J. Austral. Math. Soc. Ser. B 40(1999), 314–335

THE CONTROL PARAMETERIZATION ENHANCING TRANSFORM FOR CONSTRAINED OPTIMAL CONTROL PROBLEMS

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(Received 6 February 1997; revised 7 April 1997)

Abstract

Consider a general class of constrained optimal control problems in canonical form. Using the classical control parameterization technique, the time (planning) horizon is partitioned into several subintervals. The control functions are approximated by piecewise constant or piecewise linear functions with pre-fixed switching times. However, if the optimal control functions to be obtained are piecewise continuous, the accuracy of this approximation process greatly depends on how fine the partition is. On the other hand, the performance of any optimization algorithm used is limited by the number of decision variables of the problem. Thus, the time horizon cannot be partitioned into arbitrarily many subintervals to reach the desired accuracy. To overcome this difficulty, the switching points should also be taken as decision variables. This is the main motivation of the paper. A novel transform, to be referred to as the control parameterization enhancing transform, is introduced to convert approximate optimal control problems with variable switching times into equivalent standard optimal control problems involving piecewise constant or piecewise linear control functions with pre-fixed switching times. The transformed problems are essentially optimal parameter selection problems and hence are solvable by various existing algorithms. For illustration, two non-trivial numerical examples are solved using the proposed method.

1. Introduction

Optimal control is a mathematically challenging subject. Moreover, it has many practical applications in a wide range of disciplines such as engineering, economics and the environmental sciences, to name just a few. Numerous computational methods for various constrained optimal control problems are now available in the literature

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https://doi.org/10.1017/S0334270000010936 Published online by Cambridge University Press

(see [2, 4–6, 11–16, 20, 21] and the references cited therein).

The formulation of a typical optimal control problem may involve state variables, control variables, system parameters, an objective functional (or multiple objective functionals), state differential equations, initial and terminal state conditions, upper and lower bounds on all variables and constraints of various types. See [21] for details.

A detailed exposition of the classical control parameterization technique as a basis for solving various constrained optimal control problems numerically in a unified fashion may also be found in [21]. The classical control parameterization method is a flexible and efficient approach for a large class of optimal control problems. The central idea of the method relies on a simple and elegant approximation mechanism. The time (planning) horizon is partitioned into several subintervals and the controls are approximated by piecewise constant (or piecewise linear) functions consistent with this partition. The switching times defined by these subintervals are referred to as knots. The heights of the piecewise constant (or piecewise linear) functions are now decision variables to be optimized. Control parameterization can thus be used to approximate an optimal control problem by a finite-dimensional optimal parameter selection problem.

Since an optimal parameter selection problem can be viewed as a mathematical programming problem, the approximate problem can be readily solved by various existing optimization techniques such as those reported in [18, 19, 23] and the relevant references cited therein.

To improve the accuracy of the approximate solution thus obtained, we can refine the partition of the time horizon. This refinement is usually performed iteratively, and is terminated when a satisfactory solution is obtained. An optimal control software MISER3 [7] has been developed implementing these techniques. Their theoretical justifications can be found in [21]. Also see [3] and [9] for further results on the control parameterization technique. For an illustration of the numerical results obtained by the control parameterization technique, see [21] and the references cited therein, where many practical problems are solved.

In spite of the flexibility and the efficiency of the classical control parameterization approach, there are several numerical difficulties associated with it which are yet to be addressed.

Consider the case in which the optimal control to be obtained belongs to the class of piecewise continuous functions. The possibly finite number of discontinuity points inherited from this class of functions are referred to as switching times.

Clearly the number of, as well as the locations of, these switching times are not known in advance. The accuracy of the classical control parameterization method thus depends greatly on the choice of knot distribution. The ideal knot distribution would be to have a knot placed exactly at the location of each switching time. If one has no insight of how the switching times are distributed, a set of dense and evenly distributed knots is usually chosen in the hope that there would be a knot placed near each switching time. Hence the number of parameters in the approximate optimal parameter selection problem is usually very large. However, as the number of parameters increases, the optimization process quickly becomes much more expensive in terms of the computational time required.

Intuitively, the control parameterization method would be more effective if the switching times could be treated as decision variables to be optimized just like the control and system parameters. This would largely reduce the overall number of parameters used. However, the gradients of the cost functional and constraint functionals with respect to these switching times are known to be discontinuous (see Chapter 5 of [21] for details). Another difficulty is that the differential equations governing the dynamics of the problem would now be only piecewise continuous with the points of discontinuity (that is, the switching times) varying from one iteration to the next with the optimization process. Furthermore, the number of the decision variables would change when two or more switching times coalesce. Thus the task of integrating the differential equations accurately can be very involved. For these reasons, the gradient formulae presented in [5] and Chapter 5 of [21] were never implemented for a practical problem.

In this paper a novel transform, to be referred to as the control parameterization enhancing transform (CPET), is introduced to enhance the classical control parameterization technique. This transform involves the introduction of an additional piecewise constant control function. Using the CPET, the switching times are mapped on to an equally spaced set of knots in a new time scale. Hence, the transformed optimal control problem can be solved readily and accurately by the usual control parameterization technique. From our extensive numerical studies, CPET is far superior to the classical control parameterization technique. This is particularly true for time optimal control problems, see [10]. The remainder of the paper is organized as follows.

In Sections 2 and 3, a generic optimal control problem is defined along with the control parameterization technique. Section 4 introduces the CPET for handling the switching times. Section 5 contains a short convergence analysis in relation to previous results ([3, 21]) for the classical control parameterization method. Two examples are discussed in Sections 6 and 7.

This paper is dedicated to B. D. Craven.

2. The optimal control problem

Consider a process described by the following set of differential equations defined on $(0, T]$:

$$\dot{x}(t) = f(x(t), u(t), z), \tag{2.1}$$

https://doi.org/10.1017/S0334270000010936 Published online by Cambridge University Press

with the (possibly variable) initial conditions:

$$x(0) = x^0(z), \tag{2.2}$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^\top \in \mathbb{R}^n$, $u(t) = [u_1(t), \ldots, u_r(t)]^\top \in \mathbb{R}^r$ and $z = [z_1, \ldots, z_m]^\top \in \mathbb{R}^m$ are, respectively, the state, control and system parameter vectors. The vectors $f = [f_1, \ldots, f_n]^\top \in \mathbb{R}^n$ and $x^0 = [x_1^0, \ldots, x_n^0]^\top \in \mathbb{R}^n$ are continuously differentiable with respect to their respective arguments.

Two types of constraints on the controls and system parameters are introduced as follows:

$$h_k(z) = 0, \qquad k = 1, \ldots, N_1, \tag{2.3}$$

$$h_k(z) \geq 0, \qquad k = N_1 + 1, \ldots, N_2 \tag{2.4}$$

and

$$g_k(t) = \sum_{i=1}^{r} \alpha_{ki} u_i(t) + \beta_k = 0, \quad \forall t \in [0, T], \; k = 1, \ldots, N_3, \tag{2.5}$$

$$g_k(t) = \sum_{i=1}^{r} \alpha_{ki} u_i(t) + \beta_k \geq 0, \quad \forall t \in [0, T], k = N_3 + 1, \ldots, N_4. \tag{2.6}$$

Equations (2.3) and (2.4) involve only the system parameters, while all-time linear constraints involving only controls are included in (2.5) and (2.6). All of these constraints are independent of the state variables, and hence their gradients can be calculated directly.

All controls and system parameters are subject to upper and lower bounds as follows:

$$u_i^L \leq u_i(t) \leq u_i^U, \quad \forall t \in [0, T], \qquad i = 1, \ldots, r, \tag{2.7}$$

$$z_i^L \leq z_i \leq z_i^U \qquad i = 1, \ldots, m. \tag{2.8}$$

A measurable function $u : [0, T] \longrightarrow \mathbb{R}^r$ is called an admissible control if the constraints (2.5), (2.6) and (2.7) are satisfied. Let $\mathcal{U}$ be the class of all such admissible controls. Similarly, $z \in \mathbb{R}^m$ is called an admissible system parameter vector if the constraints (2.3), (2.4) and (2.8) are satisfied. $\mathcal{Z}$ denotes the set of all such admissible system parameter vectors. A pair $(u, z) \in \mathcal{U} \times \mathcal{Z}$ is referred to as an admissible pair. For an admissible pair $(u, z) \in \mathcal{U} \times \mathcal{Z}$, let $x(\cdot)$ denote the corresponding solution of system (2.1).

Constraints involving the state variables or nonlinear constraints in control functions can be described in the standard canonical form

$$G_k(u, z) = 0, \qquad k = 1, \ldots, N_5, \tag{2.9}$$

$$G_k(u, z) \geq 0, \qquad k = N_5 + 1, \ldots, N_6, \tag{2.10}$$

where, for each $k = 1, \ldots, N_6$,

$$G_k(\boldsymbol{u}, z) = \Phi_k(\boldsymbol{x}(T_k), z) + \int_0^{T_k} \mathscr{L}_k(\boldsymbol{x}(t), \boldsymbol{u}(t), z)\,dt. \tag{2.11}$$

Here Φ_k, $k = 1, \ldots, N_6$, and $\mathscr{L}_k$, $k = 1, \ldots, N_6$, are given real valued functions which are continuously differentiable with respect to each of their arguments, and $T_k \in (0, T]$ is referred to as the characteristic time for the k-th constraint with $T_0 = T$ by convention.

An admissible pair $(\boldsymbol{u}, z) \in \mathscr{U} \times \mathscr{Z}$ is called a feasible pair if the constraints (2.9) and (2.10) are also satisfied. Let $\mathscr{F} \times \mathscr{S}$ be the class of all feasible pairs.

The optimal control problem considered in this paper may now be stated as follows.

Subject to the dynamical system (2.1) and (2.2), find a feasible pair $(\boldsymbol{u}, z) \in \mathscr{F} \times \mathscr{S}$ such that the cost functional

$$G_0(\boldsymbol{u}, z) = \Phi_0(\boldsymbol{x}(T), z) + \int_0^{T} \mathscr{L}_0(\boldsymbol{x}(t), \boldsymbol{u}(t), z)\,dt \tag{2.12}$$

is minimized over $\mathscr{F} \times \mathscr{S}$.

Let this optimal control problem be referred to as Problem (P).

REMARK 2.1. Note that the functions defining Problem (P) do not depend explicitly on time. Let Problem (Q) denote a problem identical to Problem (P) except that the functions defining it depend explicitly on time. Then, it is well-known that by introducing an additional state variable, Problem (Q) can be easily transformed into the form of Problem (P). For details see, for example, [1]. Thus, without loss of generality, we only consider Problem (P) unless otherwise stated.

REMARK 2.2. Consider the continuous state inequality constraints defined as follows:

$$h_k(\boldsymbol{x}(t), z) \geq 0, \quad \forall t \in [0, T], k = 1, \ldots, N_S. \tag{2.13}$$

Then, by using the $\epsilon - \tau$ method given in Chapter 8 of [21], we can approximate these continuous state inequality constraints as inequality constraints in canonical form

$$\tau + \int_0^{T} \mathscr{L}_{k,\varepsilon}(x(t), z)\,dt \geq 0, \tag{2.14}$$

where

$$\mathscr{L}_{k,\varepsilon}(\boldsymbol{x}(t), z) = \begin{cases} h_k(\boldsymbol{x}(t), z), & \text{if } h_k(\boldsymbol{x}(t), z) < -\varepsilon, \\ -(h_k(\boldsymbol{x}(t), z) - \varepsilon)^2/4\varepsilon, & \text{if } -\varepsilon \leq h_k(\boldsymbol{x}(t), z) \leq \varepsilon, \\ 0, & \text{if } h_k(\boldsymbol{x}(t), z) > \varepsilon. \end{cases} \tag{2.15}$$

Under appropriate assumptions, it is shown in Lemma 8.3.3 of [21] that there exists a $\tau(\varepsilon) > 0$ such that for all τ, $0 < \tau < \tau(\varepsilon)$, if an admissible pair $(\boldsymbol{u}, z) \in \mathcal{U} \times \mathcal{Z}$ satisfies the constraints (2.14), then it also satisfies the constraints (2.13). Thus optimal control problems involving continuous state inequality constraints (2.13) can also be cast in the form of Problem (P).

REMARK 2.3. Note that many classes of optimal control problems involving, for example, terminal state constraints, interior point constraints and periodic boundary conditions, can be transformed into the form of Problem (P). For further details, see [8] and Chapters 6 and 10 of [21].

3. The classical control parameterization

We now briefly describe the control parameterization method. Essentially, each control component $u_i(t)$ is approximated by a zeroth order or first-order spline function (that is, a piecewise constant function or a piecewise linear continuous function) defined on a set of knots $\{0 = t_0^i, t_1^i, t_2^i, \ldots, t_{p_i}^i = T\}$. Note that each component may have a different set of knots and the knots are not necessarily equally spaced. For the case of piecewise constant basis functions, we write the i-th control component as a sum of basis functions with coefficients or parameters $\{\sigma_{ij}, j = 1, \ldots, p_i\}$:

$$u_i(t) = \sum_{j=1}^{p_i} \sigma_{ij} B_{ij}^{(0)}(t), \tag{3.1}$$

where $B_{ij}^{(0)}(t)$ is the indicator function for the j-th interval of the i-th set of knots defined by

$$B_{ij}^{(0)}(t) = \chi_{ij}(t) = \begin{cases} 1, & t_{j-1}^i \le t < t_j^i, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

For piecewise linear continuous basis functions, we write the i-th control component as

$$u_i(t) = \sum_{j=0}^{p_i} \sigma_{ij} B_{ij}^{(1)}(t), \tag{3.3}$$

https://doi.org/10.1017/S0334270000010936 Published online by Cambridge University Press

where $B_{ij}^{(1)}(t)$ are the witch's hat functions defined by

$$B_{i1}^{(1)}(t) = \begin{cases} (t - t_1^i)/(t_0^i - t_1^i), & t \in [t_0^i, t_1^i], \\ 0, & \text{otherwise}, \end{cases}$$

$$B_{ij}^{(1)}(t) = \begin{cases} (t - t_{j-1}^i)/(t_j^i - t_{j-1}^i), & t \in [t_{j-1}^i, t_j^i], \\ (t - t_{j+1}^i)/(t_j^i - t_{j+1}^i), & t \in [t_j^i, t_{j+1}^i], j = 1, \dots, p_i - 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.4}$$

$$B_{ip_i}^{(1)}(t) = \begin{cases} (t - t_{k_i-1}^i)/(t_{k_i}^i - t_{k_i-1}^i), & t \in [t_{p_i-1}^i, t_{p_i}^i], \\ 0, & \text{otherwise}. \end{cases}$$

These basis functions have finite support and have the added property that

$$\sigma_{ij} = u_i(t_j).$$

This latter property is important for constraints (2.7) (that is, the boundedness constraints of control) and for (2.5) and (2.6) (that is, the all-time linear constraints on control). For higher degree splines, we cannot have both small finite support and $\sigma_{ij} = u_i(t_j)$. Judging from our extensive simulation studies, it appears that the ill-conditioning inherent in optimal control problems could get worse if higher-degree splines are used to approximate the controls. Many interesting questions in this regard remain unsolved. Note the system parameters are constant for all t. Thus, with r control components, all piecewise constant, the number of such control parameters will be $p = \sum_{i=1}^{r} p_i$ and the total number of parameters will include the m system parameters to give a total of $\eta = \sum_{i=1}^{r} p_i + m$ parameters.

After the control parameterization, the cost function (2.12) as well as all the constraint functions (2.3)-(2.10) can be regarded as (somewhat implicit) functions of the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\sigma}, \boldsymbol{z})$, where (for piecewise constant controls)

$$\boldsymbol{\sigma} = [\sigma_{11}, \dots, \sigma_{1p_1}, \sigma_{21}, \dots, \sigma_{2p_2}, \dots, \sigma_{r1}, \dots, \sigma_{rp_r}]^{\top} \in \mathbb{R}^p$$
$$\boldsymbol{z} = [z_1, \dots, z_m]^{\top} \in \mathbb{R}^m$$

and

$$\boldsymbol{\theta} = [\boldsymbol{\sigma}^{\top}, \boldsymbol{z}^{\top}]^{\top} \in \mathbb{R}^{\eta}.$$

More precisely, the corresponding approximate problem may be written as

$$\min \widetilde{G}_0(\boldsymbol{\theta}) \tag{3.5}$$

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subject to

$$h_k(z) = 0, \qquad k = 1, \dots, N_1, \tag{3.6}$$

$$h_k(z) \geq 0, \qquad k = N_1 + 1, \dots, N_2, \tag{3.7}$$

$$\widetilde{g}_k(\boldsymbol{\theta}) = 0, \qquad k = 1, \dots, N_3, \tag{3.8}$$

$$\widetilde{g}_k(\boldsymbol{\theta}) \geq 0, \qquad k = N_3 + 1, \dots, N_4, \tag{3.9}$$

$$\widetilde{G}_k(\boldsymbol{\theta}) = 0, \qquad k = 1, \dots, N_5, \tag{3.10}$$

$$\widetilde{G}_k(\boldsymbol{\theta}) \geq 0, \qquad k = N_5 + 1, \dots, N_6, \tag{3.11}$$

$$u_{ij}^L < \sigma_{ij} < u_{ij}^U, \qquad j = 1, \dots, p_i;\ i = 1, \dots, r, \tag{3.12}$$

$$z_k^L < z_k < z_k^U, \qquad k = 1, \dots, m, \tag{3.13}$$

where $\widetilde{g}_k$, $k = 1, \dots, N_4$ and $\widetilde{G}_k$, $k = 0, \dots, N_6$, are obtained from (2.5), (2.6), (2.12), (2.9) and (2.10), respectively, in an obvious way.

REMARK 3.1. Note that bounds on σ_{ij} automatically put bounds on the control functions for all time as appropriate. The problem posed by (3.5)-(3.13) is now a standard constrained nonlinear mathematical programming problem which can be solved by means of a sequential quadratic programming technique. See, for example, [18, 19, 23] or the web site

http://www.mcs.anl.gov/home/otc/Guide/SoftwareGuide/ .

Like many nonlinear programming techniques, these require the analytical gradients of the cost function $\widetilde{G}_0$ as well as the constraint functions h_k, $k = 1, \dots, N_2$, $\widetilde{g}_k$, $k = 1, \dots, N_4$, $\widetilde{G}_k$,$k = 1, \dots, N_6$. However, the dependence of $\widetilde{G}_k$, $k = 0, 1, \dots, N_6$, on $\boldsymbol{\theta}$ is not explicit. Thus, their gradients need to be calculated in a somewhat roundabout way. See [7] or Chapter 5 of [21] for details.

4. The control parametrization enhancing transform

Consider the new time scale s which varies from 0 to 1. The transformation from $t \in [0, T]$ to $s \in [0, 1]$ can be defined by the differential equation

$$\frac{dt(s)}{ds} = v(s) \tag{4.1}$$

with the initial condition

$$t(0) = 0, \tag{4.2}$$

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where the scalar function $v(s)$ is called the enhancing control. It is a piecewise constant function with possible discontinuities at the pre-fixed knots $\xi_0, \ldots, \xi_M$, that is,

$$v(s) = \sum_{i=1}^{M} v_i \chi_i(s), \tag{4.3}$$

where $\chi_i(s)$ is the indicator function defined by

$$\chi_i(s) = \begin{cases} 1, & \text{if } s \in [\xi_{i-1}, \xi_i), \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,

$$t(s) = \int_0^s v(\tau)\, d\tau = \sum_{j=1}^{i-1} v_j \left(\xi_j - \xi_{j-1}\right) + v_i \left(s - \xi_{i-1}\right), \quad s \in [\xi_{i-1}, \xi_i]. \tag{4.4}$$

The optimal control function can thus be written in terms of the new time variable s, where

$$\widehat{\boldsymbol{u}}(s) = \boldsymbol{u}(t(s)), \quad \widehat{\boldsymbol{x}}(s) = [(\boldsymbol{x}(t(s)))^{\top}, t(s)]^{\top}, \quad \boldsymbol{\eta}(s) = [(\widehat{\boldsymbol{u}}(s))^{\top}, v(s)]^{\top}. \tag{4.5}$$

The equivalent transformed optimal control problem may now be stated as follows:

$$\min \widetilde{G}_0(\boldsymbol{\eta}, z) = \Phi_0(\widehat{\boldsymbol{x}}(1), z) + \int_0^1 v(s) \mathscr{L}_0(\widehat{\boldsymbol{x}}(s), \widehat{\boldsymbol{u}}(s), z)\, ds \tag{4.6}$$

subject to the dynamical system

$$\frac{d\widehat{\boldsymbol{x}}(s)}{ds} = \begin{pmatrix} v(s)\boldsymbol{f}(t(s), \widehat{\boldsymbol{x}}(s), \widehat{\boldsymbol{u}}(s), z) \\ v(s) \end{pmatrix} \tag{4.7}$$

with initial condition

$$\widehat{\boldsymbol{x}}(0) = \begin{pmatrix} \boldsymbol{x}^0 \\ 0 \end{pmatrix} \tag{4.8}$$

and subject to constraints (2.3)-(2.8) and to the canonical constraints

$$\widetilde{G}_k(\widehat{\boldsymbol{u}}, z) = 0, \quad k = 1, \ldots, N_5. \tag{4.9}$$

$$\widetilde{G}_k(\widehat{\boldsymbol{u}}, z) \geq 0, \quad k = N_5 + 1, \ldots, N_6, \tag{4.10}$$

where

$$\widetilde{G}_k(\widehat{\boldsymbol{u}}, z) = \phi_k(\widehat{\boldsymbol{x}}(\tau_k), z) + \int_0^{\tau_k = \xi_{i_k}} v(s) \mathscr{L}_k(\widehat{\boldsymbol{x}}(s), \widehat{\boldsymbol{u}}(s), z)\, ds$$

and $\tau_k = s(T_k)$ is one of the knot points in [0, 1]. For the new control $v(s)$, it is necessary that

$$v(s) \geq 0, \qquad s \in [0, 1], \tag{4.11}$$

so that the new state $t(s)$ is monotonically non-decreasing. Usually, an additional terminal state constraint is required to prevent it vanishing to zero during the optimization process. For some problems, it is necessary to impose the constraint

$$t(1) - T = 0, \tag{4.12}$$

while for others, the constraint

$$t(1) - z_j = 0 \tag{4.13}$$

is used, where z_j is some system parameter which has a lower bound of zero.

Let the corresponding transformed optimal control problem be referred to as Problem $(\widetilde{P})$.

REMARK 4.1. Consider a modified version of Problem $(\widetilde{P})$, where the differential equation (4.1) and its initial condition (4.2) are dropped, while the terminal time constraints (4.12) and (4.13) are replaced, respectively, by

$$\int_0^1 v(s)\, ds - T = 0 \tag{4.14}$$

and

$$\int_0^1 v(s)\, ds - z_j = 0. \tag{4.15}$$

Let this slightly simplified problem be referred to as Problem $(\widehat{P})$. The solution to Problem $(\widehat{P})$ can then be used to construct the solution of the original problem through solving (4.4) to obtain t as a function of s.

REMARK 4.2. Note that in the transformed problem, only the knots contribute to the discontinuities of the state differential equation. Thus, all locations of the discontinuities of the state differential equation are known and fixed during the optimization process. These locations will not change from one iteration to the next during the optimization process. Even when two or more of the original switching times coalesce, the number of these locations remains unchanged in the transformed problem. Furthermore, the gradients of the cost functional and constraint functionals with respect to the original switching times in the new transformed problem are provided by the usual gradient formulae in the classical control parametrization context.

5. Convergence analysis

The basic ideas behind the CPET method developed in Section 4 is aiming to include the switching times as parameters to be optimized and, at the same time, to avoid the numerical difficulties mentioned in Section 1. Clearly, if the optimal control is a piecewise constant function with discontinuities at $t_1, \ldots, t_M$, then, by solving the transformed problems with number of knots greater or equal to M, and by using (3.1), we obtain the exact optimal control. Similar conclusions hold if the optimal control is piecewise linear and (3.3) is used instead. For the general case, the convergence results are much harder to establish. In view of the convergence analysis given for the classical control parametrization technique in Chapter 6 of [21] and Chapter 7 to Chapter 10 for special cases, we note that the time interval is partitioned as follows.

Construct a monotonically non-decreasing sequence $\{S^p\}_{p=1}^{\infty}$ of finite subsets of $[0, T]$. For each p, let $n_p + 1$ points of S^p be defined by

$$t_0^p = 0, \quad t_{n_p}^p = T, \quad t_{k-1}^p < t_k^p, \quad k = 1, 2, \ldots, n_p.$$

Then, associated with each S^p there is the obvious partition $\mathscr{I}^p$ of $[0, T]$ defined by

$$\mathscr{I}^p = \left\{ I_k^p : k = 1, \ldots, n_p \right\},$$

where $I_k^p = [t_{k-1}^p, t_k^p)$.

We choose S^p such that S^{p+1} is a refinement of S^p and $\lim_{p\to\infty} S^p$ is dense in $[0, T]$.

With such partitions, convergence results have been obtained for classical control parametrization technique under appropriate conditions. For details, see [3] and [21].

Using the notation of [21], the approximate problem after control parametrization is denoted by Problem (P(p)). It is a finite dimensional optimization problem with n_p parameters for each original control variable. Applying this to the transformed problem of Section 4, we obtain similar approximate problems except with an additional enhancing control. This enhancing control captures the discontinuities of the optimal control if the number of knots in the partition of the new time horizon is greater than or equal to the number of discontinuities of the optimal control. Since the enhancing parameters v_i are allowed to vary, the control parametrization enhancing technique gives rise to a larger search space and hence produces a better or at least equal suboptimal cost. Note that the optimal objective function value of an approximate Problem (P(p)) with enhancing control is less than or equal to the optimal objective function value of the same problem without an enhancing control. Hence, by the squeeze theorem, since convergence has been proven for the problem without an enhancing control, the convergence for the problem with an enhancing control is guaranteed. All the corresponding convergence results obtained in Chapter 6 of [21] remain valid. In fact, the relevent corresponding convergence results obtained in [3] and for the special cases considered in Chapter 7 to Chapter 10 of [21] also hold if the control parametrization enhancing technique is used to construct approximate problems.

6. A manufacturing system example

Consider a deterministic manufacturing system over time t, with storage variables $x_i(t)$, $i = 1, \ldots, n_s$, and machine rate of processing variables $u_i(t)$, $i = 1, \ldots, n_c$. The output demand rates are functions $d_i(t)$, $i = 1, \ldots, n_d$ which are assumed constant in time for this example. The system has simplified linear dynamics

$$\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{u} - \boldsymbol{C}\boldsymbol{d}, \tag{6.1}$$

for given constant matrices $\boldsymbol{B}$, $(n_s \times n_c)$, and $\boldsymbol{C}$, $(n_s \times n_d)$. A constraint on the system is to keep storages placed before the $n_m = n_s - n_d$ machines non-negative. Label these storages from $i = 1, \ldots, n_m$. Storages from where the demand is taken, labelled $i = n_m + 1, \ldots, n_s$, are allowed to be negative but this is considered undesirable as it represents unmet demand. Machines have upper bounds on their rate of throughput. The objective of the manufacturing system is to keep storages as low as possible but to satisfy demand.

Let

$$a^+ = \max\{0, a\} = \begin{cases} a, & a \geq 0, \\ 0, & a < 0 \end{cases}$$

and

$$a^- = \max\{0, -a\} = \begin{cases} 0, & a \geq 0, \\ -a, & a < 0. \end{cases}$$

A suitable objective function for this system is

$$G_0(\boldsymbol{u}) = \int_0^T e^{-\delta t} \left[\sum_{i=1}^{n_m} (a_i x_i + \gamma_i x_i^-) + \sum_{i=n_f+1}^{n_s} (a_i x_i^+ + b_i x_i^-) + \mu \sum_{j=1}^{n_c} u_j^2 \right] dt. \tag{6.2}$$

The numbers a_i are weights attached to the storage representing a cost per unit of storage in store i, while b_i represents the cost per unit of unmet demand and is usually larger than the corresponding a_i. The γ_i are much larger than the corresponding a_i and act as a penalty on the storages placed before a machine being negative. In [22] it is shown that γ_i can be made large enough so that the corresponding x_i are never negative at a solution, after suitable smoothing of a^+ and a^-.

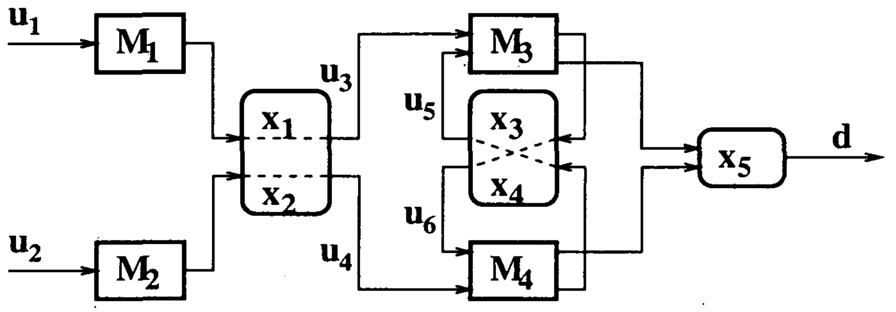

FIGURE 1. A manufacturing system.

This example (see Figure 1) has one final product with demand d, but the product can be made in one of two ways, namely by machines (M_1, M_3, M_4) or (M_2, M_4, M_3). The intermediate storages are shared, with storage x_3+x_4 limited. The costs of storage of the part-made product are the same in each combined storage. The dynamics are

$$\begin{aligned}\dot{x}_1 &= u_1 - u_3,\\ \dot{x}_2 &= u_2 - u_4,\\ \dot{x}_3 &= u_3 - u_6,\\ \dot{x}_4 &= u_4 - u_5,\\ \dot{x}_5 &= u_5 + u_6 - d.\end{aligned}$$

The state constraints are

$$\begin{aligned}x_i &\geq 0, \quad i = 1, \dots, 4,\\ x_3 + x_4 &\leq 1.\end{aligned}$$

The control constraints are

$$\begin{aligned}u_i &\geq 0, \quad i = 1, \dots, 6,\\ u_1 &\leq 1,\\ u_2 &\leq 2,\\ u_3 + u_5 &\leq 1,\\ u_4 + u_6 &\leq 2.\end{aligned}$$

The all-time state constraint $1 - x_3 - x_4 \geq 0$ is treated as a penalty in the objective function similar to the lower bound state constraints. The penalty value for these all-time state constraints is $\gamma_i = 20$. The discounting factor is zero. The costs are $a_1 = a_2 = 2$, $a_3 = a_4 = 1$ and $a_5 = 4$. The initial point in state space is $\boldsymbol{x}(0) = (1, 1, 0, 0, 1)$.

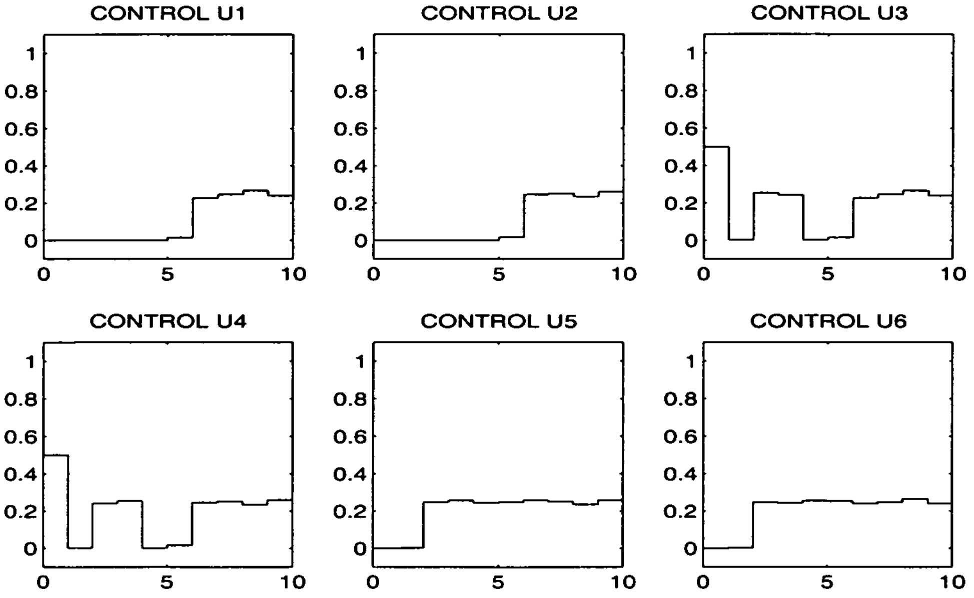

FIGURE 2. Controls for the manufacturing system, Case 1.

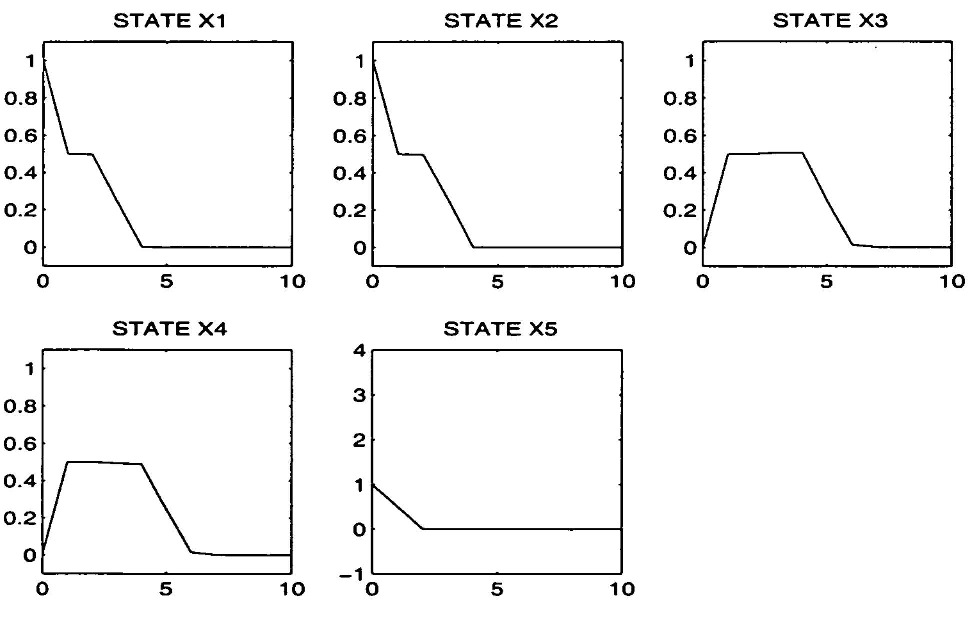

FIGURE 3. States for the manufacturing system, Case 1.

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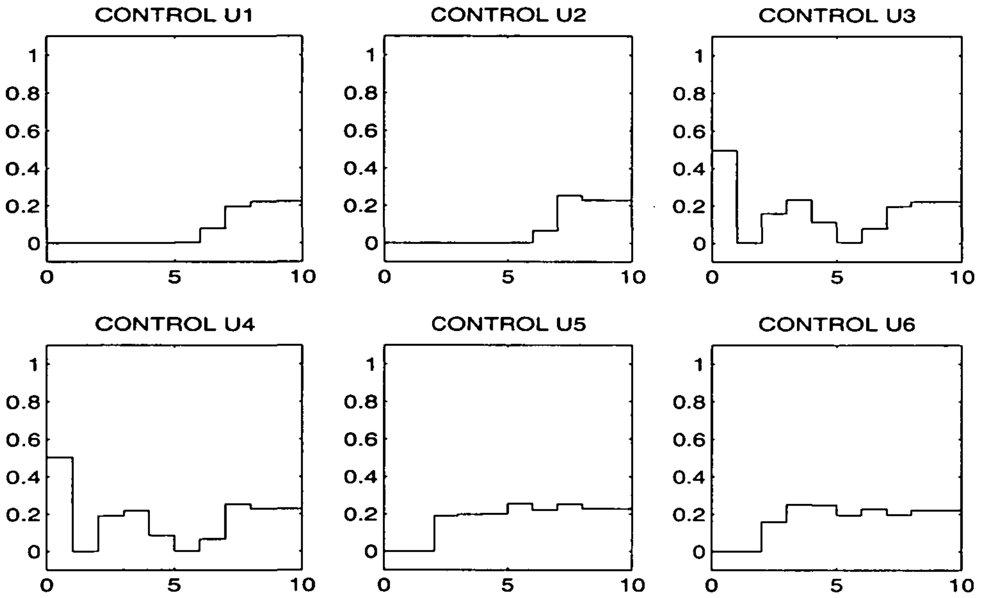

FIGURE 4. Controls for the manufacturing system, Case 2.

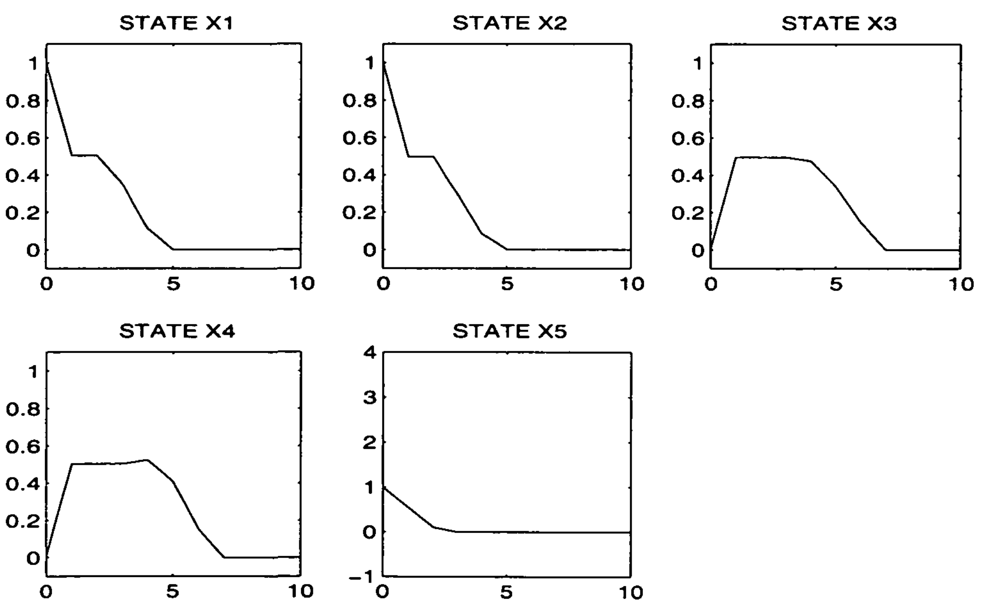

FIGURE 5. States for the manufacturing system, Case 2.

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The model is computed three times. The first is without an enhancing control, but with a value of d so that the step changes in control occur at some of the chosen fixed switching points of the control. The time interval [0, 10] is divided into ten equal intervals with the control functions parametrized by piecewise constants, making the state trajectories piecewise linear continuous. Note that the control levels are likely to change whenever a new boundary constraint of state space is reached. This case has four switching points for controls u_3 and u_4. These four switching points occur when boundaries of state space are reached. The first switching point at $t = 1$ occurs because store $x_3 + x_4$ becomes full. The second at $t = 2$ occurs because store x_5 becomes zero. The third at $t = 4$ occurs because store $x_1 + x_2$ becomes zero, while the fourth at $t = 6$ occurs because store $x_3 + x_4$ becomes empty.

Case 2 also has no enhancing control, but $d = 0.45$, a value chosen so that the times of switching (optimally) do not occur at any of the chosen fixed discontinuities of the controls. The controls are parametrized as for Case 1. Figures 4 and 5 show the effect on the control levels about the true switching points. The value of the objective is 17.37.

Case 3 uses the enhancing control with controls chosen with the required number of switches. The times of switching are chosen optimally using the enhancing control technique. This only involves 18 control parameters for $u_1, \ldots, u_6$ and 5 control parameters for the enhancing control, much less than the 60 used in Cases 1 and 2. The objective value is 16.8905, a clearly better value.

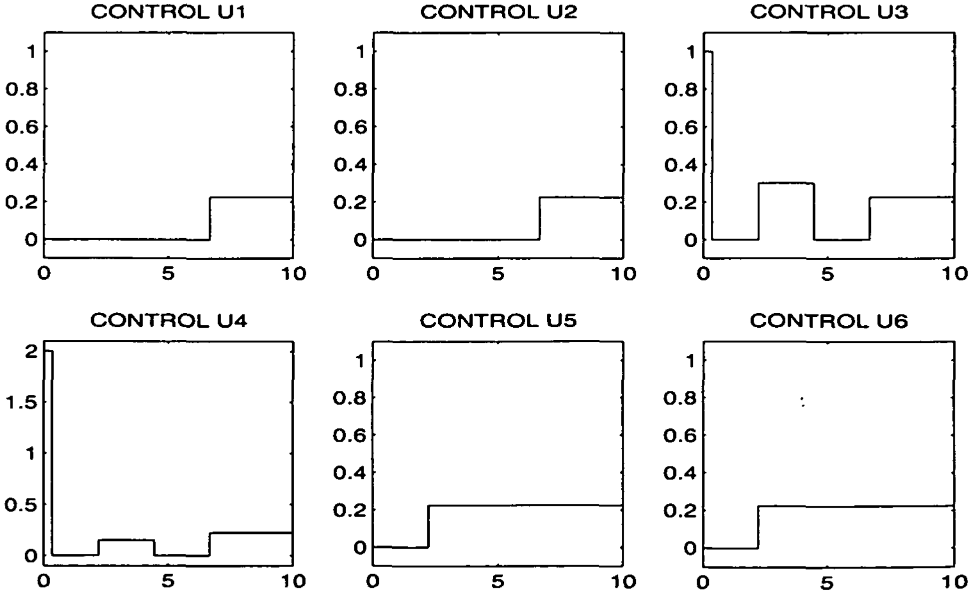

FIGURE 6. Controls for the Manufacturing System, Case 3.

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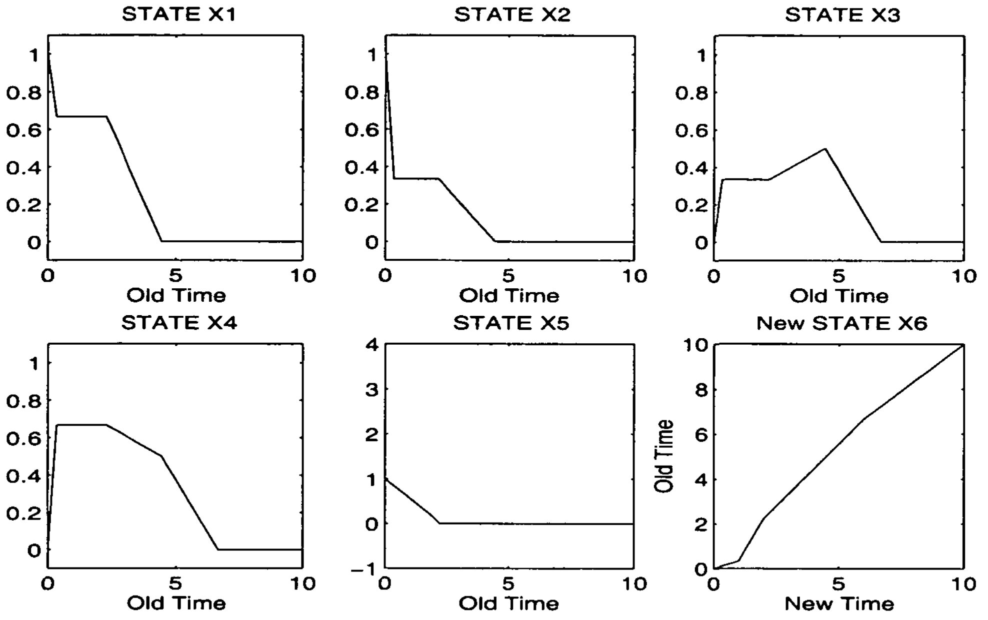

FIGURE 7. States for the Manufacturing System, Case 3.

The performance of the optimization algorithm is better for Case 3. The number of iterations to achieve the solutions in the three cases are 575, 349 and 235, respectively. Note also that Cases 1 and 2 had to be restarted (reset the Hessian) many times to achieve convergence to a similar accuracy. Case 3 only had to be restarted once.

Of some interest are the condition numbers of the three cases. The active constraint gradients have condition numbers of 1, 1 and 7, which is to be expected. However the condition numbers of the projected Hessians are 7×10^6, 7×10^6 and 800, indicating the new method is far superior in terms of algorithmic performance for this type of optimal control problem.

7. The container crane example

The dynamics of a container crane model are (see [17])

$$\begin{aligned}
\dot{x}_1(t) &= x_4(t),\\
\dot{x}_2(t) &= x_5(t),\\
\dot{x}_3(t) &= x_6(t),\\
\dot{x}_4(t) &= u_1(t) + 17.2656x_3(t),\\
\dot{x}_5(t) &= u_2(t),\\
\dot{x}_6(t) &= -\big(u_1(t) + 27.0756x_3(t) + 2x_5(t) * x_6(t)\big)/x_2(t),
\end{aligned}$$

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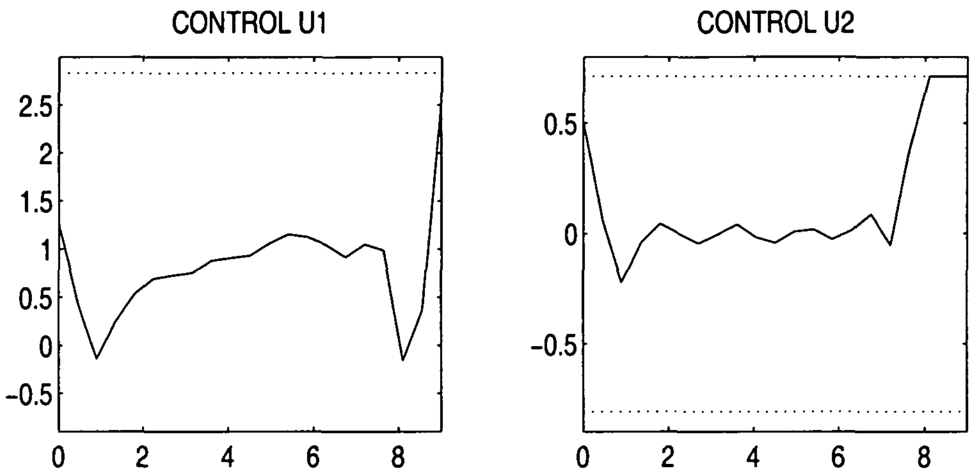

FIGURE 8. Controls for the crane problem, Case 1.

with initial and final state values

$$\begin{aligned} \boldsymbol{x}(0) &= [0, 22, 0, 0, -1, 0]^{\top}, \\ \boldsymbol{x}(9) &= [10, 14, 0, 2.5, 0, 0]^{\top}. \end{aligned}$$

Here x_1 represents a horizontal direction while x_2 represents the vertical direction and hence x_4 and x_5 are the corresponding velocities. The movement to optimize is a diagonal motion in a vertical plane, equivalent to continue lifting a container, travelling vertically at time $t = 0$, and preparing it for the horizontal movement of 2.5 units/time at $t = 9$. There are bounds on the two controls (torques of motors),

$$\begin{aligned} |u_1(t)| &\le 2.83374, \quad \forall t \in [0, 9], \\ -0.80865 \le u_2(t) &\le 0.71265, \quad \forall t \in [0, 9]. \end{aligned}$$

There are all-time state constraints (on the two translational velocities),

$$\begin{aligned} |x_4(t)| &\le 2.5, \quad \forall t \in [0, 9], \\ |x_5(t)| &\le 1.0, \quad \forall t \in [0, 9]. \end{aligned}$$

The objective is to minimise the swing of the container, represented by x_3, the position of swing, and x_6, the velocity of swing. Hence the objective function is

$$G_0 = \int_0^9 x_3^2(t) + x_6^2(t)\, dt.$$

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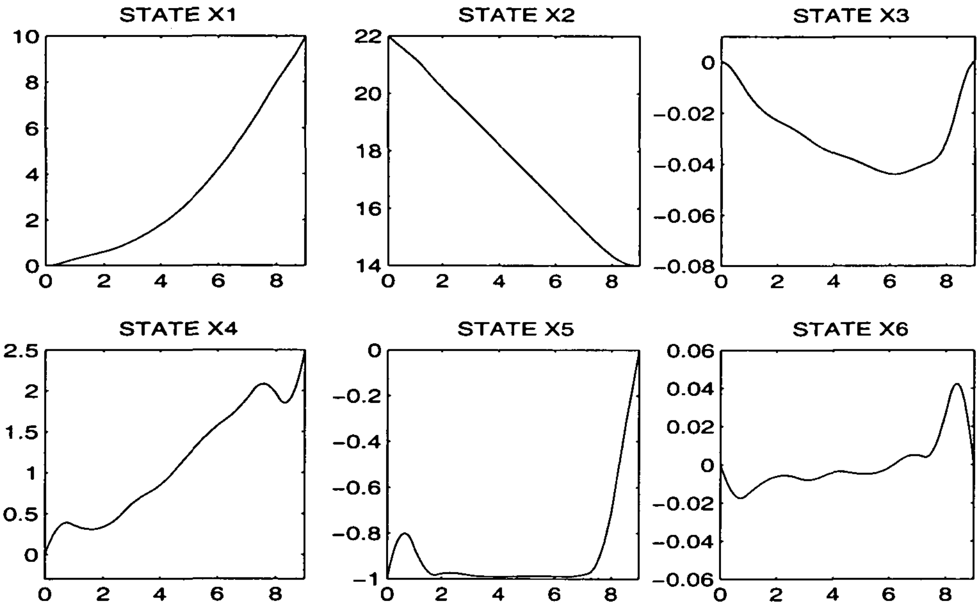

FIGURE 9. States for the crane problem, Case 1.

In the computational formulation the four linear all-time state constraints are added as penalties to the objective, and then smoothed as in [22]. The final state constraints are treated as 6 terminal state equality constraints.

The normal method of computing this example with MISER3 is to use a small number of equally spaced knots for the control parametrization, be it piecewise continuous or piecewise linear continuous, in the first instance. A solution is calculated. The control parametrization is then doubled by subdividing the knot set, the convergence accuracy increased and the problem is run again, with initial parameter values those of the previous solution. This can be repeated until the required number of parameters is attained. This procedure is followed for Figures 8 and 9, where the final parametrization has 21 equally spaced knots and where the result of the p.l.c. (piecewise linear continuous) computation is shown. See Case 1 in Table 1 for a summary of the computations performed. The p.c. (piecewise constant) computation shows a similar result. The control parametrization enhancing technique is then used to solve the same problem, where 6 knots are chosen at 0, 1, 2, 7, 8 and 9, to represent an enhancing control to parametrize time. See Case 2 in Table 1 for a summary of the computations performed. Figures 10 and 11 show the result of applying the enhancing control algorithm for p.l.c. control. In this example the controls represent torques generated by an electric motor so the p.l.c. control is more suitable. The dotted lines on the control graphs are the upper and lower bounds on the controls, displayed to

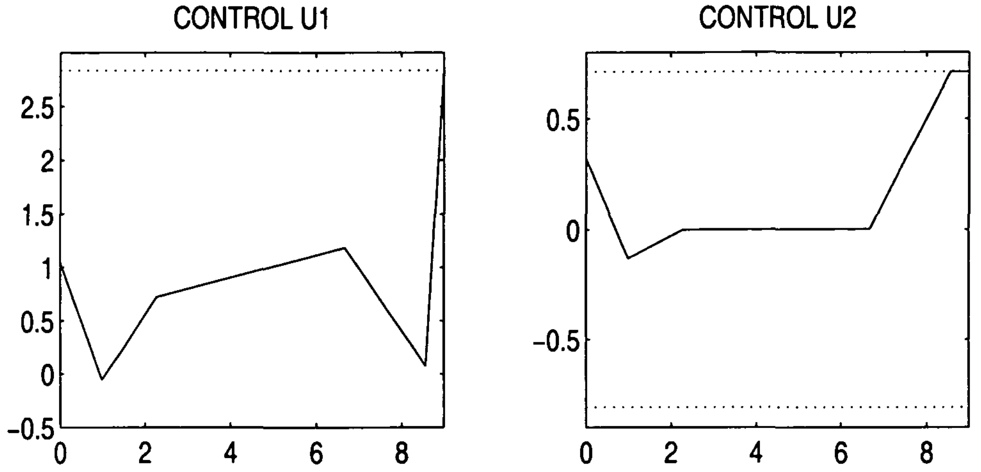

FIGURE 10. Controls for the crane problem, Case 2.

indicate when the controls come close to these bounds.

What is important about this problem is that the same objective value is achieved with far fewer switching points by a judicious placement of the switching points *via* the enhancing control.

The computations were started with initial control parameters set at zero and the accuracy requested is 10^{-7} for constraint compliance, and, 10^{-5} for the gradient stopping condition.

TABLE 1. Summary for the crane problem

Case	Number of Knots	Number of Parameters	Number of Iterations	Objective
1	21	42	105	0.01047
2	6	17	127	0.01048

The other notable property of this problem is the wide difference in control curves producing almost the same objective value.

8. Conclusions

A novel transform, known as the control parametrization enhancing transform, has been introduced to convert optimal control problems with variable switching times into equivalent standard optimal control problems. In the standard form, the problems

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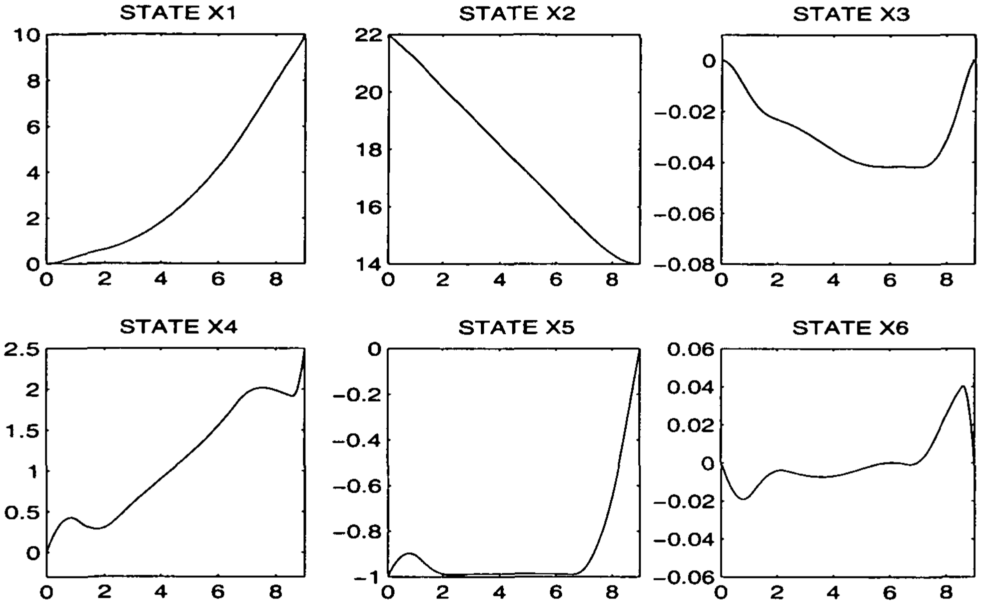

FIGURE 11. States for the crane problem, Case 2.

can be solved by the classical control parametrization method. The convergence properties of the new method are discussed and numerical examples illustrating its usefulness are given.

Acknowledgement

This paper is partially supported by a research grant from the Australian Research Council.

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