A Simple Linkage for Describing Equal Areas.

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If H (Fig. 1) be the middle point of a straight bar QP and if a straight bar OH of length one-half of QP be pin-jointed to QP at H, a simple linkage is formed, which may be called a ⊤-linkage.

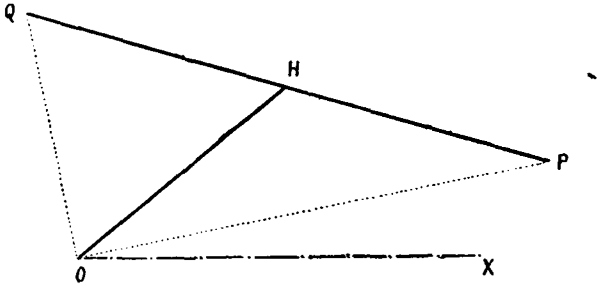

Fig. 1.

This, with an additional fixed bar guiding P or Q in a straight line through O, forms a well-known ellipsograph, or an equally well-known parallel motion. Let the ⊤-linkage be placed in the plane of the paper and let O be taken as pole, so that the linkage is free to turn about O in the plane of the paper. Then if P describe any closed curve not enclosing O, Q will describe a corresponding curve of equal area. For if the coordinates of P and Q be (r, θ) and (ρ, α) respectively, and if a be the length of OH, the equations of the transformation are

$$\text{(i)}\quad r^2 + \rho^2 = 4a^2, \quad \text{(ii)}\quad \alpha = \frac{\pi}{2} + \theta.$$

Accordingly if P describe the curve $f(r, \theta) = 0$, Q will describe the curve $f\left(\sqrt{4a^2 - \rho^2}, \alpha - \frac{\pi}{2}\right) = 0$. Multiplying (i) by $\frac{1}{2}d\theta$ and integrating on the supposition that P describes a closed curve such that PQ returns to its original position without making a circuit about O, then $\frac{1}{2}\int r^2 d\theta + \frac{1}{2}\int \rho^2 \delta\alpha = 0$ taken round the curve, so that

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if A_1 and A_2 be the areas described by P and Q respectively, $A_1 = -A_2$.

The equations of the transformed curves are most frequently complicated, but sometimes simple degenerate cases occur. Thus the family of straight lines $\theta = c$ transforms into the family of straight lines $\alpha = \frac{\pi}{2} + c$, and the family of concentric circles $r = k$ into the family of concentric circles $\rho = \sqrt{4a^2 - k^2}$, and thus the orthogonal network composed of the lines $\theta = c$ and the circles $r = k$ transforms into an equal area orthogonal network. (Figs. 2 and 3).

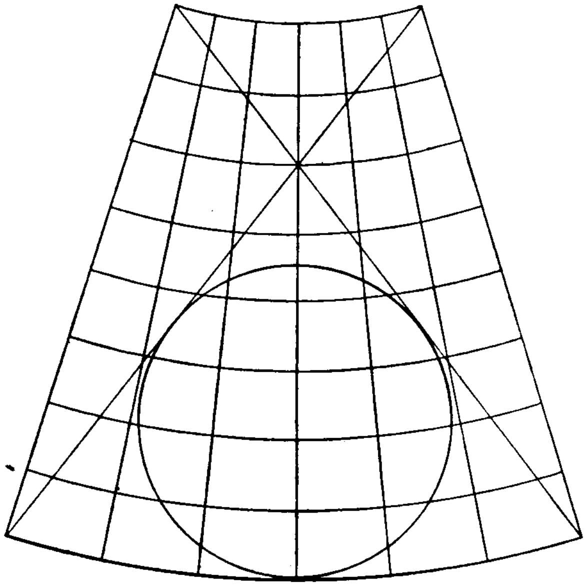

Fig. 2.

This suggested an application to some of the equal-area world-maps since the linkage deduces from any figure an infinite number of others all of equal area. Figs. 4, 5, 6, are representations of well-known equal-area world-maps, the first two by Lambert, the last by Collignon. Figs. 7, 8, 9, show equal-area maps deduced by the linkage and corrected with regard to the reversed

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orientation which arises from the fact that P and Q describe figures in opposite senses.

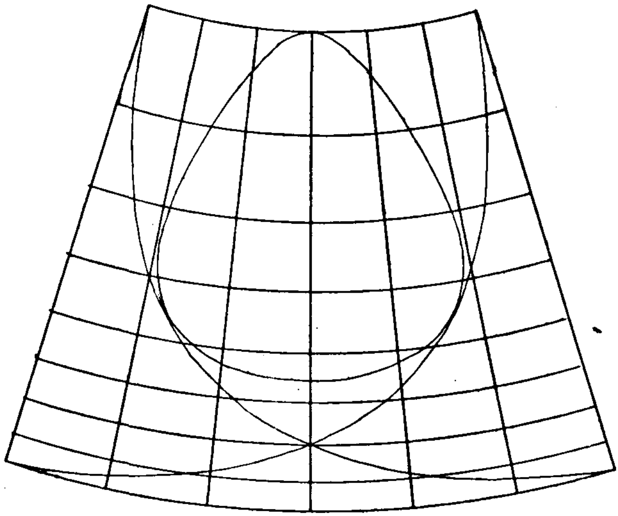

Fig. 3.

If the points P and Q describe arcs of curves instead of closed contours, then

$$\tfrac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta + \tfrac{1}{2}\int_{\alpha_1}^{\alpha_2} \rho^2 \delta\alpha = 2a^2(\theta_2 - \theta_1)$$

where θ_1, α_1 correspond to the initial, and θ_2, α_2 to the final positions of the tracing points. Thus the sum of the areas of the two sectors described by P and Q about the origin is equal to the area of the sector of a circle of radius PQ which contains the angle $\theta_2 - \theta_1$. If the linkage make a complete revolution about the pole O, PQ turns through an angle 2π, and the sum of the areas of the sectors traced by P and Q is equal to the area of a circle of radius PQ.

When one tracing point describes a curve passing through the pole an indeterminate condition arises, and a contour described twice by one tracing point may correspond to one described once by the other.

By taking a tracing point at any position S in PQ, an area can be drawn which is any required fraction of a given area A. If $QS = c$ and $SP = c'$, we have by a particular case of Holditch's Theorem that the area described by S is $A(c - c')/(c + c')$ when PQ rocks back to its original position.

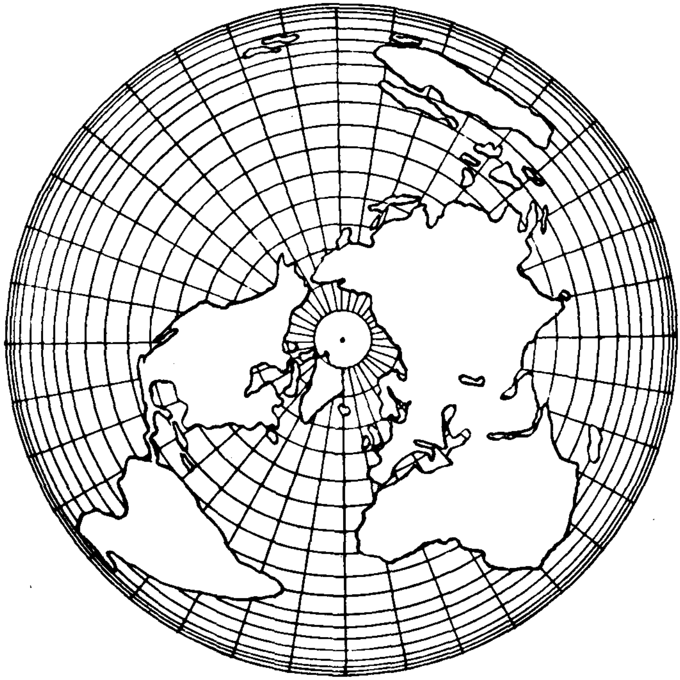

Fig. 4.

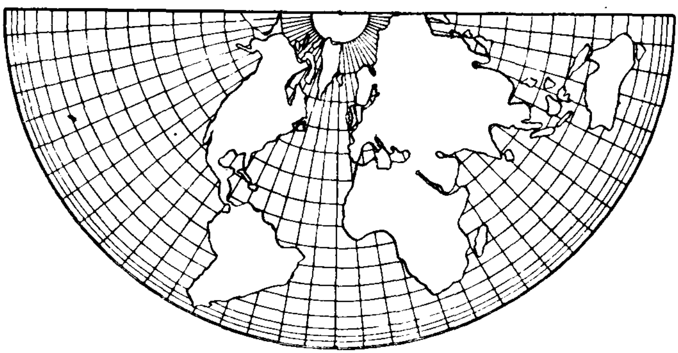

Fig. 5.

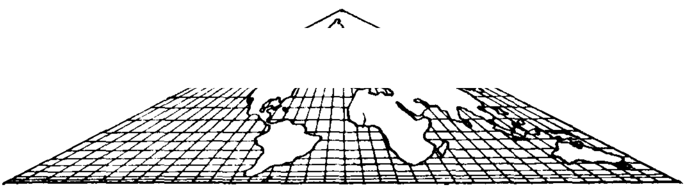

Fig. 6.

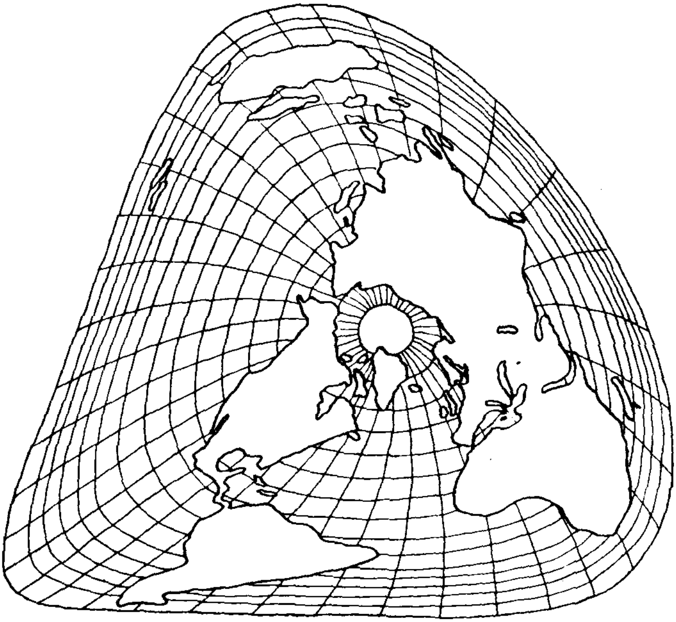

Fig. 7.

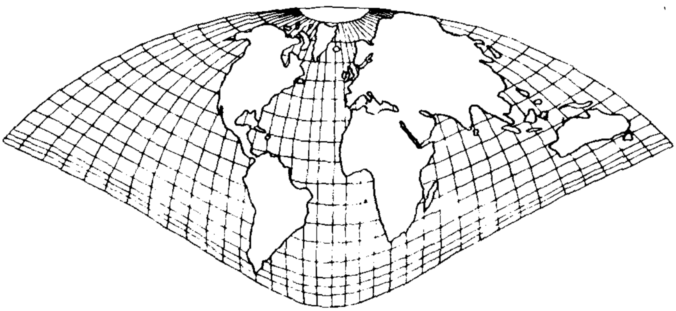

Fig. 8.

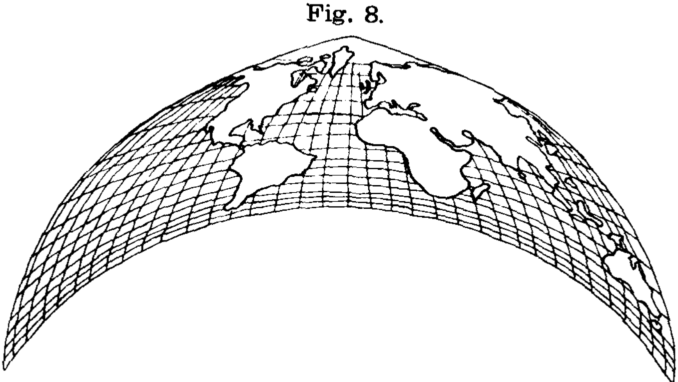

Fig. 9.

If (r, θ) and (ρ, α) be corresponding points on a curve and its transformed curve, and ϕ and Φ the angles between the radius vector and the tangent, then

$$\tan\phi = \frac{rd\theta}{dr}, \quad \tan\Phi = \frac{\rho d\alpha}{d\rho}.$$

Hence, since $rdr + \rho d\rho = 0$ and $d\alpha = d\theta$

$$\frac{\tan\phi}{\tan\Phi} = -\frac{r^2}{\rho^2}.$$

If ϕ' and Φ' be the angles for another corresponding pair of curves through these points, then

$$\frac{\tan\phi'}{\tan\Phi'} = -\frac{r^2}{\rho^2} = \frac{\tan\phi}{\tan\Phi},$$

which gives the law of intersection for the transformed network.