 2, \ldots$ *and any subproduct* C *of* B.

3. Proof of main theorems. In addition to the lemmas of the preceding section we need a result obtained by Novinger [**9**, Theorem 4.3] and independently by Taylor and Williams [**12**].

THEOREM A. *Let* E *be a Carleson set. Then there exists an outer unction* F *such that*

(a) *the zero set of* F *in* $\bar{D}$ *is* E,
(b) $F^{(p)} \in H_\infty$ *for* $p = 1, 2, \ldots,$
(c) *the zero set of* $F^{(p)}$ *in* $\bar{D}$ *contains* E *for* $p = 1, 2, \ldots$.

By expanding F in a Taylor series about points of E it is easy to see that condition (c) implies the existence of constants γ_{pq} such that

$$|F^{(p)}(z)| \leqq \gamma_{pq} \operatorname{dist}(z, E)^q \tag{14}$$

for $p, q = 0, 1, \ldots$.

We are now ready to complete the proof of Theorem 1. Define f by

$$f = BF, \tag{15}$$

where B is the Blaschke product associated with $\{z_k\}$ and F satisfies (a), (b), and (c) of Theorem A relative to the cluster set E of $\{z_k\}$. Clearly (a constant times) f satisfies (A).

In order to show that $f' \in H_1$, it suffices to show that $B'F \in H_1$. By (14) there is a constant γ such that $|F(e^{i\theta})| \leqq \gamma d(\theta, E)^\alpha$; hence, by the result of Lemma 1,

$$\int_{-\pi}^{\pi} |B'(e^{i\theta})|\,|F(e^{i\theta})|\,d\theta \leqq \gamma \int_{-\pi}^{\pi} |B'(e^{i\theta})|\,d(\theta, E)^\alpha\,d\theta < \infty.$$

This proves that $B'(e^{i\theta})F(e^{i\theta})$ is summable and justifies use of integration by parts to obtain

$$\int_{-\pi}^{\pi} e^{in\theta}B'(e^{i\theta})F(e^{i\theta})\,d\theta = -\int_{-\pi}^{\pi} e^{in\theta}B(e^{i\theta})F'(e^{i\theta})\,d\theta \\ - (n-1)\int_{-\pi}^{\pi} e^{i(n-1)\theta}B(e^{i\theta})F(e^{i\theta})\,d\theta = 0$$

for $n = 1, 2, \ldots$. Hence there exists a function in H_1 whose radial limits agree almost everywhere with the radial limits of $f'(re^{i\theta})$, and therefore f' itself belongs to H_1 [**13**, p. 203].

To complete the proof of Theorem 2, suppose that $\{z_k\}$ satisfies (1), (2), and (4) and define f in the same manner by (15). A computation by the

Leibnitz rule shows that the nth derivative of f is bounded in D since the derivatives of B and F satisfy (13) and (14), respectively.

Our insistence throughout that the cluster set of $\{z_k\}$ be a Carleson set is not superficial, for Caughran [**4**] has pointed out that an amalgam of results of Carleson and Hardy and Littlewood [**2**; **7**, Theorem 40; **6**, Theorem 2] yields the following result.

THEOREM B. *If f is analytic in D and $f' \in H_p$ $(p > 1)$ is non-zero, then the zero set in T of f is a Carleson set.*

James Caveny (private communication) has recently established the same result when $p = 1$.

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