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INVERSE SEMIGROUPS WITH CERTAIN UNIVERSAL PROPERTIES

H. LAUSCH

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1. Introduction

Free groups and free inverse semigroups are characterized by certain properties which are usually called universal. Category-theoretically they arise from left adjoints of forgetful functors which assign to each structure its underlying set. The purpose of this note is to give a construction of a wealth of inverse semigroups with certain universal properties which subsumes free groups and free inverse semigroups and incidentally elucidates some well-known constructions of free inverse semigroups (Scheiblich (1972)). Category-theoretical terminology will be freely taken from (Mac Lane (1971)).

2. The category $IS(E)$

Let E be a semilattice. $IS(E)$ will denote the category whose objects are pairs (S, θ) where S is an inverse semigroup and $\theta: E \to E(S)$ an isomorphism of semilattices, $E(S)$ denoting the semilattice of idempotents of S. A morphism in $IS(E)$ between two objects (S, θ) and (T, η) is a homomorphism $\alpha: S \to T$ of inverse semigroups such that $\eta = \theta(\alpha \mid E(S))$. In practice, we may visualize $IS(E)$ as consisting of inverse semigroups with θ labelling the idempotents and the morphisms being "idempotent-preserving" homomorphisms. As usual, T_E will denote the inverse semigroup of isomorphisms between principal ideals of E (see Munn (1966)). Let $\tau: E \to E(T_E)$ be the map defined by $e\tau =$ identity on E_e, E_e being the principal ideal of E generated by e. Then τ is an isomorphism of semilattices and hence (T_E, τ) an object in $IS(E)$.

The following propositions are easy to prove, the proofs will therefore be omitted.

PROPOSITION 2.1. *(T_E, τ) is the terminal object in $IS(E)$.*

PROPOSITION 2.2. *$IS(E)$ has all small products: if (S_i, θ_i), $i \in I$, are objects in $IS(E)$, then $(P, \pi) = \Pi\{(S_i, \theta_i) \mid i \varepsilon I\}$ is given by $P = \{\nu: I \to \cup\{S_i \mid i \in I\} \mid i\nu \in S_i, i\nu\sigma_i = j\nu\sigma_j$, for all $i, j \in I$, $\sigma_i: (S_i, \theta_i) \to (T_E, \tau)$ being* *the unique terminal morphism for* $(S_i, \theta_i)\}$, $E(P) = \{\nu \in P \mid i\nu \in E(S_i)$ *for all* $i \in I\}$ *and* $\pi: E \to E(P)$ *is defined by* $i(e\pi) = e\theta_i$. *The inverse semigroup operations on* P *are defined "pointwise".*

PROPOSITION 2.3. $IS(E)$ *has all equalizers: Let* $(S, \theta) \rightrightarrows (T, \eta)$ *be two morphisms in* $IS(E)$. *Then* $\gamma: (X, \varepsilon) \to (S, \theta)$ *is an equalizer where* $X = \{s \in S \mid s\alpha = s\beta\}$, $E(X) = E(S)$, $\varepsilon = \theta$, *and* $x\gamma = x$, *for all* $x \in X$.

COROLLARY 2.4. $IS(E)$ *has all small limits.*

3. The category Set $\downarrow$ T_E

Let Set $\downarrow T_E$ be the category of "sets on T_E" whose objects are mappings $\alpha: M \to T_E$ where M is a set and where a morphism from $\alpha: M \to T_E$ to $\beta: N \to T_E$ is defined to be a mapping $\gamma: M \to N$ such that $\alpha = \gamma\beta$. We are going to describe products and equalizers in this category.

PROPOSITION 3.1. *Set* $\downarrow T_E$ *has all small products: if* $\{\alpha_i \mid i \in I\}$ *is a set of objects in Set* $\downarrow T_E$, $\alpha_i: M_i \to T_E$, *then* $\Pi\alpha_i = \alpha: M \to T_E$ *is given by* $M = \{\nu: I \to \cup(M_i \mid i \in I) \mid i\nu \in M_i,\ i\nu\alpha_i = j\nu\alpha_j$, *for all* $i, j \in I\}$, *and* $\nu\alpha = i\nu\alpha_i$, *for some (and hence for all)* $i \in I$.

PROOF. Straightforward.

PROPOSITION 3.2. *Set* $\downarrow T_E$ *has all equalizers: let* $\mu: M \to T_E$, $\nu: N \to T_E$ *be objects in Set* $\downarrow T_E$, *and* $\alpha, \beta: M \to N$ *mappings such that* $\mu = \alpha\nu = \beta\nu$. *If* $X = \{m \in M \mid m\alpha = m\beta\}$, $\xi: X \to T_E$ *is defined by* $x\xi = x\mu$, *and* $\gamma: X \to M$ *is defined by* $x\gamma = x$, *then* $\gamma: \xi \to \mu$ *is an equalizer.*

PROOF. Obvious.

We are now ready to define a "forgetful" functor U: $IS(E) \to$ Set $\downarrow T_E$. If (S, θ) is an object of $IS(E)$ and $\sigma: (S, \theta) \to (T_E, \tau)$ is the morphism for S, let $U(S, \theta) = \sigma$, regarded as a map from S to T_E. If $\alpha: (S, \theta) \to (R, \eta)$ is a morphism in $IS(E)$, and ρ is the terminal morphism for R, then define $U\alpha = \alpha$. Indeed, $U\alpha: \sigma \to \rho$ is a morphism in Set $\downarrow T_E$ as $\alpha\rho: (S, \theta) \to (T_E, \tau)$ and hence $\alpha\rho = \sigma$ as (T_E, τ) is terminal in $IS(E)$. That U satisfies all requirements for a functor is quite plain.

4. U has a left adjoint

PROPOSITION 4.1. U *preserves all small limits.*

PROOF. We have to show that U preserves products and equalizers. Prop. 2.3 and Prop. 3.2 imply that U preserves equalizers. Let $(P, \pi) = \Pi\{(S_i, \theta_i) \mid i \in I\}$ and $\psi: (P, \pi) \to (T_E, \tau)$ the terminal morphism. If $\sigma_i: (S_i, \theta_i) \to (T_E, \tau)$ are terminal morphisms, then the domain of $\Pi\sigma_i$ in Set $\downarrow T_E$ is just the set $P = \{\nu: I \to \cup(S_i \mid i \in I) \mid i\nu \in S_i,\ i\nu\sigma_i = j\nu\sigma_j$, for all $i, j \in I\}$. Moreover $\nu\Pi\sigma_i = i\nu\sigma_i$, for some $i \in I$. Hence $\Pi\sigma_i$ is a morphism in $IS(E)$ and since both $\Pi\sigma_i$ and ψ are terminal with the same domain we have $\Pi\sigma_i = \psi$, Q.E.D.

THEOREM 4.2. *U: $IS(E) \to$ Set $\downarrow T_E$ has a left adjoint.*

PROOF. If will suffice to verify that the hypothesis of Freyd's Adjoint Functor Theorem (see e.g. Mac Lane (1971)) is satisfied. By Cor. 2.4, $IS(E)$ is small-complete and by Prop. 4.1, U preserves all small limits. All that remains is to show that the solution set condition is satisfied. Let ξ: $M \to T_E$ be an object in Set $\downarrow T_E$, (S, θ) an object in $IS(E)$, σ the terminal morphism for (S, θ), and φ: $\xi \to \sigma$ any morphism in Set $\downarrow T_E$. Then $M\varphi \cup E(S)$ generates an inverse subsemigroup $G(\varphi)$ of S, $E(G(\varphi)) = E(S)$, and $(G(\varphi), \theta)$ is an object in $IS(E)$. Every element of $G(\varphi)$ is then a finite product of elements $m\varphi$, $(m\varphi)^{-1}$, $e\theta \in E(S)$, where $m \in M$. Hence the cardinality of any such $G(\varphi)$ is bounded, given M. For θ there are as many possible choices as there are elements in the automorphism group of E. Hence taking one copy of each isomorphism class of such $(G(\varphi), \theta)$ gives a small set of objects in $IS(E)$ with terminal morphisms $\gamma\varphi$: $(G(\varphi), \theta) \to (T_E, \tau)$ say, and the set of all morphisms $\xi \to \gamma\varphi$ is a solution set. Hence U has a left adjoint.

DEFINITION. The left adjoint F: Set $\downarrow T_E \to IS(E)$ of U will be called the "free inverse E-semigroup generated by", and if α is a set over T_E, then $F(\alpha)$ will denote the free inverse E-semigroup generated by α.

The following two propositions will justify the use of the phrase "generated by".

PROPOSITION 4.3. *Let $\xi: M \to T_E$ be a set over T_E, $(\xi, UF(\xi))$ χ: (Set $\downarrow T_E) \to IS(E)\ (F(\xi), F(\xi))$ the* 1-1-*correspondence between the morphism sets arising from the adjunction, $\zeta = id_{F(\xi)}\chi^{-1}$: $\xi \to UF(\xi)$, $F(\xi) = (F, \varphi)$ (and hence $UF(\xi)$: $(F, \theta) \to (T_E, \tau)$ the terminal morphism). Then ζ regarded as a mapping from M to F is injective.*

PROOF. Let $S = Z_2 \times T_E$, $Z_2 = \{0, 1\}$ being the (additive) group of integers mod 2. Define θ: $E \to S$ by $e\theta = (0, e\tau)$; then (S, θ) is an object in $IS(E)$ with terminal morphism σ: $(S, \theta) \to (T_E, \tau)$ given by $(a, b)\sigma = b$. Let $m_1 \neq m_2 \in M$ and put $s_1 = (0, m_1\xi)$, $s_2 = (1, m_2\xi)$. Then $s_1 \neq s_2$ and $s_1\sigma = m_1\xi$, $s_2\sigma = m_2\xi$. Therefore there exists a morphism δ: $\xi \to U(S, \theta)$ with $m_1\delta = s_1$, $m_2\delta = s_2$. As δ factors through ζ, we must have $m_1\zeta \neq m_2\zeta$.

REMARK. F is generated by $M\zeta \cup E(F)$ as an inverse semigroup. This is an immediate consequence of the universal property of F (Theorem 4.2).

PROPOSITION 4.4. *Let $M = \{m\}$ be a singleton, $t \in T_E$, and $\alpha_t: M \to T_E$ the set over T_E with $m\alpha_t = t$. Then $\{F(\alpha_t) \mid t \in T_E\}$ is a generating set for the category $IS(E)$.*

PROOF. Let $\beta, \gamma: (S, \theta) \to (R, \eta)$ be two distinct morphisms in $IS(E)$. Then there exists $s \in S$ with $s\beta \neq s\gamma$. If σ is the terminal morphism for (S, θ), we claim that there is a morphism $\delta: F(\alpha_{s\sigma}) \to (S, \theta)$ with $\delta\beta \neq \delta\gamma$. As $m\alpha_{s\sigma} = s\sigma$, there is a unique $\delta: F(\alpha_{s\sigma}) \to (S, \theta)$ extending $m \to s$. If ρ is the terminal morphism for (R, η), then $s\beta\rho = s\gamma\rho = s\sigma$. Hence $\delta\beta$ extends $m \to s\beta$ and $\delta\gamma$ extends $m \to s\gamma$, therefore $\delta\beta \neq \delta\gamma$, Q.E.D.

REMARK. Obviously if $E = \{1\}$ is a singleton, then Set $\downarrow T_E$ is equivalent to the category Set, and the functor "free inverse semigroup generated by" reduces to "free group generated by the domain of". The next proposition will show that the free inverse semigroup is another special case of our construction.

PROPOSITION 4.5. *Let M by any set, F the free inverse semigroup on M, $E = E(F)$, ι the identity map on E, $\varphi: F \to T_E$ the homomorphism of inverse semigroups defined by $e(f\varphi) = f^{-1}ef$, $e \in E_{ff^{-1}}$ and $\alpha = \varphi \mid M$. Then $F(\alpha) \cong (F, \iota)$.*

PROOF. Let $\kappa: M \to F$ be the injection, then $\kappa\varphi = \alpha$, hence κ extends uniquely to a morphism $\psi: F(\alpha) \to (F, \iota)$. If $\bar{\varphi} = UF(\alpha)$, and $\zeta: \alpha \to \bar{\varphi}$ as in Prop. 4.3, then ζ extends uniquely to a homomorphism $\chi: F \to \bar{F}$ where $\bar{F}$ is given by $F(\alpha) = (\bar{F}, \lambda)$, say. Then $e \in E$ implies $e\lambda\psi = e$. The universal properties of F and $F(\alpha)$ imply that $\psi\chi$ and $\chi\psi$ are identities. Hence $e\chi = e\lambda\psi\chi = e\lambda$, thus $X: (F, \iota) \to F(\alpha)$ is a morphism in $IS(E)$. Therefore ψ is an isomorphism.

5. A description of $F(\alpha)$

It is desirable to have a normal form system for the elements of $\bar{F}$ in $F(\alpha) = (\bar{F}, \lambda)$. As the construction is a straightforward generalization of the case of a free group and is also known for the case of a free inverse semigroup, we give just a description of it without proof.

Let M be a set and $\alpha: M \to T_E$ in Set $\downarrow T_E$. By red seq (M) we denote the set of all "reduced sequences in $M \cup M^{-1}$", i.e. we choose a further set M^{-1} whose elements will be written as m^{-1}, $m \in M$, and let red seq (M) consist of all finite sequences $(m_{i_1}^{\varepsilon_1}, m_{i_2}^{\varepsilon_2}, \cdots, m_{i_k}^{\varepsilon_k})$, $m_{i_j} \in M$, $\varepsilon_j = \pm 1$, such that $m_{i_j}^{\varepsilon_j} \neq m_{i_{j+1}}^{-\varepsilon_{j+1}}$, $j = 1, \cdots, k-1$. Moreover red seq (M) shall contain the empty sequence ϕ. Let us associate with every non-empty $\Sigma \in$ red seq (M) an element $\Sigma\alpha_*$ as follows: If $\Sigma = (m_{i_1}^{\varepsilon_1}, \cdots, m_{i_k}^{\varepsilon_k})$, then $\Sigma\alpha_* = (m_{i_1}\alpha)^{\varepsilon_1} \cdots (m_{i_k}\alpha)^{\varepsilon_k}$. Let $\bar{F}(\alpha)$ be the set of all pairs (Σ, e), seq (M), $\Sigma \in$ red $e \in E(T_E)$ such that, for $\Sigma \neq \phi$, $e \leqq (\Sigma\alpha)_*^{-1}(\Sigma\alpha_*)$. Let $\Sigma^{-1} = (m_{i_k}^{-\varepsilon_k}, \cdots, m_{i_1}^{-\varepsilon_1})$, then $\Sigma^{-1} \in$ red seq (M). $\bar{F}(\zeta)$ becomes an inverse semigroup if we define $(\Sigma, e)^{-1} = (\Sigma^{-1}, (\Sigma\alpha_*)e(\Sigma\alpha_*)^{-1})$, if $\Sigma \neq \phi$, $(\phi, e)^{-1} = (\phi, e)$, and multiplication inductively on the length of Σ:

$$(\phi, e)(\phi, e') = (\phi, ee')$$

$$(\Sigma, e)(\phi, e') = (\Sigma, ee'),\ \Sigma \neq \phi$$

$$(\phi, e')(\Sigma, e) = (\Sigma, (\Sigma\alpha_*)^{-1}e'(\Sigma\alpha_*)e),\ \Sigma \neq \phi$$

(Σ, e) $(\Sigma', e') = ((\Sigma, \Sigma'), (\Sigma'\alpha_*)^{-1}e(\Sigma'\alpha_*)e')$ if $(\Sigma, \Sigma') \in$ red seq (M), and if $\Sigma = (\Sigma_1, m^{-\varepsilon})$, $\Sigma' = (m^{\varepsilon}, \Sigma_1')$, then (Σ, e) $(\Sigma', e') = (\Sigma_1, (m\alpha)^{-\varepsilon}e(m\alpha)^{\varepsilon})$ (Σ_1', e') where both Σ_1 and Σ_1' have shorter length than Σ and Σ', resp., and hence the product is defined by induction. If (T, τ) is the terminal object in $IS(E)$, then $e\bar{\lambda} = (\phi, e\tau)$ defines an isomorphism $\bar{\lambda}: E \to E(\bar{F}(\alpha))$. Altogether one can show that $F(\alpha) \cong (\bar{F}(\alpha), \bar{\lambda})$.

Remarks. The construction of the free inverse semigroup F is a special case of this construction. We see that the knowledge of $E(F)$ produces then immediately the construction of F itself. We also note that the use of the category Set $\downarrow T_E$ was somewhat arbitrary, and that under favourable conditions (with respect to the Adjoint Functor Theorem), Set could be possibly replaced by other categories. Finally, in view of the Kurosh subgroup theorem, the following problem emerges: Let $F(\alpha)$ be a free object in $IS(E)$. What condition has one to put on a subject U of $F(\alpha)$ to ensure that U is also free in $IS(E)$?

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Department of Mathematics
Monash University
Clayton, Vic. 3168, Australia.