Spherical Simplexes in n-dimensions.

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1. All points in an n-space equidistant from a fixed point (the centre) constitute what may be called a *spherical continuum of the n^{th} order*,—the continuum being of $n-1$ dimensions ($(n-1)$-dimensional spread) and of the 2nd degree. Any region of this spherical continuum bounded by n $(n-1)$-dimensional linear continua or *primes* (spaces of $n-1$ dimensions), passing through the centre shall be called a *spherical simplex of the n^{th} order*. This spherical simplex is bounded by n faces, spherical simplexes of the $(n-1)^{\text{th}}$ order, each of which in turn is bounded by $n-1$ spherical simplexes of the $(n-2)^{\text{th}}$ order, and so on till we reach spherical triangles, arcs and lastly points, the vertices. The total number of spherical simplexes of different orders connected with one of the n^{th} order is $2^n - 2$. The n spherical continua of the $(n-1)^{\text{th}}$ order which contain the faces of the spherical simplex of the n^{th} order determine a set of 2^n spherical simplexes of the same order, 2^{n-1} pairs, the two spherical simplexes of a pair being symmetrically situated with respect to the centre and therefore congruent.[1]

Let every $n-1$ of the n primes intersect one another in lines which meet the spherical continuum in the vertices denoted by the numerals $1, 2, \ldots, n$; and let $(12\ldots m)$ denote a spherical simplex of the m^{th} order lying on the spherical continuum $K(12\ldots m)$ of the same order in which the m-dimensional linear continuum $P(12\ldots m)$ passing through the centre intersects $K(12\ldots m)$; also let $(\overline{12\ldots m}, u\ldots v, y\ldots w)$ denote the angle between the spaces $P(12\ldots mu\ldots v)$ and $P(12\ldots my\ldots w)$ in the space $P(1\ldots mu\ldots vy\ldots w)$ in which they lie.

Without loss of generality we shall regard the radius as of unit length. Then let

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) & \ldots & \cos(1m) \\ \cos(12) & 1 & \cos(23) & \ldots & \cos(2m) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \cos(1m) & \cos(2m) & \ldots & \ldots & 1 \end{vmatrix} \equiv \Delta(12\ldots m) \equiv \sin^2(12\ldots m),$$

[1] The subject has been studied by Schläfli in the second chapter of his book *Theorie der vielfachen Kontinuitat*. The field of his investigation is different from that of the present paper and the method adopted is totally different.

Also *vide* Coolidge, *Non Euclidean Geometry*.

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denoting the square of $m!$ times the content of the join of the points 1, 2, .., m and the centre.[1]

Also, as in the ordinary geometry, with each face are associated two *poles*; and hence 2^n spherical simplexes can be formed, having for their vertices poles of these faces; and among them there is one $(1'2'..n')$ whose vertices lie on the same side of the corresponding faces of the original spherical simplex as the latter's opposite vertices. Now $(1..n-1)$ and $(1..n-2.n)$ intersect one another in $(1..n-2)$. In $P(1..n-1)$ and $P(1..n-2\,n)$ let Op and Oq be drawn respectively, through the centre O, perpendiculars to $P(1..n-2)$ to meet $K(1..n)$ in p and q; then (pq) measures the angle $(\overline{1..n-2}, n-1, n)$. And the faces of $(1'..n')$ of which $1, 2,.., n-2$ are the poles intersect one another in $((n-1)'n')$; let this arc intersect $K(1..n-1)$ and $K(1..n-2\,n)$ in p and q respectively. Then it may be seen that $(pq)+((n-1)'n')=\pi$, or $((n-1)'n')=\pi-(\overline{1..n-2}, n-1, n)$; similarly for others. So the relation between the spherical simplexes is of a dual character; and this gives rise to a duality between theorems relating to a spherical simplex.

In what follows we shall suppose that all di-spherical simplexes, *i.e.* circular arcs, (mn) are less than π. Also we shall use N to denote the set of numbers from 1 to n, N_p to denote all of these but the number p, N_{pq} all but p and q and so on.

2. Consider the perpendicular h_p let fall from p on $P(N_p)$. Equating the values of h_p we have the $n-1$ quantities

(1) $\sin(pq)\sin(\bar{q}, p, N_{pq})$ all equal to one another for the same value of p. Again, since $h_p = \dfrac{\sin(N)}{\sin(N_p)}$, equating the values of $\sin(N)$ we have the $n-1$ quantities

(2) $\sin(pq)\sin(\bar{p}, q, N_{pq})$ equal to one another for the same value of p.

Also, from above,

(3) $$\frac{\sin(N_p)}{\sin(N_q)} = \frac{\sin(\bar{p}, q, N_{pq})}{\sin(\bar{q}, p, N_{pq})}.$$

In general, let M denote any set of m numbers ($m < n$) containing a particular number p, and consider the perpendiculars let fall from

[1] This is in accordance with the nomenclature used by Prof. von Staudt, who has called the function $\sqrt{\Delta(123)}$ the sine of the solid angle that the spherical triangle subtends at the centre of the sphere. *Crelle* **24** (1842), 252.

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p on $P(M_p)$'s. Then for all values of M the $\dfrac{(n-1)!}{(m-1)!\,(n-m)!}$ quantities

(4) $\dfrac{\sin(M)}{\sin(M_p)}\sin(\overline{M}_p,\ p,\ N_M)$ are equal to one another for the same value of p. It follows that, for a particular value of M denoting a set of m numbers $p\,q\,r\,..$, the m quantities

(5) $\dfrac{\sin(N_i)}{\sin(M_i)}\sin(\overline{M}_i\,.\,i,\ N_M)$ are equal to one another for $i = p, q, r, \ldots$, obtained by equating the values of $\sin(N)$.

In particular, when $m = n-1$, the $n-1$ quantities

(6) $\dfrac{\sin(N_q)}{\sin(N_{pq})}\sin(\overline{N}_{pq},\ p,\ q)$ are equai for the same value of p.

Consequently we have the $n(n-1)/2$ quantities

(7) $\dfrac{\sin(\overline{N}_{pq},\ p,\ q)}{\sin(N_{pq})\,\Pi\sin(N_j)}$ equal to one another and to $\dfrac{\sin(N)}{\Pi\sin(N_i)}$, where Π denotes the continued product for

$j = 1, 2, \ldots, p-1, p+1, \ldots, q-1, q+1, \ldots, n$ and $i = 1, 2, \ldots, n$.

It immediately follows that

(8) $\sin(N) = \dfrac{\sin(\overline{N}_{pq},\ p,\ q)}{\sin(N_{pq})}\sin(N_p)\sin(N_q)$, for all values of p and q.[1]

Again substituting in (6) the value of $\sin(N_q)$ from (8) which is

$$\sin(N_q) = \frac{\sin(\overline{N}_{pqr},\ p,\ r)}{\sin(N_{pqr})}\sin(N_{pq})\sin(N_{qr}),$$

we have

(9) $$\frac{\sin(\overline{N}_{pq},\ p,\ q)}{\sin(\overline{N}_{pqr},\ p,\ q)} = \frac{\sin(\overline{N}_{pr},\ p,\ r)}{\sin(\overline{N}_{pqr},\ p,\ r)} = \frac{\sin(\overline{N}_{qr},\ q,\ r)}{\sin(\overline{N}_{pqr},\ q,\ r)}.$$

Similarly substituting the values of $\dfrac{\sin(N_q)}{\sin(N_{pq})}$ from (1) in (6), we

[1] Schläfli, *loc. cit* §20 (4), who has given only one of these forms. The formula is also proved by the consideration of a parallelochesm of the nth order whose content is equal to the product of the contents of two of its adjacent faces (parallelochesms of $(n-1)$th order) multiplied by the sine of the angle between the faces and divided by the content of the parallelochesm of $(n-2)$th order in which the faces intersect.

have, for fixed values of q and r, the $(n-2)$ quantities

(10) $\sin(\overline{q, r, N_{pqr}}) \sin(\overline{N_{pq}, p, q})$ equal for all values of p.

Proceeding in this way we may have a number of formulae connecting the elements of a spherical simplex, *e.g.* equating corresponding terms from (1), (4), (6) we shall have a number of other formulae. But we have given above only the more fundamental which we shall have occasion to use hereafter. It may be seen that from (6), when $n=4$,

$$(11)\quad \frac{\sin(\overline{12}, 3, 4)}{\sin(12)} \cdot \frac{\sin(\overline{34}, 1, 2)}{\sin(34)} = \frac{\sin(\overline{13}, 2, 4)}{\sin(13)} \cdot \frac{\sin(\overline{24}, 1, 3)}{\sin(24)}$$

$$= \frac{\sin(\overline{14}, 2, 3)}{\sin(14)} \cdot \frac{\sin(\overline{23}, 1, 4)}{\sin(23)}$$

$$= \frac{\sin^2(1234)}{\sin(123)\sin(124)\sin(134)\sin(234)}.$$

§3. From (8), $\cos^2(\overline{N_{pq}}, p, q) = \dfrac{\Delta(N_p)\Delta(N_q) - \Delta(N_{pq})\Delta(N)}{\Delta(N_p)\Delta(N_q)}$

$$= \frac{\Delta^2\begin{pmatrix} p \\ q \end{pmatrix} N_{pq} \Big)}{\Delta(N_p)\Delta(N_q)},$$

where $\Delta\left(\begin{smallmatrix} n-1 \\ n \end{smallmatrix} 12 \ldots n-2\right)$

$$= \begin{vmatrix} \cos(n-1\,n) & \cos(1\,n-1) & \cos(2\,n-1) & \ldots\ldots & \cos(n-2\,n-1) \\ \cos(1n) & 1 & \cos(12) & \ldots\ldots & \cos(1\,n-2) \\ \cos(2n) & \cos(12) & 1 & \ldots\ldots & \cos(2\,n-2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cos(n-2\,n) & \cos(1\,n-2) & \cos(2\,n-2) & \ldots\ldots & 1 \end{vmatrix}$$

It may therefore be seen that

$$\Delta\begin{pmatrix} p \\ q \end{pmatrix} N_{pq}\Big) = \frac{1}{\Delta(N_{pqr})}\left[\Delta(N_{pq})\Delta({}^{p}{}_{q}N_{pqr}) - \Delta({}^{p}{}_{r}N_{pqr})\Delta({}^{q}{}_{r}N_{pqr})\right]$$

$$= \frac{\sin^2(N_{pq})\sin(N_{pr})\sin(N_{qr})}{\sin^2(N_{pqr})}$$

$$\times\left[\cos(\overline{N_{pqr}}, p, q) - \cos(\overline{N_{pqr}}, p, r)\cos(\overline{N_{pqr}}, q, r)\right].$$

Accordingly, for all $n-2$ values of r

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(12), (13) $\cos(\overline{N}_{pq}, p, q)$

$$=\frac{\dfrac{\sin^2(N_{pq})}{\sin(N_{pqr})}\cdot\sin(N_r)\cot(\overline{N}_{pqr}, p, q)-\sin(N_p)\sin(N_q)\cot(N_{pqr}, p, r)\cot(\overline{N}_{pqr}, q, r)}{\sin(N_p)\sin(N_q)}$$

$$=\frac{\cos(\overline{N}_{pqr}, p, q)-\cos(\overline{N}_{pqr}, p, r)\cos(\overline{N}_{pqr}, q, r)}{\sin(\overline{N}_{pqr}, p, r)\sin(\overline{N}_{pqr}, q, r)}.$$[1]

Also, since $\Delta\binom{p}{q}N_{pq}) = \sin(N)\sin(N_{pq})\cot(\overline{N}_{pq}, p, q)$, we have

(14) $\sin(N) = \sin(N) = \Delta(^p_q N_{pq})\tan(\overline{N}_{pq}, p\,q)\operatorname{cosec}(N_{pq})$, for all values of p and q.

Moreover, as in the ordinary geometry, if the three angles in the numerator of (13) be denoted by A, B, C and their sum by $2S$, we shall express the usual $2\sqrt{\{\sin S\sin(S-A)\sin(S-B)\sin(S-C)\}}$ by $\sin(\overline{N}_{pqr}, pqr)$ and regard the angle as the spherical trihedroidal angle at (N_{pqr}) formed by the spherical continua $K(N_{pq})$, $K(N_{pr})$ and $K(N_{qr})$. Then,

$$\sin(N)=\frac{\sin(N_p)\sin(N_q)\sin(\overline{N}_{pqr}, pqr)}{\sin(N_{pq})\sin(\overline{N}_{pqr}, p, r)\sin(\overline{N}_{pqr}, q, r)},\text{ from (8)}$$

(15) $$=\frac{\sin(N_{pq})\sin(N_{pr})\sin(N_{qr})}{\sin^2(N_{pqr})}\cdot\sin(\overline{N}_{pqr}, pqr),\text{ from (8}a)$$

$$=\sin(N_{pqr})\sin(N_{pqs})\sin(N_{prs})\sin(N_{qrs})\sin(\overline{N}_{pqrs}, r, s)$$
$$\sin(\overline{N}_{pqrs}, q, s)\sin(\overline{N}_{pqrs}, p, s)\sin(\overline{N}_{pqr}, pqr)\,/\,\sin^3(N_{pqrs})$$

Proceeding thus it may be seen ultimately, taking $p=n$, $q=n-1$, .., that

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[1] Owing to the importance of formula (13) we append another proof by the method of projection: Let a be the projection of n on $P(N_n)$, b and c the projections of a on $P(N_{n\,n-2})$ and $P(N_{n\,n-1})$ respectively, d the projection of b or c on $P(N_{n\,n-1\,n-2})$, e the projection of c on bd and f that of a on ce.

Then $db=de+eb$. But $db=nd\cos(\overline{N}_{n\,n-1\,n-2}, n-1, n)$;

$de=cd\cos(\overline{N}_{n\,n-1\,n-2}, n-2, n-1)=nd\cos(\overline{N}_{n\,n-1\,n-2}, n-2, n)\cos(\overline{N}_{n\,n-1\,n-2}, n-2, n-1)$;

$eb=fa=ca\cos(\tfrac{1}{2}\pi-(\overline{N}_{n\,n-1\,n-2}, n-2, n-1))$

$=nc\cos(\overline{N}_{n\,n-1}, n-1, n)\sin(\overline{N}_{n\,n-1\,n-2}, n-2, n-1)$

$=nd\sin(\overline{N}_{n\,n-1\,n-2}, n-2, n)\sin(\overline{N}_{n\,n-1\,n-2}, n-2, n-1)\cos(\overline{N}_{n\,n-1}, n-1, n)$.

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(16) $\sin(N) = \sin(12)\sin(13)\ldots\sin(1n)\sin(\overline{1},2,3)\sin(\overline{1},2,4)$
$\ldots\ldots\sin(\overline{1},2,n)\sin(\overline{12},3,4)\ldots\sin(\overline{12},3,n)\sin(\overline{123},4,5)$
$\ldots\ldots\sin(\overline{1\ldots n-4},n-3,n)\sin(\overline{1\ldots n-3},n-2,n-1,n).$

We shall express the coefficient of $\sin(12)\ldots\sin(1n)$ by $\sin(\overline{1},23\ldots n)$ and regard the angle as a spherical polyhedroidal angle at 1 formed by the arcs (12), .. , ($1n$).

§4. From (13),

$$\cos(\overline{N}_{pq},p,q)+\cos(\overline{N}_{pr},p,r)\cos(\overline{N}_{qr},q,r)$$
$$=\frac{\sin^2(\overline{N}_{pqr},pqr)\cos(\overline{N}_{pqr},p,q)}{\sin^2(\overline{N}_{pqr},p,q)\sin(\overline{N}_{pqr},p,r)\sin(\overline{N}_{pqr},q,r)}$$
$$=\frac{\sin^2(N)\sin^4(N_{pqr})\cos(\overline{N}_{pqr},p,q)}{\sin^2(N_{pq})\sin^2(N_{pr})\sin^2(N_{qr})\sin^2(\overline{N}_{pqr},p,q)\sin(\overline{N}_{pqr},p,r)\sin(\overline{N}_{pqr},q,r)}$$
$$=\frac{\sin^2(N)\sin(N_{pqr})\cos(\overline{N}_{pqr},p,q)}{\sin(N_p)\sin(N_q)\sin(N_r)\sin(\overline{N}_{pqr},p,q)}$$
$$=\cos(\overline{N}_{pqr},p,q)\sin(\overline{N}_{pr},p,r)\sin(\overline{N}_{qr},q,r).$$

Therefore

$$(17)\quad \cos(\overline{N}_{pqr},p,q)=\frac{\cos(N_{pq},p,q)+\cos(\overline{N}_{pr},p,r)\cos(\overline{N}_{qr},q,r)}{\sin(\overline{N}_{pr},p,r)\sin(\overline{N}_{qr},q,r)}.$$

Moreover, as in the ordinary geometry, if the three angles in the numerator of (17) be denoted by A, B, C and their sum by $2S$, we shall express the usual $2\sqrt{\{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)\}}$ by $\sin(p'q'r')$, in accordance with what has been said at the end of §1. It may then be seen that

$$(18)\quad \sin^2(N)=\frac{\sin(N_p)\sin(N_q)\sin(N_r)\sin(p'q'r')}{\sin(N_{pqr})}.$$
$$=\frac{\sin(p'q'r')}{\sin^4(N_{pqr})}.\sin^2(N_{pq})\sin^2(N_{pr})\sin^2(N_{qr})\sin(\overline{N}_{pqr},p,q)$$
$$\sin(\overline{N}_{pqr},p,r)\sin(\overline{N}_{pqr},q,r)$$

Proceeding thus it may be seen ultimately, taking $p=n$, $q=n-1$, .. that

(19) $\sin(N)=\sin(12)\sin(13)\ldots\sin(1n)\sin(\overline{1},2,3)\ldots\sin(\overline{1},2,n)$
$\sin(\overline{12},3,4)\ldots\sin(\overline{12},3,n)\ldots\sin(\overline{1\ldots n-4},n-3,n)\sin^2((n-2)'$
$(n-1)'n')\,/\,\sin(\overline{1\ldots n-2},n-1,n)\sin(\overline{1\ldots n-3\,n-1},n-2,n)$
$\sin(\overline{1\ldots n-3\,n},n-2,n-1).$

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Comparing (16) with (19),

(20) $\sin^2(p'q'r') = \sin(\overline{N}_{pq}, p, q) \sin(\overline{N}_{pr}, p, r) \sin(\overline{N}_{qr}, q, r) \sin(\overline{N}_{pqr}, pqr)$

Also from (18), for all sets of values of p, q, r, the quantities

(21) $\dfrac{\sin(p'q'r')}{\sin(N_{pqr}) \, \Pi \sin(N_i)}$ are equal, Π denoting the continued product for $i \neq p, q, r$.

§ 5. (i) The arc (pu) drawn from a vertex p to meet the opposite face (N_p) orthogonally in u is an altitude of (N).

Since in this case $\dfrac{\sin(N_q u)}{\sin(N_{pq} u)} = \dfrac{\sin(N_{qr} u)}{\sin(N_{pqr} u)} = \ldots . = \sin(pu)$, we have

$\sin(pu) = \dfrac{\sin(N_q)}{\sin(N_{pq})} \sin(\overline{N}_{pq}, p, q)$, from (6) and $= \sin(pq) \sin(\overline{q}, p, N_{pq})$, from (1). Accordingly

(22) $\sin(pu) = \dfrac{\sin(N)}{\sin(N_p)} = \sin(pq) \sin(\overline{q}, p, r) \sin(\overline{qr}, p, s) \ldots \sin(\overline{N}_{pt}, p, t)$

(ii) Let u be any point on (N_p). Then from (13),

$$\sin(\overline{N}_{pqr}, q, r) \cos(\overline{N}_{pqr}, p, u)$$
$$= \sin(\overline{N}_{pqr}, q, u) \cos(\overline{N}_{pqr}, p, r) + \sin(\overline{N}_{pqr}, r, u) \cos(\overline{N}_{pqr}, p, q).$$

Or

(23) $\sin(N_p) \sin(N_{qr} u) \cot(\overline{N}_{pqr}, p, u) =$
$\sin(N_q) \sin(N_{pr} u) \cot(N_{pqr}, p, r) + \sin(N_r) \sin(N_{pq} u) \cot(\overline{N}_{pqr}, p, q)$.

(iii) Let u be any point in (qr). If we multiply the known formula in the ordinary geometry

$$\sin(23) \cos(1u) = \sin(2u) \cos(13) + \sin(3u) \cos(12)$$

successively by sines of the altitudes from (4) to (23), from (5) to (234) and so on, we shall have

$$\cos(pu) \sin(N_p) = \cos(pq) \sin(N_{pq} u) + \cos(pr) \sin(N_{pr} u).$$

Now divide both sides by $\sin(pu) \sin(N_p)$ and multiply and divide the right side by $\sin(\overline{u}, p, N_{pq})$ and apply (3) to the numerator.

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We shall have

$$\cot(pu)$$

$$= \frac{1}{\sin(N)}\{\cos(pq)\sin(N_q)\sin(\overline{p,u,N_{pq}}) + \cos(pr)\sin(N_r)\sin(\overline{p,u,N_{pr}})\}$$

$$= \frac{\sin(N_{qr})}{\sin(\overline{N_{qr},q,r})}\left\{\frac{\cos(pq)\sin(\overline{p,u,N_{pq}})}{\sin(N_r)} + \frac{\cos(pr)\sin(\overline{p,u,N_{pr}})}{\sin(N_q)}\right\}$$

$$= \frac{\sin(Nqrs)}{\sin(\overline{N_{qr},q,r})}\left\{\frac{\cos(pq)\sin(\overline{p,u,N_{pq}})}{\sin(N_{rs})\sin(\overline{N_{qrs},s,q})} + \frac{\cos(pr)\sin(\overline{p,u,N_{pr}})}{\sin(N_{qs})\sin(\overline{N_{qrs},s,r})}\right\}$$

Proceeding thus we have ultimately, taking

$$p = 1,\ q = n,\ r = n-1,\ s = n-2, \ldots$$

(24) $\sin(\overline{1..n-2,\ n-1,\ n})\cot(1u)$

$$= \frac{\cot(1n-1)\sin(\overline{1,u,2\,..\,n-2\,n})}{\sin(\overline{1,2,n-1})\sin(\overline{12,3,n-1})\,..\,\sin(\overline{1\,..\,n-3,n-2,n-1})}$$

$$+ \frac{\cot(1n)\sin(\overline{1,u,2..n-1})}{\sin(\overline{1,2,n})\sin(\overline{12,3,n})..\sin(\overline{1..n-3,n-2,n})}.$$

§ 6. The content of a simplex (linear) with $n+2$ vertices in a space of n dimensions vanishes. Therefore the relation between the arcs joining a point $n+1$ with n other points N on $K(N)$ is

(25) $\sin(12..n+1) = 0$. Or, squaring,

$$\Sigma\cos^2(i,n+1)\,\Delta(N_i) - \Delta(N) - 2\,\Sigma\cos(i,n+1)\cos(j,n+1)\,\Delta\binom{i}{j}N_{ij} = 0,$$

$$i, j = 1, 2, \ldots, n;\ i \neq j$$

or

$$\Sigma\cos^2(i,n+1)\sin^2(N_i) - \sin^2(N)$$
$$= 2\,\Sigma\cos(i,n+1)\cos(j,n+1)\sin(N_i)\sin(N_j)\cos(\overline{N_{ij},i,j}).$$

Now if the n points N lie on a spherical continuum of $(n-1)^{\text{th}}$ order, $\sin(N) = 0$ and it is seen that the above relation reduces to

(26) $\Sigma\cos(i\,n+1)\sin(N_i) = 0$,

which is the relation between the arcs joining any point $n+1$ with n other points N on a spherical continuum of $(n-1)^{\text{th}}$ order.

Again if the arcs $(n+1\,i)$ be produced to meet the spherical continuum of $(n-1)^{\text{th}}$ order of which $n+1$ is the pole in points $1', 2', \ldots, n'$, then

(27) $\Sigma \sin(i\,i') \sin(N_i) = 0$,

which is the relation between the arcs drawn perpendicular to one spherical continuum of $(n-1)^{\text{th}}$ order from n points on another of the same order.

Also, if in (25) we substitute $\dfrac{\overline{p\,n+2}^2 + \overline{q\,n+2}^2 - \overline{pq}^2}{2\overline{p\,n+2}\;\overline{q\,n+2}}$ for $\cos(pq)$, where $\overline{pq}$ is the distance joining p and q, we obtain

$$(28)\quad \begin{vmatrix} 0 & 1 & 1 & 1 \ldots\ldots & 1 \\ 1 & 0 & \overline{12}^2 & \overline{13}^2 \ldots\ldots & \overline{1n+2}^2 \\ 1 & \overline{12}^2 & 0 & \overline{23}^2 \ldots\ldots & \overline{2\,n+2}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \overline{1\,n+2}^2 & \overline{2\,n+2}^2 & \ldots\ldots\ldots\ldots & 0 \end{vmatrix} = 0$$

which is the identical relation connecting the distances between $n+2$ points in an n-space.

And if in (25) we substitute $1 - \dfrac{\overline{pq}^2}{2R^2}$ for $\cos(pq)$ and V denotes the content of the simplex with the $n+1$ points as vertices, we obtain

$$(29)\quad (-1)^n\, 2^{n+1} (Rn!\,V)^2 = \begin{vmatrix} 0 & \overline{12}^2 & \overline{13}^2 \ldots & \overline{1n+1}^2 \\ \overline{12}^2 & 0 & \overline{23}^2 \ldots & \overline{2\,n+1}^2 \\ \ldots & \ldots & \ldots & \ldots \\ \overline{1n+1}^2 & \overline{2n+1}^2 & \ldots\ldots\ldots & 0 \end{vmatrix},$$

giving the radius of a spherical continuum of n^{th} order circumscribing a simplex whose vertices are the $n+1$ points.

Moreover, if the $n+1$ points lie on a spherical continuum of $(n-1)^{\text{th}}$ order, $V=0$ and substituting $2R\sin\frac{1}{2}(pq)$ for $\overline{pq}$ in (29) we have

$$(30)\quad \begin{vmatrix} 0 & \sin^2\frac{1}{2}(12) & \sin^2\frac{1}{2}(13) \ldots\ldots \sin^2\frac{1}{2}(1n+1) \\ \sin^2\frac{1}{2}(12) & 0 & \sin^2\frac{1}{2}(23) \ldots\ldots \sin^2\frac{1}{2}(2\,n+1) \\ \ldots & \ldots & \ldots \\ \sin^2\frac{1}{2}(1\,n+1) & \sin^2\frac{1}{2}(2\,n+1) \ldots\ldots\ldots\ldots & 0 \end{vmatrix} = 0,$$

which is the relation between the arcs joining $n+1$ points on a spherical continuum of $(n-1)^{\text{th}}$ order.

Finally, put $(1n+1) = (2\,n+1) = \ldots = (n\,n+1) = r$ in (25); r is then the spherical radius of the small (in the ordinary sense) spherical continuum of $(n-1)^{\text{th}}$ order circumscribing (N). Accordingly,

$$(31)\quad \sec^2 r \sin^2(N) = \Sigma \sin^2(N_i) - 2\,\Sigma \sin(N_i)\sin(N_j)\cos(\overline{N_{ij}}, i, j).$$

Also from (29) we have

$$\begin{vmatrix} 0 & 1 & 1 & 1 \ldots\ldots & 1 & 1 \\ 1 & 0 & \overline{12}^2 & \overline{13}^2 \ldots\ldots & \overline{1n+1}^2 & R^2 \\ 1 & \overline{12}^2 & 0 & \overline{23}^2 \ldots\ldots & \overline{2\,n+1}^2 & R^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \overline{1n+1}^2 & \overline{2\,n+1}^2 & \ldots\ldots & 0 & R^2 \\ 1 & R^2 & R^2 & \ldots\ldots & R^2 & 0 \end{vmatrix} = 0$$

This, on substitution of $2R\sin\frac{1}{2}(pq)$ for $\overline{pq}$ and R for $(p\,n+1)$, gives us

$$(32)\quad \sin^2 r \begin{vmatrix} 0 & 1 & 1 & 1 \ \ldots\ldots & 1 \\ 1 & 0 & \sin^2\frac{1}{2}(12) & \sin^2\frac{1}{2}(13)\ldots\sin^2\frac{1}{2}(1n) \\ 1 & \sin^2\frac{1}{2}(12) & 0 & \sin^2\frac{1}{2}(23)\ldots\sin^2\frac{1}{2}(2n) \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \sin^2\frac{1}{2}(1n) & \sin^2\frac{1}{2}(2n) & \ldots\ldots & 0 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 0 & \sin^2\frac{1}{2}(12) & \sin^2\frac{1}{2}(13)\ldots\sin^2\frac{1}{2}(1n) \\ \sin^2\frac{1}{2}(12) & 0 & \sin^2\frac{1}{2}(23)\ldots\sin^2\frac{1}{2}(2n) \\ \ldots & \ldots & \ldots \\ \sin^2\frac{1}{2}(1n) & \sin^2\frac{1}{2}(2n) & \ldots\ldots \quad 0 \end{vmatrix}.$$