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Spectral Asymptotics of Laplacians Associated with One-dimensional Iterated Function Systems with Overlaps

Sze-Man Ngai

Abstract. We set up a framework for computing the spectral dimension of a class of one-dimensional self-similar measures that are defined by iterated function systems with overlaps and satisfy a family of second-order self-similar identities. As applications of our result we obtain the spectral dimension of important measures such as the infinite Bernoulli convolution associated with the golden ratio and convolutions of Cantor-type measures. The main novelty of our result is that the iterated function systems we consider are not post-critically finite and do not satisfy the well-known open set condition.

1 Introduction

Let μ be a continuous, positive, finite Borel measure on $\mathbb{R}$ with support $\operatorname{supp}(\mu) \subseteq [a, b]$. Define the standard Dirichlet form on $L^2((a, b), \mu)$,

$$\mathcal{E}(u, v) := \int_a^b u'(x)v'(x)\,dx,$$

with domain $\operatorname{Dom}(\mathcal{E})$ equal to the Sobolev space

$$H_0^1(a, b) := \{u \in L^2((a, b), dx) : u' \in L^2((a, b), dx), u(a) = u(b) = 0\}$$

(*Dirichlet boundary condition*) or

$$H^1(a, b) := \{u \in L^2((a, b), dx) : u' \in L^2((a, b), dx)\}$$

(*Neumann boundary condition*). It is well known that in either case $\operatorname{Dom}(\mathcal{E})$ is dense in $L^2((a, b), \mu)$, and the quadratic form $\mathcal{E}$ is closed. Hence one can define a Dirichlet (resp. Neumann) Laplace operator Δ_μ^D (resp. Δ_μ^N) called the *Dirichlet* (resp. *Neumann*) *Laplacian with respect to* μ, by

$$(1.1) \qquad \mathcal{E}(u, v) = \int_a^b (-\Delta_\mu u)v\,d\mu, \qquad \text{for all } v \in C_c^\infty(a, b),$$

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where $\Delta_\mu = \Delta_\mu^D$ or Δ_μ^N, and $C_c^\infty(a,b)$ is the space of compactly supported C^∞ functions on (a,b).

It is known (see, *e.g.*, [BNT, HLN]) that there exists an orthonormal basis $\{u_n\}_{n=1}^\infty$ of $L^2((a,b),\mu)$ consisting of eigenfunctions of $-\Delta_\mu$. The eigenvalues $\lambda_n = \lambda_n(-\Delta_\mu)$, $n \geq 1$, are simple and satisfy $\lim_{n\to\infty} \lambda_n = \infty$.

Define the *eigenvalue counting function* for $-\Delta_\mu$ as

$$N(\lambda, -\Delta_\mu) := \#\{n : \lambda_n \leq \lambda\},$$

where $\#A$ denotes the cardinality of a set A, and $\mu|_E$ denotes the restriction of the measure μ to a subset $E \subseteq \mathbb{R}$. The *lower* and *upper spectral dimensions* of μ are defined, respectively, as

$$\underline{\dim}_s(\mu) := \varliminf_{\lambda\to\infty} \frac{2\ln N(\lambda, -\Delta_\mu)}{\ln \lambda} \quad \text{and} \quad \overline{\dim}_s(\mu) := \varlimsup_{\lambda\to\infty} \frac{2\ln N(\lambda, -\Delta_\mu)}{\ln \lambda}.$$

If $\underline{\dim}_s(\mu) = \overline{\dim}_s(\mu)$, we call the common value the *spectral dimension* of μ and denote it by $\dim_s(\mu)$.

Since $\dim(H^1(a,b)/H_0^1(a,b)) = 2$ (see Remark 2.4), we have (see [K, Theorem 4.1.7 and Corollary 4.1.8]),

$$\lambda_n(-\Delta_\mu^N) \leq \lambda_n(-\Delta_\mu^D) \leq \lambda_{n+2}(-\Delta_\mu^N),$$

and thus

$$N(\lambda, -\Delta_\mu^D) \leq N(\lambda, -\Delta_\mu^N) \leq N(\lambda, -\Delta_\mu^D) + 2, \quad \lambda \geq 0.$$

Consequently, $N(\lambda, -\Delta_\mu^D)$ and $N(\lambda, -\Delta_\mu^N)$ behave the same asymptotically as $\lambda \to \infty$; moreover, $\underline{\dim}_s(\mu)$, $\overline{\dim}_s(\mu)$, and $\dim_s(\mu)$ are the same for $-\Delta_\mu^D$ and $-\Delta_\mu^N$. Therefore, we need only consider the Dirichlet Laplacian.

Throughout the rest of this paper, unless otherwise specified, Δ_μ denotes the Dirichlet Laplacian Δ_μ^D. The Dirichlet eigenvalues $\lambda_n = \lambda_n(-\Delta_\mu)$ satisfy

$$0 < \lambda_1 < \lambda_2 < \cdots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty. \tag{1.2}$$

The asymptotic behavior of $N(\lambda, -\Delta_\mu)$ and the computation of $\dim_s(\mu)$ are interesting problems and have been studied extensively. For the case where μ is Lebesgue measure, Δ_μ is the standard Laplacian, and (a,b) is replaced by a bounded smooth or fractal subset in $\mathbb{R}^d$, we refer the the reader to [W, L, LP, F2] and the references therein.

We are mainly interested in the case where μ is a self-similar measure. Let $\{S_i\}_{i=1}^m$, $m \geq 2$, be an *iterated function system (IFS)* of contractive similitudes of the form

$$S_i(x) = r_i R_i x + b_i, \quad i = 1, \ldots, m, \tag{1.3}$$

where $0 < r_i < 1$, R_i is an orthogonal transformation, and $b_i \in \mathbb{R}^d$. To each such IFS, there corresponds a unique compact set $K \subseteq \mathbb{R}^d$ such that

$$K = \bigcup_{i=1}^m S_i(K)$$

(see [H, F1]). K is called the *self-similar set* (or *attractor*) defined by $\{S_i\}_{i=1}^m$. To each set of probability weights $\{w_i\}_{i=1}^m$, *i.e.*, $w_i > 0$ and $\sum_{i=1}^m w_i = 1$, there corresponds a unique probability measure, called a *self-similar measure*, satisfying the identity

$$\mu = \sum_{i=1}^{m} w_i \mu \circ S_i^{-1}. \tag{1.4}$$

Moreover, $\operatorname{supp}(\mu) = K$. Kigami and Lapidus [KL] obtained the spectral dimension for a fractal Laplacian defined on a *post-critically finite* (PCF) self-similar structure with a regular harmonic structure and a Bernoulli measure (see [K]). The reader is referred to [K, S] for more results in this area. M. Solomyak and Verbitsky [SV], Naimark and M. Solomyak [NS1, NS2] computed the spectral dimension for self-similar measures defined by IFS's satisfying the *open set condition* (OSC). Recall that an IFS $\{S_i\}_{i=1}^m$ satisfies the open set condition if there exists a nonempty bounded open set U such that $\bigcup_{i=1}^m S_i(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \varnothing$ if $i \neq j$ (see [H, F1]). The OSC is a separation condition; an IFS that does not satisfy the OSC is said to have *overlaps*. It is still an open question whether the PCF condition implies the OSC; we refer the reader to [DL] for a partial result.

The PCF condition and the OSC are key conditions in studying the analysis and geometry of fractals. Since the late 1980s, differential operators on fractals defined by IFS's that satisfy these conditions, such as the Sierpinski gasket and Sierpinski carpet, have been constructed and studied extensively using both analytic and probabilistic approaches (see [K,S] and the references therein). Meanwhile, despite difficulties due to overlaps, some geometric and measure-theoretic properties of IFS's that do not satisfy these conditions have also been obtained (see, *e.g.*, [So, LN2, NW, JY, LN4]). However, not much is known concerning the analytic aspect of such fractals, and this has motivated the present work.

There are many interesting IFS's that do not satisfy the PCF condition and the OSC; some of them have been studied for a long time. In fact, the family of IFS's

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1 - \rho), \qquad 1/2 < \rho < 1,$$

has been studied very extensively. Since $S_1[0, 1] \cap S_2[0, 1] = [1 - \rho, \rho]$, the IFS's are not PCF and do not satisfy the OSC. The self-similar measures μ_ρ defined by the IFS with $1/2 < \rho < 1$, together with equal probability weights $w_1 = w_2 = 1/2$, have been studied since the 1930's and are still not completely understood (see [PSS]). In particular, the characterization of ρ for which μ_ρ is absolutely continuous or singular is still an open problem. Erdős [E] showed in the 1930's that if ρ^{-1} is a *Pisot number* (*i.e.*, an algebraic integer > 1 whose algebraic conjugates all lie inside the unit circle), then μ_ρ is singular. On the other hand, B. Solomyak [So, PS1] used a *transversality condition* to show that for Lebesgue almost all $\rho \in (1/2, 1)$, μ_ρ is absolutely continuous. Lau and the author [LN2] introduced the *weak separation property*, a separation condition weaker than the OSC, to study the multifractal decomposition of the class of measures with ρ^{-1} being a Pisot number. The most well-known Pisot number is the golden ratio $(\sqrt{5} + 1)/2$. In this paper we will compute the spectral dimension for μ_ρ where ρ is the reciprocal of the golden ratio.

In this paper we will formulate a set of conditions under which the spectral dimension of certain self-similar measures on $\mathbb{R}$ can be computed. Second-order self-similar identities play a crucial role in these conditions. The notion of second-order self-similar identities was first introduced by Strichartz *et al.* [STZ] to study the density of self-similar measures. Let $\{S_i\}_{i=1}^m$ and μ be given by (1.3) and (1.4) respectively. Define

$$T_i(x) = r^{n_i}x + d_i, \quad i = 0, 1, \ldots, L,$$

where n_i is a positive integer and $d_i \in \mathbb{R}^d$. We say that μ satisfies a family of *second-order self-similar identities* (or simply *second-order identities*) with respect to $\{T_i\}_{i=0}^L$ if

(i) $\operatorname{supp}(\mu) \subseteq \bigcup_{i=0}^L T_i(\operatorname{supp}(\mu))$, and

(ii) for each $A \subseteq \operatorname{supp}(\mu)$ and $0 \le i, j \le L$, $\mu(T_i \circ T_j A)$ can be expressed as a linear combination of $\{\mu(T_k A) : k = 0, 1, \ldots, L\}$ as

$$\mu(T_i \circ T_j A) = \sum_{k=0}^{L} c_k \mu(T_k A),$$

where $c_k = c_k(i, j)$ are independent of A.

For our purposes, we assume that $\{T_i\}_{i=0}^L$ satisfies the OSC. Second-order self-similar identities were also employed by Lau and the author [LN2, LN3] to obtain the multifractal L^q-spectra and justify the multifractal formalism for measures such as the infinite Bernoulli convolution associated with the golden ratio and the 3-fold convolution of the Cantor measure.

For an IFS satisfying a family of second-order identities, we formulate a set of conditions under which we can derive a formula that yields the spectral dimension of the Laplacian defined by an associated self-similar measure (see Theorem 1.1). Using this set-up, we obtain the spectral dimension of the Laplacians defined by the well-known infinite Bernoulli convolution associated with the golden ratio, and convolutions of Cantor-type measures (see Theorems 1.2 and 1.3). Our results differ from similar ones in the literature (see *e.g.,* [KL, NS1, NS2]) in that the IFS's we study are not PCF and do not satisfy the OSC.

To state our main results, we consider one-dimensional IFS's consisting of equicontractive similitudes of the form

$$S_i(x) = \rho x + b_i, \qquad i = 1, \ldots, m, \tag{1.5}$$

where $0 < \rho < 1$ and $0 = b_1 < b_2 < \cdots < b_m$. Note that $\operatorname{supp}(\mu) \subseteq [0, b]$, where $b = b_m/(1 - \rho)$. Let μ be the self-similar measure corresponding to probability weights $\{w_i\}_{i=1}^m$. Our approach is to use second-order identities to derive a system of functional equations for the eigenvalue counting functions on suitable subsets of $[a, b]$, and then apply the vector-valued renewal theorem proved by Lau *et al.* [LWC]. Define

$$T_i(x) = \rho_i x + d_i =: \rho^{n_i}x + d_i, \qquad i = 0, 1, \ldots, L, \tag{1.6}$$

where n_i is a positive integer and $d_i \in \mathbb{R}$. We will assume that $\{T_i\}_{i=0}^{L}$ is a nonoverlapping (*i.e.*, satisfying the OSC) family with respect to which μ satisfies a family of second-order identities. We will also assume that $\{T_i\}_{i=0}^{L}$ can be partitioned into two subfamilies $\{T_i\}_{i\in\mathcal{I}_0}$ and $\{T_i\}_{i\in\mathcal{I}_1}$; each is equicontractive, with contraction ratios $\rho^{\widetilde{n}}$ and $\rho^{\overline{n}}$, respectively. For convenience, we assume, by rearranging indices if necessary, that $\mathcal{I}_1 = \{1,\ldots,K\}$. Under additional conditions on the similitudes and the second-order identities (see conditions (C1), (C2), and (C3) in Section 3), we can derive a vector renewal equation of the form

$$\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}, \tag{1.7}$$

where $\alpha \geq 0$,

$$\mathbf{f} = \mathbf{f}^{(\alpha)}(t) = [f_1^{(\alpha)}(t),\ldots,f_p^{(\alpha)}(t)], \quad t \in \mathbb{R},$$

$$f_i^{(\alpha)}(t) := e^{-\alpha t} N\Big(e^t, -\Delta_{\mu|_{T_i[0,b]}}\Big), \quad i = 1,\ldots,p,$$

$$\mathbf{M}_\alpha = [\mu_{ji}^{(\alpha)}] \text{ is a } p\times p \text{ matrix of Radon measures on } \mathbb{R}, \text{ and}$$

$$\mathbf{z} = \mathbf{z}^{(\alpha)}(t) = [z_1^{(\alpha)}(t),\ldots,z_p^{(\alpha)}(t)] \text{ is some error function}$$

(see derivation of Theorem 3.3). Besides second-order identities, a key ingredient in deriving (1.7) involves the unitary equivalence of operators on Hilbert spaces; we use some ideas of M. Solomyak and Verbitsky [SV].

Let

$$\mathbf{M}_\alpha(\infty) := \Big[\mu_{ji}^{(\alpha)}(\mathbb{R})\Big]_{j,i=1}^{p}. \tag{1.8}$$

For each $i = 1,\ldots,p$ and $\alpha \geq 0$, define

$$F_i(\alpha) := \sum_{j=1}^{p} \mu_{ji}^{(\alpha)}(\mathbb{R}), \qquad D_i := \{\alpha \geq 0 : F_i(\alpha) < \infty\}, \qquad \tilde{\alpha}_i := \inf D_i. \tag{1.9}$$

By using a slightly modified vector-valued renewal theorem in [LWC], we obtain the following main theorem.

Theorem 1.1 *Let μ be a self-similar measure defined by an IFS as in* (1.5)*, and let Δ_μ denote Δ_μ^D or Δ_μ^N. Assume that μ satisfies a family of second-order identities with respect to an IFS of the form* (1.6) *so that conditions* (C1), (C2), *and* (C3) *in Section* 3 *hold. Let $\mathbf{M}_\alpha(\infty)$, $F_i(\alpha)$, and $\tilde{\alpha}_i$ be defined as in* (1.8) *and* (1.9)*. Assume that for each $i = 1,\ldots,p$, $\lim_{\alpha\to\tilde{\alpha}_i^+} F_i(\alpha) > 1$.*

(a) *There exists a unique $\alpha > 0$ such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to* 1.

(b) *If we assume, in addition, that for the unique α in* (a)*, there exists $\sigma > 0$ such that for all $i = 1,\ldots,p$, $z_i^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$. Then $\dim_s(\mu) = 2\alpha$. Moreover, if $\mathbf{M}_\alpha(\infty)$ is irreducible, then there exist constants $C_1, C_2 > 0$ such that for λ sufficiently large,*

$$C_1\lambda^\alpha \leq N\big(\lambda, -\Delta_{\mu|_{[0,b]}}\big) \leq C_2\lambda^\alpha.$$

When applying Theorem 1.1 it is necessary, and usually difficult, to show that, as $t \to \infty$, the error terms $z_i^{(\alpha)}(t)$ have order $o(e^{-\sigma t})$ for some $\sigma > 0$. We will obtain such estimates for the infinite Bernoulli convolution associated with the golden ratio and the class of convolutions of Cantor-type measures. We use some techniques developed by Lau and the author [LN1, LN3] for computing the L^q-spectrum and Hausdorff dimension of the measure. We also obtain some new estimates for these measures.

The infinite Bernoulli convolution associated with the golden ratio, defined below, is one of the most fundamental examples of a self-similar measure defined by an IFS with overlaps:

$$S_1(x) = \rho x, \quad S_2(x) = \rho x + (1-\rho), \quad \rho = \frac{\sqrt{5}-1}{2}, \quad w_1 = w_2 = \frac{1}{2}. \tag{1.10}$$

The corresponding self-similar identity is

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}, \tag{1.11}$$

with $\operatorname{supp}(\mu) = [0,1]$. By a result of Erdős [E], μ is singular. The L^2-dimension for μ was first computed by Lau [La1, La2]. The L^q-spectrum for $q \geq 0$, the multifractal formalism in the corresponding region, the L^∞-dimension, and the Hausdorff dimension of μ were obtained by Lau and the author [LN1, LN2]. The L^q-spectrum for $q < 0$ was obtained by Feng [Fe], and the multifractal formalism in the corresponding region was justified by Feng [Fe] and Feng and Olivier [FO].

This paper contributes to the study of this infinite Bernoulli convolution by computing its spectral dimension. It is shown in [STZ] that by defining

$$\begin{aligned} T_0(x) &= S_1S_1(x) = \rho^2 x \\ T_1(x) &= S_1S_2S_2(x) = S_2S_1S_1(x) = \rho^3 x + \rho^2 \\ T_2(x) &= S_2S_2(x) = \rho^2 x + \rho, \end{aligned} \tag{1.12}$$

one can obtain the following second-order identities for μ:

$$\begin{bmatrix} \mu(T_0T_iA) \\ \mu(T_1T_iA) \\ \mu(T_2T_iA) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_0A) \\ \mu(T_1A) \\ \mu(T_2A) \end{bmatrix}, \quad i = 0,1,2,\ A \subseteq [0,1],$$

where

$$M_0 = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 0 \end{bmatrix}, \quad M_1 = \frac{1}{4}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_2 = \frac{1}{8}\begin{bmatrix} 0 & 4 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

For any integer $k \geq 0$ and any index $J = (j_1, \dots, j_k) \in \mathcal{J}_0^k := \{0,2\}^k$, let

$$c_J = \frac{1}{4}[0,1,0]M_J \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2 \cdot 4^{k+1}}\begin{bmatrix} 1, 1 \end{bmatrix} P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{1.13}$$

where $M_J := M_{j_1} \cdots M_{j_k}$, $P_J := P_{j_1} \cdots P_{j_k}$,

$$P_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then it is shown in [LN1, Proposition 2.1(i)] that for all $A \subseteq [0,1]$,

$$\mu(T_1 T_J T_1 A) = c_J \mu(T_1 A). \tag{1.14}$$

By using Theorem 1.1 and by estimating the error term, we obtain the following.

Theorem 1.2 *Let μ be the infinite Bernoulli convolution associated with the golden ratio as defined in* (1.10) *and* (1.11), *let c_J be defined as in* (1.13), *and let Δ_μ denote Δ_μ^D or Δ_μ^N. Then there exists a unique positive real number satisfying*

$$\sum_{k=0}^{\infty} \sum_{J \in \mathcal{J}_0^k} (\rho^{2k+3} c_J)^\alpha = 1.$$

Moreover, $\dim_s(\mu) = 2\alpha$, *and there exist constants $C_1, C_2 > 0$ such that for all λ sufficiently large,*

$$C_1 \lambda^\alpha \le N\left(\lambda, -\Delta_\mu\right) \le C_2 \lambda^\alpha.$$

Numerical approximations by taking k up to 20 yield $\dim_s(\mu) \approx 0.998\cdots$.

Convolutions of Cantor-type measures provide an interesting family of self-similar measures that satisfy a family of second-order identities. Let

$$S_0(x) = \frac{1}{m}x, \qquad S_1(x) = \frac{1}{m}x + \frac{m-1}{m}, \tag{1.15}$$

where $m \ge 3$ is an odd integer. The attractor of this IFS is a Cantor-type set. Let ν_m be the self-similar measure defined by the IFS (1.15) with probability weights $p_0 = p_1 = 1/2$. The m-fold convolution of ν_m^{*m} is the self-similar measure defined by the following IFS with overlaps:

$$S_i(x) = \frac{1}{m}x + \frac{m-1}{m}i, \quad i = 0, 1, \dots, m, \tag{1.16}$$

together with probability weights

$$w_i := \frac{1}{2^m}\binom{m}{i}, \quad i = 0, 1, \dots, m.$$

That is,

$$\mu_m = \sum_{i=0}^{m} \frac{1}{2^m}\binom{m}{i} \mu_m \circ S_i^{-1}, \tag{1.17}$$

with $\operatorname{supp}(\mu_m) = [0, m]$. Define

$$T_i(x) = \frac{1}{m}x + i, \qquad i = 0, 1, \dots, m-1. \tag{1.18}$$

It is shown in [LN3, Proposition 2.1] that μ_m satisfies a family of second-order identities with respect to this IFS $\{T_i\}_{i=0}^{m-1}$ (see (6.2)).

The main reason to restrict m to an odd integer is that conditions (C2) and (C3) in Section 3 hold (see [LN3]). These conditions are essential in deriving (1.7).

The L^q-dimensions of the 3-fold convolution for positive integers q were obtained by Fan, Lau, and the author [FLN]. The L^q-spectrum for $q > 0$, the justification of the multifractal formalism in the corresponding region, and the Hausdorff dimension of the μ_m were obtained by Lau and the author [LN3]. For $q < 0$, the L^q-spectrum for the 3-fold convolution was obtained by Lau and Wang [LW], and a modified multifractal formalism in the corresponding region was proved by Feng *et al.* [FLW]. One of our objectives in this paper is to compute $\dim_s(\mu_m)$.

Define

$$c_{i,J} = [w_{i+1}, w_i]\, P_J \begin{bmatrix} w_0 \\ w_m \end{bmatrix}, \qquad i = 1, \dots, m-2, \quad J \in \mathcal{I}_0^k = \{0, m-1\}^k, \tag{1.19}$$

where $P_J = P_{j_1} \cdots P_{j_k}$,

$$P_0 = \begin{bmatrix} w_0 & 0 \\ w_m & w_{m-1} \end{bmatrix}, \quad \text{and} \quad P_{m-1} = \begin{bmatrix} w_1 & w_0 \\ 0 & w_m \end{bmatrix}. \tag{1.20}$$

Theorem 1.3 *Let μ_m be the m-fold convolution of the Cantor-type measure ν_m, let $c_{i,J}$, $i = 1, \dots, m-2$, be defined as in (1.19), and let Δ_μ denote Δ_μ^D or Δ_μ^N. Then there exists a unique positive real number α satisfying*

$$\frac{1}{m^\alpha} \sum_{i=1}^{m-1} w_i^\alpha + \sum_{k=0}^{\infty} \frac{1}{m^{(k+2)\alpha}} \sum_{i=1}^{m-2} \sum_{J \in \mathcal{I}_0^k} c_{i,J}^\alpha = 1.$$

Moreover, $\dim_s(\mu_m) = 2\alpha$, *and there exist constants $C_1, C_2 > 0$ such that for all λ sufficiently large,*

$$C_1 \lambda^\alpha \le N(\lambda, -\Delta_{\mu_m}) \le C_2 \lambda^\alpha.$$

For the 3-fold convolution, numerical approximations by taking k up to 18 in the above formula yield $\dim_s(\mu_3) \approx 0.997\cdots$.

Even though for each of the our examples, numerical result shows that $\dim_s(\mu)$ is close to 1, we can show that it is actually strictly less than 1.

Corollary 1.4 *For the measures μ in Theorems* 1.2 *and* 1.3, *we have* $\dim_s(\mu) < 1$.

This paper is organized as follows. In Section 2 we establish some essential properties concerning the unitary equivalence and the eigenvalue counting functions of operators. In Section 3 we formulate a set of conditions and under which we derive the functional equation (1.7). Section 4 is devoted to the proof of Theorem 1.1. In Section 5 we compute the spectral dimension of the infinite Bernoulli convolution associated with the golden ratio and prove Theorem 1.2. Finally, in Section 6 we compute the spectral dimension of the convolutions of Cantor-type measures and prove Theorem 1.3 and Corollary 1.4.

2 Unitarily Equivalent Operators

Unitary equivalence of operators plays an important role in deriving the renewal equation (1.7) and in estimating the error term $\mathbf{z}$. In this section we establish some basic properties concerning unitary equivalence of Laplace operators, especially those defined by self-similar measures.

Let $(H_1, \|\cdot\|_1)$ and $(H_2, \|\cdot\|_2)$ be Hilbert spaces. Recall that a surjective linear operator $\varphi\colon H_1 \to H_2$ is *unitary* if $\|\varphi x\|_2 = \|x\|_1$ for all $x \in H_1$. Let T_1, T_2 be linear operators on H_1 and H_2, respectively. T_1 and T_2 are said to be *unitarily equivalent*, denoted $T_1 \approx T_2$, if there exists a unitary operator $\varphi\colon H_1 \to H_2$ such that $\varphi\,\mathrm{Dom}(T_1) = \mathrm{Dom}(T_2)$ and $\varphi T_1 x = T_2 \varphi x$ for all $x \in \mathrm{Dom}(T_1)$. The second condition means that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Dom}(T_1) & \xrightarrow{\varphi} & \mathrm{Dom}(T_2) \\ \Big\downarrow{\scriptstyle T_1} & & \Big\downarrow{\scriptstyle T_2} \\ H_1 & \xrightarrow{\varphi} & H_2. \end{array}$$

Note that u is a λ-eigenvector of T_1 if and only if φu is a λ-eigenvector of T_2. In particular, unitarily equivalent operators have the same set of eigenvalues.

Let $S\colon [a,b] \to [c,d]$ be a contractive similitude with contraction ratio ρ such that $S[a,b] = [c,d]$, $S(a) = c$, and $S(b) = d$. Let ν be a continuous positive finite Borel measure on $[a,b]$ with $\mathrm{supp}(\nu) \subseteq [a,b]$. We will compare the Dirichlet Laplacians $\Delta_{\nu|_{[a,b]}}$ and $\Delta_{\nu\circ S^{-1}|_{[c,d]}}$. To simplify notation we let $\Delta_\nu := \Delta_{\nu|_{[a,b]}}$ and $\Delta_{\nu\circ S^{-1}} := \Delta_{\nu\circ S^{-1}|_{[c,d]}}$.

Lemma 2.1 *Let $S\colon \mathbb{R} \to \mathbb{R}$ be a contractive similitude, with contraction ratio ρ, such that $S[a,b] = [c,d]$, $S(a) = c$, and $S(b) = d$. Let ν be a continuous positive finite Borel measure on $[a,b]$ with $\mathrm{supp}(\nu) \subseteq [a,b]$. Define $\varphi\colon L^2((c,d), \nu\circ S^{-1}) \to L^2((a,b),\nu)$ by $\varphi(u) = u \circ S$. Then*

(a) *φ is unitary;*

(b) *$\varphi(\mathrm{Dom}(-\Delta_{\nu\circ S^{-1}})) = \mathrm{Dom}(-\Delta_\nu)$; furthermore, for all $u \in \mathrm{Dom}(-\Delta_{\nu\circ S^{-1}})$,*

$$\varphi\left(-\Delta_{\nu\circ S^{-1}}(u)\right) = \tfrac{1}{\rho}\big(-\Delta_\nu(\varphi(u))\big). \tag{2.1}$$

Proof (a) φ is clearly a linear surjection. Moreover, it is unitary because

$$\|\varphi(u)\|^2_{L^2((a,b),\nu)} = \int_a^b |u(Sx)|^2\, d\nu = \int_{S(a,b)} |u(x)|^2\, d\nu\circ S^{-1} = \|u\|^2_{L^2((c,d),\nu\circ S^{-1})}.$$

(b) It is known (see *e.g.*, [BNT]) that $u \in \mathrm{Dom}(-\Delta_{\nu\circ S^{-1}})$ if and only if $u \in L^2((c,d), \nu\circ S^{-1})$, $u(c) = u(d) = 0$, and there exists $f \in C[c,d]$, satisfying $f = -\Delta_{\nu\circ S^{-1}}(u)$, such that

$$u'(y) = u'(c) + \int_c^y f\, d\nu\circ S^{-1}, \qquad c \le y \le d,$$

i.e.,

$$u'(y) = u'(c) + \int_{S^{-1}(c,y)} f \circ S \, d\nu, \qquad c \le y \le d.$$

Let $y = S(x)$. Then the above equation becomes

$$u'(S(x)) = u'(S(a)) + \int_a^x f \circ S \, d\nu, \qquad a \le x \le b,$$

i.e.,

$$(\varphi(u))'(x) = (\varphi(u))'(a) + \int_a^x \rho\varphi(f) \, d\nu, \qquad a \le x \le b.$$

By part (a), $\varphi(u) \in L^2((a,b),\nu)$. Also,

$$\varphi(u)(a) = u(S(a)) = u(c) = 0, \qquad \varphi(u)(b) = u(S(b)) = u(d) = 0,$$

and $\rho\varphi(f) \in C[a,b]$. Thus, $\varphi(u) \in \mathrm{Dom}(-\Delta_\nu)$ and hence $\varphi(\mathrm{Dom}(-\Delta_{\nu\circ S^{-1}})) \subseteq \mathrm{Dom}(-\Delta_\nu)$. The reverse inclusion can be established similarly. Furthermore,

$$-\Delta_\nu(\varphi(u)) = \rho\varphi(f) = \rho\varphi(-\Delta_{\nu\circ S^{-1}}(u)),$$

which yields (2.1). ∎

Proposition 2.2 *Assume the same hypotheses as in Lemma* 2.1.

(a) *Then* $-\Delta_{\nu\circ S^{-1}|_{[c,d]}} \approx \frac{1}{\rho}(-\Delta_{\nu|_{[a,b]}})$.

(b) *If, in addition,* $\nu|_{[c,d]} = w\nu \circ S^{-1}$ *on* $[c,d]$ *for some constant* $w > 0$, *then*

$$-\Delta_{\nu|_{[c,d]}} \approx \tfrac{1}{\rho w}\left(-\Delta_{\nu|_{[a,b]}}\right).$$

Proof Part (a) follows directly from Lemma 2.1 and the definition of unitary equivalence. To prove (b), we first show that

$$-\Delta_{\frac{1}{w}\nu|_{[c,d]}}(u) = w(-\Delta_{\nu|_{[c,d]}}(u)) \tag{2.2}$$

for all $u \in \mathrm{Dom}(-\Delta_{\frac{1}{w}\nu|_{[c,d]}}) = \mathrm{Dom}(-\Delta_{\nu\circ S^{-1}})$. In fact, for all $v \in L^2((c,d), \frac{1}{w}\nu) = L^2((c,d),\nu)$,

$$\int_c^d \left(-\Delta_{\frac{1}{w}\nu|_{[c,d]}}(u)\right) v \, d(\frac{1}{w}\nu) = \int_c^d u'(x)v'(x)\,dx,$$

which implies that

$$\int_c^d \left(-\Delta_{\frac{1}{w}\nu|_{[c,d]}}(u)\right) v \, d\nu = w\int_c^d \left(-\Delta_{\nu|_{[c,d]}}(u)\right) v \, d\nu.$$

Thus, (2.2) follows. Now, by part (a),

$$\tfrac{1}{\rho}(-\Delta_{\nu|_{[a,b]}}) \approx -\Delta_{\nu\circ S^{-1}|_{[c,d]}} = -\Delta_{\frac{1}{w}\nu|_{[c,d]}} = w(-\Delta_{\nu|_{[c,d]}}).$$

Thus, part (b) follows. ∎

For two continuous positive finite Borel measures μ and ν on $[a,b]$, we say that $\mu \le \nu$ on $[a,b]$ if $\mu(E) \le \nu(E)$ for any Borel measurable subset $E \subseteq [a,b]$.

Proposition 2.3 *Let μ, ν be continuous positive finite Borel measures on $[a,b]$ and assume that there exists some constant $w > 0$ such that $\mu \le w\nu$ on $[a,b]$. Then for any $n \ge 1$,*

$$\lambda_n(-\Delta_\mu) \ge \frac{1}{w}\lambda_n(-\Delta_\nu).$$

Proof Let L be any finite-dimensional subspace of

$$H_0^1(a,b) = \mathrm{Dom}\big((-\Delta_{\mu|_{[a,b]}})^{1/2}\big) = \mathrm{Dom}\big((-\Delta_{\nu|_{[a,b]}})^{1/2}\big).$$

Then

$$\begin{aligned}\lambda_{-\Delta_\mu}(L) :&= \sup\Big\{ \int_a^b (u')^2\,dx \Big/ \int_a^b |u|^2\,d\mu : u \in L, u \neq 0 \Big\}\\ &\ge \frac{1}{w}\sup\Big\{ \int_a^b (u')^2\,dx \Big/ \int_a^b |u|^2\,d\nu : u \in L, u \neq 0 \Big\}\\ &=: \frac{1}{w}\lambda_{-\Delta_\nu}(L).\end{aligned}$$

By the variational formula,

$$\begin{aligned}\lambda_n(-\Delta_\mu) &= \inf\big\{\lambda_{-\Delta_\mu}(L) : L \text{ is a subspace of } H_0^1(a,b),\ \dim(L) = n\big\}\\ &\ge \frac{1}{w}\inf\big\{\lambda_{-\Delta_\nu}(L) : L \text{ is a subspace of } H_0^1(a,b),\ \dim(L) = n\big\}\\ &= \frac{1}{w}\lambda_n(-\Delta_\nu).\end{aligned}$$

■

Let $a < b$, and let $\mathcal{P} = \{a_i\}_{i=0}^{n+1}$ be a partition of $[a,b]$ satisfying

$$a = a_0 < a_1 < \cdots < a_n < a_{n+1} = b.$$

Let

$$\mathcal{F} = \mathcal{F}(\mathcal{P}) := \{u \in H_0^1(a,b) : u(a_i) = 0 \text{ for all } i = 0, 1, \dots, n+1\}.$$

Then $\mathcal{F}$ is a closed subspace of $H_0^1(a,b)$. Define a relation $\sim$ on $H_0^1(a,b)$, induced by $\mathcal{F}$, by $u \sim v$ if and only if $u - v \in \mathcal{F}$. Then $\sim$ is an equivalence relation on $H_0^1(a,b)$. Define the quotient space

$$H_0^1(a,b)/\mathcal{F} := \{[u] : u \in H_0^1(a,b)\},$$

where $[u]$ is the equivalence class of $H_0^1(a,b)$ defined by $\sim$. Define addition and scalar multiplication on $H_0^1(a,b)/\mathcal{F}$ as usual. For each $i = 1, \dots, n$, let f_i be a function in $H_0^1(a,b)$ that satisfies

$$f_i(a_j) = \delta_{ij}, \qquad i, j = 1, \dots, n,$$

where δ_{ij} is the Kronecker delta. Such an f_i clearly exists. It is straightforward to prove the following:

$$H_0^1(a,b)/\mathcal{F} = \operatorname{span}\{[f_i] : i = 1,\dots,n\}, \quad \dim\big(H_0^1(a,b)/\mathcal{F}\big) = n.$$

Remark 2.4 A similar argument shows that $\dim(H^1(a,b)/H_0^1(a,b)) = 2$.

Let $-\Delta^{\mathcal{F}}_{\mu|_{[a,b]}}$ be the Laplacian defined as in (1.1) with $\operatorname{Dom}(\mathcal{E}) = \mathcal{F}$, and let $N(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{[a,b]}}) := \#\{n : \lambda_n(-\Delta^{\mathcal{F}}_{\mu|_{[a,b]}}) \le \lambda\}$ denote the associated *eigenvalue counting function*. If $\mathcal{F} = H_0^1(a,b)$, $N(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{[a,b]}})$ reduces to $N(\lambda, -\Delta_{\mu|_{[a,b]}})$. It follows from above that

$$N\big(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{[a,b]}}\big) = \sum_{i=0}^{n} N\big(\lambda, -\Delta_{\mu|_{[a_i,a_{i+1}]}}\big). \tag{2.3}$$

Moreover, it follows from the variational formula (see, e.g., [K, Theorem 4.1.7 and Corollary 4.18] or [D]) that

$$\begin{aligned} N\big(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{[a,b]}}\big) &\le N\big(\lambda, -\Delta_{\mu|_{[a,b]}}\big) \le N\big(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{[a,b]}}\big) + n \\ &= N\big(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{[a,b]}}\big) + \#\mathcal{P} - 2. \end{aligned} \tag{2.4}$$

Now let $\{S_i\}_{i=1}^m$ be an IFS of contractive similitudes on $\mathbb{R}$, and let μ be an associated self-similar measure. Fix $1 \le i \le m$ and suppose $S_i[a,b] = [c,d]$ with $S_i(a) = c$ and $S_i(b) = d$. We will compare the eigenvalues of $-\Delta_{\mu|_{[c,d]}}$, $-\Delta_{\mu\circ S_i^{-1}|_{[c,d]}}$, and $-\Delta_{\mu|_{[a,b]}}$.

Proposition 2.5 *Let $\{S_i\}_{i=1}^m$, $m \ge 2$, be an IFS of contractive similitudes on $\mathbb{R}$, and let μ be the self-similar measure associated with positive probability weights $\{w_i\}_{i=1}^m$. Fix $1 \le i \le m$ and suppose $S_i[a,b] = [c,d]$, $S_i(a) = c$, $S_i(b) = d$, and $\mu[a,b] > 0$. Then*

(a) *for any $n \ge 1$, $\lambda_n(-\Delta_{\mu|_{[c,d]}}) \le \frac{1}{w_i}\lambda_n(-\Delta_{\mu\circ S_i^{-1}|_{[c,d]}})$;*
(b) *$N(\lambda, -\Delta_{\mu|_{[c,d]}}) \ge N(w_i\lambda, -\Delta_{\mu\circ S_i^{-1}|_{[c,d]}}) - 1$;*
(c) *$N(\lambda, -\Delta_{\mu|_{[c,d]}}) \ge N(\rho_i w_i\lambda, -\Delta_{\mu|_{[a,b]}}) - 1$;*
(d) *if $a < c < d < b$, then*

$$N\big(\lambda, -\Delta_{\mu|_{[c,d]}}\big) \le N\big(\lambda, -\Delta_{\mu|_{[a,b]}}\big) \le N\Big(\frac{1}{\rho_i w_i}\lambda, -\Delta_{\mu|_{[c,d]}}\Big) + 1.$$

Proof (a) Since $\mu = \sum_{i=1}^m w_i\mu\circ S_i^{-1}$ on $[c,d]$, $\mu\circ S_i^{-1} \le \frac{1}{w_i}\mu$ on $[c,d]$. The stated inequality follows from Proposition 2.3.

(b) Let $\lambda_n := \lambda_n(-\Delta_{\mu|_{[c,d]}})$, and let $N(\lambda, -\Delta_{\mu|_{[c,d]}}) = n$ so that $\lambda_n \le \lambda < \lambda_{n+1}$. Then by part (a), $w_i\lambda_{n+1} \le \lambda_{n+1}(-\Delta_{\mu\circ S_i^{-1}|_{[c,d]}})$. Hence,

$$N\big(\lambda, -\Delta_{\mu|_{[c,d]}}\big) + 1 \ge N\big(w_i\lambda_{n+1}, -\Delta_{\mu\circ S_i^{-1}|_{[c,d]}}\big) \ge N\big(w_i\lambda, -\Delta_{\mu\circ S_i^{-1}|_{[c,d]}}\big),$$

which yields the desired inequality.

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(c) Note that by Proposition 2.2, we have

$$N\big(w_i\lambda, -\Delta_{\mu\circ S_i^{-1}|_{[c,d]}}\big) = N\big(w_i\lambda, \tfrac{1}{\rho_i}(-\Delta_{\mu|_{[a,b]}})\big) = N\big(\rho_i w_i\lambda, -\Delta_{\mu|_{[a,b]}}\big).$$

The assertion follows by combining this with part (b).

(d) Let $\mathcal{F} := \mathcal{F}(a, c, d, b)$. Then by (2.3) and (2.4),

$$N\big(\lambda, -\Delta_{\mu|_{[a,c]}}\big) + N\big(\lambda, -\Delta_{\mu|_{[c,d]}}\big) + N\big(\lambda, -\Delta_{\mu|_{[d,b]}}\big) = N\big(\lambda, -\Delta^{\mathcal{F}}_{\mu|_{[a,b]}}\big) \leq N\big(\lambda, -\Delta_{\mu|_{[a,b]}}\big).$$

The first inequality follows; the second one follows directly from part (c). ■

3 Derivation of the Functional Equation

In this section we formulate a set of conditions under which we can derive the renewal equation (1.7). Let $\{S_i\}_{i=1}^m$, $m \geq 2$, and $\{T_i\}_{i=0}^L$ be defined as in (1.5) and (1.6), respectively. Fix a set of positive probability weights $\{w_i\}_{i=1}^m$, and let μ be the corresponding self-similar measure. Then $\operatorname{supp}(\mu) \subseteq [0, b]$, where $b = b_m/(1-\rho)$. We assume that $\{T_i\}_{i=0}^L$ is a non-overlapping family, $\operatorname{supp}(\mu) \subseteq \bigcup_{i=0}^L T_i[0, b] \subseteq [0, b]$, and μ satisfies the following family of second-order identities. For $A \subseteq [0, b]$,

$$\begin{bmatrix}\mu(T_0T_iA)\\ \vdots\\ \mu(T_LT_iA)\end{bmatrix} = M_i \begin{bmatrix}\mu(T_0A)\\ \vdots\\ \mu(T_LA)\end{bmatrix}, \qquad i = 0, 1, \dots, L,$$

where M_i is an $L \times L$ constant matrix.

For a finite sequence of indices $I = (i_1, \dots, i_n)$ we let $|I| = n$ denote the *length* of I, and for $1 \leq k \leq n$, we let $I|_k := (i_1, \dots, i_k)$ denote the initial segment of I consisting of the first k indices. If $J = (j_1, \dots, j_\ell)$ is another index, we let $IJ := (i_1, \dots, i_n, j_1, \dots, j_\ell)$ denote the concatenation of I and J.

For $I = (i_1, \dots, i_n) \in \{1, \dots, m\}^n$, we use the standard notation

$$S_I = S_{i_1} \cdots S_{i_n} = S_{i_1} \circ \cdots \circ S_{i_n}, \quad w_I := w_{i_1} \cdots w_{i_n}, \ \textit{etc.}$$

Also, for $I \in \{0, 1, \dots, L\}^k$, we let $\rho_I := \rho_{i_1} \cdots \rho_{i_k} = \rho^{n_{i_1} + \cdots + n_{i_k}}$.

We assume that $\{T_i\}_{i=0}^L$ can be partitioned into two subcollections, $\{T_i\}_{i\in\mathcal{I}_0}$ and $\{T_i\}_{i\in\mathcal{I}_1}$, each being equicontractive, with

$$n_i = \begin{cases} \widetilde{n} & \text{if } i \in \mathcal{I}_0,\\ \overline{n} & \text{if } i \in \mathcal{I}_1.\end{cases}$$

Let $\mathcal{I} := \mathcal{I}_0 \cup \mathcal{I}_1 = \{0, 1, \dots, L\}$ and assume for convenience that $\mathcal{I}_1 = \{1, \dots, K\}$. For $k \geq 0$, we denote by $\mathcal{I}_0^k$ the k-fold Cartesian product $\mathcal{I}_0 \times \cdots \times \mathcal{I}_0$ ($\mathcal{I}_0^0 := \varnothing$); $\mathcal{I}_1^k$ is similarly defined. Moreover, we assume that the following conditions (C1), (C2), and (C3) are satisfied.

(C1) For $i \in \mathcal{I}_1$, $T_i[0,b]$ contains an interval of the form $S_{j_1}\cdots S_{j_k}[0,b]$, $1 \le j_i \le m$.

Furthermore, there exists a positive integer ω (chosen to be the smallest) and a subset $\mathbf{I} \subseteq \{(i_1,\dots,i_\omega) : i_j \in \mathcal{I}_1 \text{ for } 1 \le j \le \omega\}$ such that for $A \subseteq [0,b]$, conditions (C2) and (C3) below are satisfied.

(C2) Suppose $\omega \ge 2$ and assume that for some $k \in \{2,\dots,\omega\}$, the set $\mathbf{I}_k^0$ defined below is nonempty:

$$\mathbf{I}_k^0 := \Big\{ I = (i_1,\dots,i_k) : i_1 \in \mathcal{I}_1 \text{ and } k \text{ is the smallest integer such that } I \text{ is not an initial segment of any member of } \mathbf{I} \Big\}.$$

Then for each $I \in \mathbf{I}_k^0$ there exists an index $j = j(I) \in \mathcal{I}_1$, and a constant $c(I,j)$, depending only on I and j, such that

$$\mu(T_I A) = c(I,j)\mu(T_j A), \quad A \subseteq [0,b].$$

(C3) Suppose $I = (i_1,\dots,i_\omega) \in \mathbf{I}$. Then for each $J = (j_1,\dots,j_k) \in \mathcal{I}_0^k$, $k \ge 0$ and for each $\ell \in \mathcal{I}_1$, there exists an index $j = j(I,J,\ell) \in \mathcal{I}_1$ and a constant $c(I,J,\ell,j)$, depending only on I, J, ℓ, and j, such that

$$\mu(T_{IJ\ell}A) = c(I,J,\ell,j)\mu(T_j A), \quad A \subseteq [0,b].$$

These conditions are similar to those in [LN3], which are formulated to compute the L^q-spectrum $\tau(q)$ of a self-similar measure. Condition (C1) is slightly more restrictive, and (C2) and (C3) are less restrictive. (C1) ensures that $\mu(T_i[0,b]) > 0$ for all $i \in \mathcal{I}_1$.

The infinite Bernoulli convolution associated with the golden ratio, defined in (1.11), together with $\{T_i\}_{i=0}^2$ defined in (1.12), provides a basic example of an IFS that satisfies (C1), (C2), and (C3) (see [LN1]). Another family of examples can be obtained by taking convolutions of the Cantor-type measures, as defined in (1.17), together with the $\{T_i\}_{i=0}^{m-1}$ defined in (1.18). Details for the threefold convolution are given in [LN3]; we will not repeat them here.

Proposition 3.1 *Let $I \in \mathbf{I}_k^0$, $j = j(I)$ and $c(I,j)$ be as in condition* (C2). *Suppose* (C2) *holds. Then*

$$-\Delta_{\mu|_{T_I[0,b]}} \approx \frac{1}{\rho_I \rho_j^{-1} c(I,j)} \big(-\Delta_{\mu|_{T_j[0,b]}}\big).$$

Proof Let $B \subseteq T_I[0,b]$, and let $A \subseteq [0,b]$ such that $B = T_I A$. Then by condition (C2),

$$\mu(B) = \mu(T_I A) = c(I,j)\mu(T_j A) = c(I,j)\mu(T_j T_I^{-1} B).$$

Thus, $\mu = c(I,j)\mu \circ (T_I T_j^{-1})^{-1}$ on $T_I[0,b]$. Now the assertion follows by applying Proposition 2.2(b) with $S := T_I T_j^{-1} \colon T_j[0,b] \to T_I[0,b]$. ∎

Proposition 3.2 *Let $I \in \mathbf{I}$, $J \in \mathcal{I}_0^k$, $k \geq 0$, $\ell \in \mathcal{I}_1$, and let $j = j(I, J, \ell) \in \mathcal{I}_1$ and $c(I, J, \ell, j)$ be as in condition* (C3). *Suppose condition* (C3) *holds. Then*

$$-\Delta_{\mu|_{T_I T_J T_\ell [0,b]}} \approx \frac{1}{\rho_{IJ} c(I, J, \ell, j)} \big(-\Delta_{\mu|_{T_j [0,b]}} \big) .$$

Proof Let $B \subseteq T_{IJ\ell}[0, b]$, and let $A \subseteq [0, b]$ such that $B = T_{IJ\ell}A$. It follows from condition (C3) that

$$\mu(B) = \mu(T_{IJ\ell}A) = c(I, J, \ell, j)\mu(T_j A) = c(I, J, \ell, j)\mu(T_j T_{IJ\ell}^{-1} B).$$

That is, $\mu = c(I, J, \ell, j)\mu \circ (T_{IJ\ell} T_j^{-1})^{-1}$ on $T_{IJ\ell}[0, b]$. The assertion now follows from Proposition 2.2(b) by letting $S := T_{IJ\ell} T_j^{-1} \colon T_j[0, b] \to T_{IJ\ell}[0, b]$. ■

It follows from Proposition 2.5(c)(d) that the asymptotic behavior of $N(\lambda, -\Delta_{\mu|_{[0,b]}})$ is controlled by that of $N(\lambda, -\Delta_{\mu|_{T_i[0,b]}})$ for any $i \in \mathcal{I}_1$; more precisely, for $I = (i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$ such that $S_I[0, b] \subseteq T_i[0, b]$,

$$\begin{aligned} N\big(\rho^k \overline{w}_I \lambda, -\Delta_{\mu|_{[0,b]}} \big) - 1 \leq N\big(\lambda, -\Delta_{\mu|_{S_I[0,b]}} \big) &\leq N\big(\lambda, -\Delta_{\mu|_{T_i[0,b]}} \big) \\ &\leq N\big(\lambda, -\Delta_{\mu|_{[0,b]}} \big) , \end{aligned}$$

where $\overline{w}_I := \sum \{ w_J : S_J = S_I, |J| = k \}$. Thus it suffices to study $N(\lambda, -\Delta_{\mu|_{T_i[0,b]}})$ for $i \in \mathcal{I}_1$.

Fix $i \in \mathcal{I}_1$ and let

$$\mathbf{I}_k^0(i) := \{ I \in \mathbf{I}_k^0 : I|_1 = i \} \quad \text{and} \qquad \mathbf{I}(i) := \{ I \in \mathbf{I} : I|_1 = i \}.$$

We will derive a functional equation for $N(\lambda, -\Delta_{\mu|_{T_i[0,b]}})$. For each integer $n \geq 1$, we define a partition $\mathcal{P}_n = \mathcal{P}_n(i)$ of $T_i[0, b]$ as follows:

$$\mathcal{P}_1 := \{ T_i(0),\ T_i(b) \} \quad \mathcal{P}_2 := \mathcal{P}_1 \cup \{ T_{i i_2}(x) : x = 0 \text{ or } b,\ i_2 \in \mathcal{I} \}.$$

For $2 \leq n \leq \omega$, define

$$\mathcal{P}_n := \mathcal{P}_{n-1} \cup \big\{ T_{i i_2 \ldots i_n}(x) : x = 0 \text{ or } b, (i, i_2, \ldots, i_k) \notin \mathcal{I}_0^k \text{ for } 2 \leq k \leq n-1,\ i_n \in \mathcal{I} \big\} .$$

For $n > \omega$, write $n = \omega + k + 1$, where $k \geq 0$, and define

$$\mathcal{P}_n := \mathcal{P}_{n-1} \cup \big\{ T_{IJi_n}(x) : x = 0 \text{ or } b, I \in \mathbf{I}(i), J \in \mathcal{I}_0^k, i_n \in \mathcal{I} \big\} .$$

Then $\mathcal{P}_n$, $n \geq 2$, are end-points of the subintervals generated by the following proce-

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dure. First, replace T_i by subintervals of the form $T_{ii_2}[0,b]$, $i_2 \in \mathcal{I}$. If $(i,i_2) \in \mathbf{I}_2^0(i)$, keep $T_{ii_2}[0,b]$; otherwise, replace it by subintervals of the form $T_{ii_2i_3}[0,b]$. Repeat the procedure until the length of an index is ω. For each $I \in \mathbf{I}(i)$, replace $T_I[0,b]$ by subintervals of the form $T_{Ii_{\omega+1}}[0,b]$. If $i_{\omega+1} \in \mathcal{I}_1$, keep the corresponding subinterval; otherwise, replace it by subintervals of the form $T_{Ii_{\omega+1}i_{\omega+2}}[0,b]$. Continue.

Let $\mathcal{F}_n := \mathcal{F}(\mathcal{P}_n)$, $n \geq 1$. Then, if $\mathbf{I}_k^0 \neq \varnothing$ for some $k \in \{2,\dots,\omega\}$, we have

$$\begin{aligned}
&N\big(\lambda, -\Delta^{\mathcal{F}_\omega}_{\mu|_{T_i[0,b]}}\big)\\
&\quad= \sum_{(i,i_2)\in\mathbf{I}_2^0(i)} N\big(\lambda, -\Delta_{\mu|_{T_{ii_2}[0,b]}}\big) + \sum_{(i,i_2,i_3)\in\mathbf{I}_3^0(i)} N\big(\lambda, -\Delta_{\mu|_{T_{ii_2i_3}[0,b]}}\big) + \cdots\\
&\qquad \cdots + \sum_{(i,i_2,\dots,i_\omega)\in\mathbf{I}_\omega^0(i)} N\big(\lambda, -\Delta_{\mu|_{T_{ii_2\dots i_\omega}[0,b]}}\big) + \sum_{(i,i_2,\dots,i_\omega)\in\mathbf{I}(i)} N\big(\lambda, -\Delta_{\mu|_{T_{ii_2\cdots i_\omega}[0,b]}}\big)\\
&\quad= \sum_{k=2}^{\omega}\sum_{I\in\mathbf{I}_k^0(i)} N\big(\lambda, -\Delta_{\mu|_{T_I[0,b]}}\big) + \sum_{(i,i_2,\dots,i_\omega)\in\mathbf{I}(i)} N\big(\lambda, -\Delta_{\mu|_{T_{ii_2\cdots i_\omega}[0,b]}}\big).
\end{aligned}$$

By Proposition 3.1, for $k = 2,\dots,\omega$ and $I \in \mathbf{I}_k^0(i)$,

$$\begin{aligned}
N\big(\lambda, -\Delta_{\mu|_{T_I[0,b]}}\big) &= N\Big(\lambda, \frac{1}{\rho_I\rho_{j(I)}^{-1}c(I,j(I))}\big(-\Delta_{\mu|_{T_{j(I)}[0,b]}}\big)\Big)\\
&= N\Big(\rho_I\rho_{j(I)}^{-1}c(I,j(I))\lambda, -\Delta_{\mu|_{T_{j(I)}[0,b]}}\Big).
\end{aligned}$$

Also, for each $I \in \mathbf{I}(i)$, we can subdivide $T_I[0,b]$, in the way described above, into subintervals of the forms $T_{IJ\ell}[0,b]$ or $T_{IJ}[0,b]$, where $J \in \mathcal{I}_0^k$ and $\ell \in \mathcal{I}_1$. Hence, for $n > \omega$,

$$\begin{aligned}
N\big(\lambda, -\Delta^{\mathcal{F}_n}_{\mu|_{T_i[0,b]}}\big) &= \sum_{k=2}^{\omega}\sum_{I\in\mathbf{I}_k^0(i)} N\big(\rho_I\rho_{j(I)}^{-1}c(I,j(I))\lambda, -\Delta_{\mu|_{T_{j(I)}[0,b]}}\big)\\
&\quad+ \sum_{k=0}^{n-\omega-1}\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{I}_0^k}\sum_{\ell\in\mathcal{I}_1} N\big(\lambda, -\Delta_{\mu|_{T_{IJ\ell}[0,b]}}\big)\\
&\quad+ \sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{I}_0^{n-\omega}} N\big(\lambda, -\Delta_{\mu|_{T_{IJ}[0,b]}}\big).
\end{aligned}$$

Consequently, by applying Proposition 3.2 to the second summation above and then using (2.4), we get

$$(3.1)\quad N\big(\lambda, -\Delta_{\mu|_{T_i[0,b]}}\big) = \sum_{k=2}^{\omega}\sum_{I\in\mathbf{I}_k^0(i)} N\big(\rho_I\rho_{j(I)}^{-1}c(I,j(I))\lambda, -\Delta_{\mu|_{T_{j(I)}[0,b]}}\big)$$
$$+\sum_{k=0}^{\infty}\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^k}\sum_{\ell\in\mathcal{J}_1} N\big(\rho_{IJ}c(I,J,\ell,j)\lambda, -\Delta_{\mu|_{T_{j(I,J,\ell)}[0,b]}}\big)$$
$$-\sum_{k=n-\omega}^{\infty}\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^k}\sum_{\ell\in\mathcal{J}_1} N\big(\rho_{IJ}c(I,J,\ell,j)\lambda, -\Delta_{\mu|_{T_{j(I,J,\ell)}[0,b]}}\big)$$
$$+\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^{n-\omega}} N\big(\lambda, -\Delta_{\mu|_{T_{IJ}[0,b]}}\big) + \varepsilon_n(i),$$

where $0 \le \varepsilon_n(i) \le \#\mathcal{P}_n - 2 \le \#\mathcal{P}_n$.

For each $i \in \mathcal{J}_1$, define $f_i(t) = f_i^{(\alpha)}(t) := e^{-\alpha t}N(e^t, -\Delta_{\mu|_{T_i[0,b]}})$. Note that if we let $\lambda = e^t$, then for any $\beta > 0$,

$$(3.2)\qquad e^{-\alpha t}N\big(\beta\lambda, -\Delta_{\mu|_{T_i[0,b]}}\big) = e^{-\alpha(t+\ln\beta)}e^{\alpha\ln\beta}N\big(e^{t+\ln\beta}, -\Delta_{\mu|_{T_i[0,b]}}\big)$$
$$= \beta^\alpha f_i(t+\ln\beta).$$

Multiplying both sides of (3.1) by $e^{-\alpha t}$ and using (3.2), we get, for all $n > m$,

(3.3)

$$f_i(t) = \sum_{k=2}^{\omega}\sum_{I\in\mathbf{I}_k^0(i)} \big(\rho_I\rho_{j(I)}^{-1}c(I,j(I))\big)^{\alpha} f_{j(I)}\big(t+\ln(\rho_I\rho_{j(I)}^{-1}c(I,j(I)))\big)$$
$$+\sum_{k=0}^{\infty}\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^k}\sum_{\ell\in\mathcal{J}_1}\big(\rho_{IJ}c(I,J,\ell,j)\big)^{\alpha} f_{j(I,J,\ell)}\big(t+\ln(\rho_{IJ}c(I,J,\ell,j))\big)$$
$$-\sum_{k=n-\omega}^{\infty}\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^k}\sum_{\ell\in\mathcal{J}_1}\big(\rho_{IJ}c(I,J,\ell,j)\big)^{\alpha} f_{j(I,J,\ell)}\big(t+\ln(\rho_{IJ}c(I,J,\ell,j))\big)$$
$$+\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^{n-\omega}} e^{-\alpha t}N\big(e^t, -\Delta_{\mu|_{T_{IJ}[0,b]}}\big) + e^{-\alpha t}\varepsilon_n(i).$$

Since $\lambda_1(-\Delta_{\mu|_{T_i[0,b]}}) > 0$ for all $i \in \mathcal{J}_1$ (see (1.2)), there exists $t_o \in \mathbb{R}$ such that

$$(3.4)\qquad f_i(t) = 0 \quad \text{for all } t < t_o \text{ and all } i \in \mathcal{J}_1.$$

Now for each $t \in \mathbb{R}$, let $n_t = n_t(i)$ be the smallest integer such that

$$(3.5)\qquad t + \max\Big\{\ln(\rho_{IJ}c(I,J,\ell,j)) : I \in \mathbf{I}(i),\, J \in \mathcal{J}_0^{n_t-\omega},\, \ell \in \mathcal{J}_1\Big\} < t_o.$$

Then the third summation in (3.3) vanishes and thus we get

$$
\begin{aligned}
f_i(t) = {} & \sum_{k=2}^{\omega} \sum_{I\in \mathbf{I}_k^0(i)} \big(\rho_I\rho_{j(I)}^{-1}c(I,j(I))\big)^{\alpha} f_{j(I)}\big(t+\ln(\rho_I\rho_{j(I)}^{-1}c(I,j(I)))\big)\\
&+\sum_{k=0}^{\infty}\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^k}\sum_{\ell\in\mathcal{J}_1}\big(\rho_{IJ}c(I,J,\ell,j)\big)^{\alpha} f_{j(I,J,\ell)}\big(t+\ln(\rho_{IJ}c(I,J,\ell,j))\big)\\
&+\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^{n_t-\omega}} e^{-\alpha t}N\big(e^t,-\Delta_{\mu|_{T_{IJ}[0,b]}}\big)+e^{-\alpha t}\varepsilon_{n_t}(i),
\end{aligned}
\tag{3.6}
$$

where $0\le \varepsilon_{n_t}(i)\le \#\mathcal{P}_{n_t}$.

If $\omega<2$ or $\mathbf{I}_k^0=\varnothing$ for all $2\le k\le\omega$, the first summation in (3.6) vanishes. This is the case for the infinite Bernoulli convolution associated with the golden ratio (see Section 5).

Let $\#\mathcal{J}_1=p$. For $i\in\mathcal{J}_1$, let $\mu_{ji}^{(\alpha)}$, $j\in\mathcal{J}_1$, be the discrete measure such that

$$\mu_{ji}^{(\alpha)}\big(-\ln(\rho_I\rho_{j(I)}^{-1}c(I,j(I)))\big) = \big(\rho_I\rho_{j(I)}^{-1}c(I,j(I))\big)^{\alpha}$$

if $I\in\mathbf{I}_k^0(i)$, $2\le k\le\omega$, and $j(I)=j$, and

$$\mu_{ji}^{(\alpha)}\big(-\ln(\rho_{IJ}c(I,J,\ell,j))\big) = \big(\rho_{IJ}c(I,J,\ell,j)\big)^{\alpha}$$

if $I\in\mathbf{I}(i)$, $J\in\mathcal{J}_0^k$, $\ell\in\mathcal{J}_1$, $k\ge 0$, and $j(I,J,\ell)=j$.

Let

$$\mathbf{f}:=[f_1^{(\alpha)}(t),\dots,f_p^{(\alpha)}(t)] \qquad\text{and}\qquad \mathbf{M}_\alpha:=\Big[\mu_{ji}^{(\alpha)}\Big]_{j,i=1}^{p}. \tag{3.7}$$

We have just finished proving the following.

Theorem 3.3 *Let μ be a self-similar measure defined by an IFS as in* (1.5). *Assume that μ satisfies a family of second-order identities with respect to an IFS of the form* (1.6) *so that conditions* (C1), (C2), *and* (C3) *in Section* 3 *hold. Let $\mathbf{f},\mathbf{M}_\alpha$ be defined as in* (3.7), *and let n_t be defined as in* (3.5). *Then $\mathbf{f}$ satisfies the vector-valued renewal equation $\mathbf{f}=\mathbf{f}*\mathbf{M}_\alpha+\mathbf{z}$, where $\mathbf{z}=\mathbf{z}^{(\alpha)}(t)=[z_1^{(\alpha)}(t),\dots,z_p^{(\alpha)}(t)]$,*

$$z_i^{(\alpha)}(t)=\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^{n_t-\omega}} e^{-\alpha t}N\big(e^t,-\Delta_{\mu|_{T_{IJ}[0,b]}}\big)+e^{-\alpha t}\varepsilon_{n_t}(i)\quad \textit{for } 1\le i\le p,$$

and $0\le\varepsilon_{n_t}(i)\le\#\mathcal{P}_{n_t}$.

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4 Proof of Theorem 1.1

We need a vector-valued renewal theorem in Lau *et al.* [LWC]. We introduce some terminology and refer the reader to [LWC] for any unexplained terms. Let $\mathbf{F}$ be a matrix-valued Radon measure that vanishes on $(-\infty, 0)$, *i.e.,*

$$\mathbf{F} = \begin{bmatrix} F_{11} & \dots & F_{1n} \\ \vdots & & \vdots \\ F_{n1} & \dots & F_{nn} \end{bmatrix},$$

where $F_{ij}(x) = \mu_{ij}(-\infty, x]$ and each μ_{ij} is a Radon measure (*i.e.,* positive Borel regular measure) on $\mathbb{R}$ that vanishes on $(-\infty, 0)$. Let $\mathbf{F}(\infty) := [F_{ij}(\infty)]$, and let $\mathbf{m} = [m_{ij}] = [\int_0^\infty x\, dF_{ij}]$ be the *moment matrix*. We say that each column of $\mathbf{F}$ is *nondegenerate* at 0 if

$$\sum_{i=1}^{n} F_{ij}(0) < \sum_{i=1}^{n} F_{ij}(\infty) \quad \text{for } 1 \le j \le n.$$

In this case, there exists some $\delta > 0$ such that the vector

$$\left[\sum_{i=1}^{n} \big(F_{i1}(\infty) - F_{i1}(\delta) \big), \dots, \sum_{i=1}^{n} \big(F_{in}(\infty) - F_{in}(\delta) \big) \right]$$

is coordinatewise positive. For any *path* $\gamma = (i_1, \dots, i_k)$ with $i_j \in \{1, \dots, n\}$, we use the notation

$$\mu_\gamma := \mu_{i_1 i_2} * \mu_{i_2 i_3} * \cdots * \mu_{i_{k-1} i_k}.$$

Such a γ is a called a *cycle* if $i_1 = i_k$ and a *simple cycle* if it is a cycle and $i_1, \dots, i_{k-1}$ are distinct. We denote by $\mathbb{R}_{\mathbf{F}}$ the closed subgroup of $(\mathbb{R}, +)$ generated by

$$\bigcup \big\{ \operatorname{supp}(\mu_\gamma) : \ \gamma \text{ is a simple cycle on } \{1, \dots, n\} \big\}.$$

The following theorem has been modified from [LWC, Theorem 4.3] to suit our purposes. The proof is similar. First, the original condition that each entry of $\mathbf{F}$ is nondegenerate at 0 is replaced by the slightly weaker condition that each column of $\mathbf{F}$ is nondegenerate at 0. Second, the condition that $\mathbf{z}$ vanishes on $(-\infty, 0)$ is replaced by the condition that $\mathbf{z}$ vanishes on $(-\infty, x_o)$ for some $x_o \in \mathbb{R}$. Last, the continuity of $\mathbf{f}$ is replaced by Borel measurability (see [LWC, Remark 3.2]).

Theorem 4.1 (Lau *et al.* [LWC]) *Let $\mathbf{F}$ be an $n \times n$ matrix-valued Radon measure defined on $\mathbb{R}$ that vanishes on $(-\infty, 0)$ and assume that each column of $\mathbf{F}$ is nondegenerate at 0. Suppose $\mathbf{F}(\infty)$ is irreducible and has maximal eigenvalue 1. Let $\mathbf{U} = \sum_{k=0}^\infty \mathbf{F}^{*k}$ and let $\mathbf{z}$ be a directly Riemann integrable function on $\mathbb{R}$ that vanishes on $(-\infty, x_o)$ for some $x_o \in \mathbb{R}$. Then $\mathbf{f} = \mathbf{z} * \mathbf{U}$ is a bounded Borel measurable solution of*

$$\mathbf{f}(x) = (\mathbf{f} * \mathbf{F})(x) + \mathbf{z}(x), \qquad x \in \mathbb{R},$$

and it is unique in the class of Borel measurable solutions that vanish on $(-\infty, x_o)$. Furthermore, the following hold:

(a) *If* $\mathbb{R}_{\mathbf{F}} = \mathbb{R}$, *then*

$$\lim_{x\to\infty} \mathbf{f}(x) = \left(\int_{-\infty}^{\infty} \mathbf{z}(t)\,dt\right)\mathbf{A},$$

where

$$\mathbf{A} = \frac{1}{\gamma}\begin{bmatrix} u_1v_1 & \dots & u_1v_n \\ \vdots & & \vdots \\ u_nv_1 & \dots & u_nv_n \end{bmatrix}, \quad \gamma = [v_1,\dots,v_n]\begin{bmatrix} m_{11} & \dots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \dots & m_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

and $\mathbf{u} = [u_1,\dots,u_n]$, $\mathbf{v} = [v_1,\dots,v_n]$ *are the unique, normalized, positive right and left 1-eigenvectors of* $\mathbf{F}(\infty)$, *respectively.* ($\mathbf{A} = \mathbf{0}$ *if one of the* m_{ij} *is* ∞.)

(b) *If* $\mathbb{R}_{\mathbf{F}} = \langle\lambda\rangle$ *for some* $\lambda > 0$, *then for each* $x > 0$,

$$\lim_{k\to\infty}\left[f_1(x + a_{11} + k\lambda),\dots,f_n(x + a_{1n} + k\lambda)\right] = \left(\sum_{k=-\infty}^{\infty}\mathbf{z}(x + k\lambda)\right)\mathbf{A},$$

where $a_{1j} \in \operatorname{supp}(\mu_{\gamma(1,j)})$ *and* $\gamma(1,j)$ *is any path from* 1 *to* j *such that* $\mu_{\gamma(1,j)} \neq 0$.

Proof of Theorem 1.1 (a) First, we observe the fact that each $F_i(\alpha)$ is a strictly decreasing, continuous, positive function of α, which tends to 0 as $\alpha \to \infty$ and exceeds 1 when $\alpha \searrow \tilde{\alpha}_i$; moreover, $F_i(0) = \infty$. Thus there exists a unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is 1 (see [Mi, Theorem 2.1]). This proves part (a).

(b) For the rest of the proof we let α be the unique number obtained in the proof of (a) above. We first show that the moments satisfy $0 \le m_{ji}^{(\alpha)} < \infty$ for all $i, j = 1,\dots,K$ and

$$0 < \sum_{j=1}^{K} m_{ji}^{(\alpha)} < \infty. \tag{4.1}$$

Since $m_{ji}^{(\alpha)} \ge 0$, it suffices to prove (4.1). In fact, the assumption $\lim_{\alpha\to\tilde{\alpha}_i^+} F_i(\alpha) > 1$ implies that there exists $\epsilon > 0$ such that $F_i(\alpha - \epsilon) < \infty$. Thus,

$$\begin{aligned} 0 < \sum_{j=1}^{K} m_{ji}^{(\alpha)} &= \sum_{k=2}^{\omega}\sum_{I\in\mathbf{I}_k^0(i)}\left(\rho_I\rho_{j(I)}^{-1}c(I,j(I))\right)^{\alpha-\epsilon+\epsilon}\left|\ln(\rho_I\rho_{j(I)}^{-1}c(I,j(I)))\right| \\ &\quad + \sum_{k=0}^{\infty}\sum_{I\in\mathbf{I}(i)}\sum_{J\in\mathcal{J}_0^k}\sum_{\ell\in\mathcal{J}_1}\left(\rho_{IJ}c(I,J,\ell,j)\right)^{\alpha-\epsilon+\epsilon}\left|\ln(\rho_{IJ}c(I,J,\ell,j))\right|. \end{aligned}$$

By using the fact that $\lim_{t\to 0^+} t^\epsilon \ln t = 0$, we have

$$0 < \sum_{j=1}^{K} m_{ji}^{(\alpha)} \le CF_i(\alpha - \epsilon) < \infty \quad \text{for some constant } C > 0,$$

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proving (4.1).

It follows from the derivation of the renewal equation (3.6) that the columns of $\mathbf{M}_\alpha$ are nondegenerate at 0. Also, notice that $\mathbf{z}^{(\alpha)}(t) > \mathbf{0}$ for all t sufficiently large.

From Theorem 3.3 we have $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$, where, by assumption, $\mathbf{z}$ is directly Riemann integrable on $\mathbb{R}$. We now divide the proof of $\dim_s(\mu) = 2\alpha$ into the following two cases.

Case 1. $\mathbf{M}_\alpha(\infty)$ *is irreducible.*

Using the above observations and Theorem 4.1, we obtain constants $C_1, C_2 > 0$ such that

$$0 < C_1 \leq \varliminf_{t\to\infty} f_i(t) \leq \varlimsup_{t\to\infty} f_i(t) \leq C_2 < \infty, \qquad i = 1, \dots, p,$$

which implies that $\dim_s(\mu) = 2\alpha$, and, moreover, for $\lambda = e^t$ sufficiently large,

$$C_1\lambda^\alpha \leq N\Big(\lambda, -\Delta_{\mu|_{T_i[0,b]}}\Big) \leq C_2\lambda^\alpha.$$

Case 2. $\mathbf{M}_\alpha(\infty)$ *is reducible.*

First, we notice that if $\beta < \alpha$, then each nonzero entry of $\mathbf{M}_\beta(\infty)$ is strictly less than the corresponding entry of $\mathbf{M}_\alpha(\infty)$. It follows that the maximal eigenvalue of $\mathbf{M}_\beta(\infty)$ is less than 1 (see [Mi, Corollary II.2.2] for the irreducible case; the reducible case follows by considering the irreducible blocks in the normal form [Mi, Lemma VI.1.1] of these matrices). Hence by [LWC, Remark 4.4 and Theorem 4.5], $\lim_{t\to\infty} f_i^{(\beta)}(t) = 0 \qquad$ for $i = 1, \dots, p$. Thus, $\overline{\dim}_s(\mu) \leq 2\alpha$.

On the other hand, since the maximal eigenvalue of $\mathbf{M}_\alpha(\infty)$ is 1, by [LWC, Theorem 4.5], there exists some $i \in \{1, \dots, p\}$ such that

$$\varliminf_{t\to\infty} f_i^{(\alpha)}(t) > 0.$$

Consequently, $\underline{\dim}_s(\mu) \geq 2\alpha$. This completes the proof. ■

5 Infinite Bernoulli Convolution Associated with the Golden Ratio

In this section we compute the spectral dimension of the infinite Bernoulli convolution associated with the golden ratio.

Let $\{S_1, S_2\}$ and μ be defined as in (1.10) and (1.11), respectively. The IFS is not PCF and does not satisfy the OSC, because the intersection $S_1[0,1] \cap S_2[0,1]$ is the interval $[\rho^2, \rho]$.

Let $\{T_i\}_{i=1}^3$ be defined as in (1.12), and let $\rho_1 = \rho_3 := \rho^2$, $\rho_2 := \rho^3$. It is known (see [LN1]) that conditions (C1), (C2), and (C3) are satisfied by letting $\mathcal{J}_0 := \{0, 2\}$, $\mathcal{J}_1 := \{1\}$, $\mathbf{I} = \{(1)\}$ and thus $\omega = 1$. We also note that $\mathcal{P}_1 = \{T_1(0), T_1(1)\}$ and for all $n \geq 2$,

$$\mathcal{P}_n = \mathcal{P}_{n-1} \cup \{T_{1J1}(x) : J \in \mathcal{J}_0^{n-2},\ x = 0 \text{ or } 1\}.$$

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Hence, $\#\mathcal{P}_n = 2 + \sum_{i=1}^{n-1} 2^i = 2^n$. Since $p = \#\mathcal{I}_1 = 1$, the vector-valued renewal equation (1.7) reduces to the following scalar-valued one:

$$f(t) = \sum_{k=0}^{\infty} \sum_{J \in \mathcal{I}_0^k} (\rho^{2k+3} c_J)^\alpha f\big(t + \ln(\rho^{2k+3} c_J)\big) + z^{(\alpha)}(t),$$

where c_J is defined in (1.13),

$$z^{(\alpha)}(t) := \sum_{J \in \mathcal{I}_0^{n_t - 1}} e^{-\alpha t} N\big(e^t, -\Delta_{\mu|_{T_{1J}[0,1]}}\big) + e^{-\alpha t} \varepsilon_{n_t}, \tag{5.1}$$

and $\#\varepsilon_{n_t} \le \#\mathcal{P}_{n_t} = 2^{n_t}$.

We need to show that there exists some $\sigma > 0$ such that $z^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$. To this end we will develop several lemmas; the final estimate will be given in Lemma 5.8.

To estimate $\sum_{J \in \mathcal{I}_0^{n_t - 1}} e^{-\alpha t} N(e^t, -\Delta_{\mu|_{T_1 T_J[0,1]}})$, we divide the sum into two parts, namely,

$$\mathcal{I}_0^{n_t-1}(0) := \{J \in \mathcal{I}_0^{n_t-1} : j_{n_t-1} = 0\} \quad \text{and} \quad \mathcal{I}_0^{n_t-1}(2) := \{J \in \mathcal{I}_0^{n_t-1} : j_{n_t-1} = 2\}.$$

Since the two parts are symmetric, it suffices to consider the case $J \in \mathcal{I}_0^{n_t-1}(0)$. Our main idea is to express $N(e^t, -\Delta_{\mu|_{T_1 T_J[0,1]}})$ in terms of

$$N\big(C_1(J)e^t, -\Delta_{\mu|_{T_{20}[0,1]}}\big), \quad N\big(C_2(J)e^t, -\Delta_{\mu|_{T_1[0,1]}}\big) \quad \text{and} \quad N\big(C_3(J)e^t, -\Delta_{\mu|_{T_{02}[0,1]}}\big),$$

where $C_1(J), C_2(J)$, and $C_3(J)$ depend on J and thus on t. Each of these terms can be shown to be bounded by some constant independent of t.

For $J \in \mathcal{I}_0^{n_t-1}(0)$, write $J = (j_1, \dots, j_{n-1}) = (J', 0)$, where $J' \in \mathcal{I}_0^{n_t-2}$. We divide $T_{1J}[0,1]$ into the following three nonoverlapping subintervals:

$$T_{1J0}[0,1], \qquad T_{1J1}[0,1], \qquad T_{1J2}[0,1],$$

and study the corresponding Laplacian on them. We begin with the easiest case.

Case 1. $T_{1J1}[0,1]$.

Lemma 5.1 *Let $J = (J', 0) \in \mathcal{I}_0^{n_t-1}(0)$. Then*

$$-\Delta_{\mu|_{T_{1J1}[0,1]}} \approx \frac{1}{\rho_{1J} c_J}\big(-\Delta_{\mu|_{T_1[0,1]}}\big).$$

Proof Let $A \subseteq T_{1J1}[0,1]$, and let $B \subseteq T_1[0,1]$ such that $A = T_{1J}B$. Since $T_1^{-1}B \subseteq [0,1]$, by (1.14),

$$\mu(A) = \mu(T_{1J1}T_1^{-1}B) = c_J\mu(B) = c_J\mu(T_{1J}^{-1}A),$$

i.e., $\mu = c_J \mu \circ T_{1J}^{-1}$ on $T_{1J1}[0,1]$. The result now follows by applying Proposition 2.2(b) with $S := T_{1J} : T_1[0,1] \to T_{1J1}[0,1]$. ■

Case 2. $T_{1J2}[0,1]$.

We first prove the following.

Lemma 5.2 *For any μ-measurable subset $A \subseteq T_{1J'02}[0,1]$,*

$$c_{J'}\mu \circ T_{1J'}^{-1}(A) \le \mu(A) \le 2c_{J'}\mu \circ T_{1J'}^{-1}(A).$$

Proof Let $A = T_{1J'02}B$, where $B \subseteq [0,1]$. Then $\mu(A) = \mu(T_{1J'1}T_1^{-1}T_{02}B)$. Note that $T_{02}(B) \subseteq [\rho^3, \rho^2]$, and therefore

$$T_1^{-1}T_{02}B \subseteq T_1^{-1}[\rho^3, \rho^2] = [-\rho, 0].$$

Hence by [LN1, Proposition 2.1(ii)],

$$c_{J'}\mu(T_{02}B) \le \mu(A) \le 2c_{J'}\mu(T_{02}B),$$

and the asserted inequalities follow. ■

Lemma 5.3 *Let $J = (J', 0) \in \mathcal{J}_0^{n_t - 1}(0)$. Then*

(a) $-\Delta_{\mu\circ T_{1J'}^{-1}|_{T_{1J'02}[0,1]}} \approx \frac{1}{\rho_{1J'}}(-\Delta_{\mu|_{T_{02}[0,1]}})$;

(b) $\lambda_n(-\Delta_{\mu|_{T_{1J'02}[0,1]}}) \ge \frac{1}{2c_{J'}}\lambda_n(-\Delta_{\mu\circ T_{1J'}^{-1}|_{T_{1J'02}[0,1]}})$.

Proof (a) follows by using Proposition 2.2(b) with

$$S = T_{1J'} \colon T_{02}[0,1] \to T_{1J'02}[0,1].$$

(b) follows by combining Proposition 2.3 with Lemma 5.2. ■

Finally, we consider the following.

Case 3. $T_{1J0}[0,1]$.

We need the following estimate for the measure μ.

Lemma 5.4 *For any μ-measurable subset $A \subseteq T_{1J0}[0,1]$,*

$$\mu(A) \le 4c_J\mu \circ (T_{1J}T_2^{-1})^{-1}(A).$$

Proof Let $A \subseteq T_{1J0}[0,1]$. Then there exists $B \subseteq T_0[0,1]$ such that $A = T_{1J}B = T_{1J}T_2^{-1}T_2B$. Thus, by applying the second-order identities (see [LN1]), we have

$$\mu(A) = \mu(T_{1J'0}B) = [0,1,0]M_{J'}\begin{bmatrix}\mu(T_{00}B)\\ \mu(T_{10}B)\\ \mu(T_{20}B)\end{bmatrix} = [0,1,0]M_J\begin{bmatrix}\mu(T_0B)\\ \mu(T_1B)\\ \mu(T_2B)\end{bmatrix}. \tag{5.2}$$

We compare the values of $\mu(T_0B)$, $\mu(T_1B)$, and $\mu(T_2B)$. It follows by applying the self-similar identity (1.11) that

$$\mu(T_0B) = \tfrac{1}{4}\mu(B), \quad \mu(T_1B) = \tfrac{1}{8}\mu(B+\rho) + \tfrac{1}{4}\mu(B), \quad \mu(T_2B) = \tfrac{1}{4}\mu(B+\rho) + \tfrac{1}{4}\mu(B).$$

Hence,

$$\mu(T_2B) \geq \mu(T_1B) \geq \mu(T_0B). \tag{5.3}$$

Putting (5.3) into (5.2), we have

$$\mu(A) \leq [0,1,0]M_J \begin{bmatrix} \mu(T_2B) \\ \mu(T_2B) \\ \mu(T_2B) \end{bmatrix} = 4c_J\mu(T_2B) = 4c_J\mu\big((T_{1J}T_2^{-1})^{-1}A\big),$$

which completes the proof. ■

Lemma 5.5 *Let $J \in \mathcal{I}_0^{n_t-1}$.*

(a) $-\Delta_{\mu\circ(T_{1J}T_2^{-1})^{-1}|_{T_{1J0}[0,1]}} \approx \frac{1}{\rho_{1J}\rho_2^{-1}}(-\Delta_{\mu|_{T_{20}[0,1]}})$;

(b) $\lambda_n(-\Delta_{\mu|_{T_{1J0}[0,1]}}) \geq \frac{1}{4c_J}\lambda_n(-\Delta_{\mu\circ(T_{1J}T_2^{-1})^{-1}|_{T_{1J0}[0,1]}})$.

Proof (a) follows by applying Proposition 2.2(b) with $S := T_{1J}T_2^{-1} : T_{20}[0,1] \to T_{1J0}[0,1]$. (b) follows by combining Proposition 2.3 with Lemma 5.4. ■

Equipped with the above lemmas, we are now ready to estimate $z^{(\alpha)}(t)$. We will use frequently the simple fact that if $\lambda_n := \lambda_n(-\Delta_\nu)$ and $\lambda_n^* := \lambda_n^*(-\Delta_{\nu^*})$ and there exists a constant $c > 0$ such that $\lambda_n \geq (1/c)\lambda_n^*$ for all $n \geq 1$, then $N(\lambda, -\Delta_\nu) \leq N(c\lambda, -\Delta_{\nu^*})$ for all $\lambda > 0$. In fact, for any $\lambda > 0$, let $n \in \mathbb{N}$ be the unique integer such that $\lambda_n \leq \lambda < \lambda_{n+1}$. Then

$$\lambda_n^* \leq c\lambda_n \leq c\lambda < c\lambda_{n+1}.$$

Hence, $N(\lambda, -\Delta_\nu) = n \leq N(c\lambda, -\Delta_{\nu^*})$.

Lemma 5.6 *There exists a constant $C > 0$ such that*

$$\sum_{J\in\mathcal{I}_0^{n_t-1}} e^{-\alpha t}N\big(e^t, -\Delta_{\mu|_{T_{1J}[0,1]}}\big) \leq Ce^{-\alpha t}2^{n_t}.$$

Proof For $J \in \mathcal{I}_0^{n_t-1}(0)$, by using (2.3), (2.4), and the lemmas above, we have

$$\begin{aligned} N\big(e^t, -\Delta_{\mu|_{T_{1J}[0,1]}}\big) &\leq \sum_{k=0}^{2} N\big(e^t, -\Delta_{\mu|_{T_{1Jk}[0,1]}}\big) + 2 \\ &\leq N\big(4c_Je^t, -\Delta_{\mu\circ(T_{1J}T_2^{-1})^{-1}|_{T_{1J0}[0,1]}}\big) && \text{(Lemma 5.5(b))} \\ &\quad + N\big(\rho_{1J}c_Je^t, -\Delta_{\mu|_{T_1[0,1]}}\big) && \text{(Lemma 5.1)} \\ &\quad + N\big(2c_{J'}e^t, -\Delta_{\mu\circ T_{1J'}^{-1}|_{T_{1J'02}[0,1]}}\big) && \text{(Lemma 5.3(b))} \\ &\quad + 2. \end{aligned}$$

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We now use Lemma 5.5(a), Lemma 5.3(a), and that $\rho_0 = \rho_2 = \rho^2$ and $\rho_1 = \rho^3$ to conclude that

$$
\begin{aligned}
N\big(e^t, -\Delta_{\mu|_{T_{1J}[0,1]}}\big) &\le N\big(4\rho^{2n_t-1}c_J e^t, -\Delta_{\mu|_{T_{20}[0,1]}}\big) && \text{(Lemma 5.5(a))}\\
&\quad + N\big(\rho^{2n_t+1}c_J e^t, -\Delta_{\mu|_{T_1[0,1]}}\big)\\
&\quad + N\big(2\rho^{2n_t-1}c_{J'} e^t, -\Delta_{\mu|_{T_{02}[0,1]}}\big) && \text{(Lemma 5.3(a))}\\
&\quad + 2.
\end{aligned} \tag{5.4}
$$

For each $t \in \mathbb{R}$, the n_t defined in (3.5) is the smallest positive integer such that

$$t + \ln(4\rho^{2n_t+1}c_J) \le t_o \quad \text{for all } J \in \mathcal{J}_0^{n_t-1},$$

where t_o is defined in (3.4). Equivalently,

$$4\rho^{2n_t+1}c_J e^t \le e^{t_o} \quad \text{for all } J \in \mathcal{J}_0^{n_t-1}.$$

It follows that each term on the right-hand side of (5.4) is bounded by some constant independent of J and t. Hence, there exists a constant $C > 0$ such that

$$N\big(e^t, -\Delta_{\mu|_{T_{1J}[0,1]}}\big) \le C \quad \text{for all } t \in \mathbb{R} \text{ and } J \in \mathcal{J}_0^{n_t-1}(0).$$

By symmetry, the same holds for all $J \in \mathcal{J}_0^{n_t-1}(2)$, and the lemma follows. ∎

For the infinite Bernoulli convolution we consider in this section, equation (1.9) becomes

$$F(\alpha) = \sum_{k=0}^{\infty} \sum_{J \in \mathcal{J}_0^k} \big(\rho^{2k+3}c_J\big)^{\alpha}, \qquad D = \big\{\alpha \in \mathbb{R} : F(\alpha) < \infty\big\}, \qquad \tilde{\alpha} = \inf D. \tag{5.5}$$

Clearly, $F(\alpha)$ is a strictly decreasing continuous function of α on D. Moreover, $[1, \infty) \subseteq D \subseteq (0, \infty)$. The first inclusion follows from [LN1, Proposition 2.4(i)]; the second inclusion is obvious. Thus $0 \le \tilde{\alpha} \le 1$. The following lemma shows that the hypothesis $\lim_{\alpha \to \tilde{\alpha}^+} F(\alpha) > 1$ in Theorem 1.1 is satisfied.

Lemma 5.7 *Let $F(\alpha)$ and $\tilde{\alpha}$ be defined as in (5.5). Then $F(\alpha) \to \infty$ as $\alpha \searrow \tilde{\alpha}$. Moreover, $F(\tilde{\alpha}) = \infty$.*

Proof First, we need to strengthen [LN1, Proposition 2.5] slightly by proving the following claim. Let $\{a_k\}$ be a submultiplicative sequence of nonnegative numbers, *i.e.*, $a_{m+k} \le a_m a_k$ for all m, k. Suppose there exists some k_o such that $a_{k_o} < 1$. Then there exists some $r \in [0, 1)$ and a constant $C > 0$ such that $a_k \le Cr^k$ for all k. In fact, for all $k > k_o$,

$$a_k \le a_{k_o} a_{k-k_o} \le a_{k_o}^2 a_{k-2k_o} \le \cdots \le C a_{k_o}^{k/k_o} = Cr^k,$$

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where $r = a_{k_o}^{1/k_o}$ if $a_{k_0} \neq 0$. If $a_{k_0} = 0$, then $a_k = 0$ for all $k \geq k_0$, and the result is obvious.

Next, by using the definition of c_J in (1.13), we have

$$F(\alpha) = \sum_{k=0}^{\infty} \frac{\rho^{(2k+3)\alpha}}{(2 \cdot 4^{k+1})^{\alpha}} \sum_{J \in \mathcal{J}_0^k} \left([1,1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{\alpha}.$$

Let

$$s_k(\alpha) := \sum_{J \in \mathcal{J}_0^k} \left([1,1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{\alpha}.$$

It is proved in [LN1, Proposition 2.5] that the sequence $\{s_k(\alpha)\}_k$ is submultiplicative, and thus $R(\alpha) := \lim_{k\to\infty} \sqrt[k]{s_k(\alpha)}$ exists in $\mathbb{R}$. Moreover, if we let $s_k(\alpha) = a_k(\alpha) R(\alpha)^k$, then $\{a_k(\alpha)\}_k$ is also submultiplicative and $\lim_{k\to\infty} \sqrt[k]{a_k(\alpha)} = 1$. By using the claim above, we have $a_k(\alpha) \geq 1$ for all k. Thus,

$$F(\alpha) = \frac{\rho^{3\alpha}}{8^{\alpha}} \sum_{k=0}^{\infty} a_k(\alpha) \left(\frac{\rho^{2\alpha} R(\alpha)}{4^{\alpha}} \right)^k \geq \frac{\rho^{3\alpha}}{8^{\alpha}} \sum_{k=0}^{\infty} \left(\frac{\rho^{2\alpha} R(\alpha)}{4^{\alpha}} \right)^k.$$

We note that $R(\alpha)$ is continuous. In fact, it is proved in [LN1, Proposition 2.4] that for all $J \in \mathcal{J}_0^k$, $c_J \leq 1/(4(4\rho)^k)$, and thus $[1,1]P_J[1,1]^t \leq 2\rho^{-k}$. Hence, for any $\epsilon > 0$,

$$\begin{aligned} \bigl| R(\alpha+\epsilon) - R(\alpha) \bigr| &\leq \lim_{k\to\infty} \left| \sqrt[k]{s_k(\alpha)} \sqrt[k]{\sup_{J \in \mathcal{J}_0^k} \left([1,1] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{\epsilon}} - \sqrt[k]{s_k(\alpha)} \right| \\ &= R(\alpha) |\rho^{-\epsilon} - 1|. \end{aligned}$$

Next, let $g(\alpha) := (\rho^{2\alpha} R(\alpha))/4^{\alpha}$. Then g is a continuous function of α. The following identity shows that $g(\alpha)$ is decreasing:

$$\left(\sqrt[k]{\frac{\rho^{3\alpha} a_k(\alpha)}{8^{\alpha}}}\, g(\alpha) \right)^k = \sum_{J \in \mathcal{J}_0^k} (\rho^{2k+3} c_J)^{\alpha}, \quad k \geq 0. \tag{5.6}$$

Moreover, (5.6) shows that

$$\lim_{\alpha\to 0} g(\alpha) = 2 \quad \text{and} \quad \lim_{\alpha\to\infty} g(\alpha) = 0.$$

Now it is easy to see that $F(\alpha)$ converges if and only if $\rho^{2\alpha} R(\alpha)/4^{\alpha} < 1$. Furthermore, $\tilde{\alpha}$ satisfies

$$\rho^{2\tilde{\alpha}} R(\tilde{\alpha})/4^{\tilde{\alpha}} = 1, \quad F(\tilde{\alpha}) = \frac{\rho^{3\tilde{\alpha}}}{8^{\tilde{\alpha}}} \sum_{k=0}^{\infty} a_k(\tilde{\alpha}) = \infty,$$

and $F(\alpha) \to \infty$ as $\alpha \searrow \tilde{\alpha}$. ∎

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For the rest of this section we let α be the unique positive number satisfying

$$\sum_{k=0}^{\infty}\sum_{J\in\mathcal{J}_0^k}\left(\rho^{2k+3}c_J\right)^{\alpha}=1. \tag{5.7}$$

Such an α exists by Lemma 5.7.

Before proving the next lemma, we recall that the *lower L^∞-dimension* of a compactly supported finite Borel measure ν on $\mathbb{R}^d$ is defined as

$$\underline{\dim}_\infty(\nu)=\varliminf_{\delta\to 0^+}\frac{\ln(\sup_x \nu(B_\delta(x)))}{\ln\delta},$$

where $B_\delta(x)$ is the δ-ball with center x, and the supremum is taken over all $x\in\operatorname{supp}(\nu)$.

Lemma 5.8 *Let α be defined as in* (5.7), *and let $z^{(\alpha)}(t)$ be defined as in* (5.1). *Then there exists some $\sigma>0$ such that $e^{-\alpha t}2^{n_t}=o(e^{-\sigma t})$ as $t\to\infty$. Consequently, $z^{(\alpha)}(t)=o(e^{-\sigma t})$ as $t\to\infty$.*

Proof We first obtain a lower estimate for α by using some known results in [LN1]. Note that α is the unique q coordinate of the intersection of the curve $y=\tau(q)$ and the line $y=-q$ in the (q,y)-plane, where $\tau(q)$ is defined by (see [LN1])

$$\sum_{k=0}^{\infty}(\rho^{2k+3})^{-\tau(q)}\sum_{J\in\mathcal{J}_0^k}c_J^q=1.$$

Also, note that by the strict concavity of $\tau(q)$ for $q>0$, the line in the (q,y)-plane with slope $\dim_{\mathrm{H}}(\mu)=\tau'(1)$ and passing through $(1,0)$ is strictly above the curve of $\tau(q)$ (except at the point $(1,0)$). Let α_1 be the q value at the intersection of the lines

$$y=\dim_{\mathrm{H}}(\mu)(q-1)\quad\text{and}\quad y=-q.$$

Then

$$\alpha>\alpha_1=\frac{\dim_{\mathrm{H}}(\mu)}{\dim_{\mathrm{H}}(\mu)+1}.$$

Moreover, by using the strict concavity of $\tau(q)$ for $q>0$, we have

$$\dim_{\mathrm{H}}(\mu)>\underline{\dim}_\infty(\mu).$$

Using the fact that the function $f(x)=x/(x+1)$ is strictly increasing for $x>0$, together with the explicit formula $\underline{\dim}_\infty(\mu)=-\ln(4\rho)/(2\ln\rho)$ in [LN1, Theorem 4.4], we have

$$\alpha>\alpha_1=\frac{\dim_{\mathrm{H}}(\mu)}{\dim_{\mathrm{H}}(\mu)+1}>\frac{\underline{\dim}_\infty(\mu)}{\underline{\dim}_\infty(\mu)+1}=\frac{\ln(4\rho)}{\ln(4/\rho)}. \tag{5.8}$$

Next, we will obtain an upper estimate for 2^{n_t}. Since n_t is the smallest positive integer such that $t + \ln(\rho^{2n_t+1} c_J) \le t_o$ for all $J \in \mathcal{J}_0^{n_t-1}$, there exists $J_o \in \mathcal{J}_0^{n_t-2}$ such that

$$\rho^{2n_t-1} c_{J_o} > e^{t_o - t}.$$

Also, by [LN1, Proposition 2.4], $c_{J_o} \le 1/(4(4\rho)^{n_t-2})$. Hence,

$$n_t < \frac{\ln(4\rho) - t_o + t}{\ln(4/\rho)}.$$

Thus,

$$2^{n_t} \le \exp\left(\frac{(\ln(4\rho) - t_o)\ln 2}{\ln(4/\rho)}\right) \exp\left(\frac{\ln 2}{\ln(4/\rho)} t\right) =: C \exp\left(\frac{\ln 2}{\ln(4/\rho)} t\right). \tag{5.9}$$

Combining (5.8) and (5.9), we get

$$e^{-\alpha t} 2^{n_t} \le C \exp\left(\frac{-\ln(2\rho)}{\ln(4/\rho)} t\right).$$

The first part of the lemma follows by noting that $\ln(2\rho)/\ln(4/\rho) \approx 0.1134 > 0$. Combining this with Lemma 5.6 yields the second part. ■

Proof of Theorem 1.2 Combine Lemmas 5.7 and 5.8, and Theorem 1.1.

6 Convolutions of Cantor-type Measures

In this section we compute the spectral dimension of convolutions of Cantor-type measures. The attractor of the IFS in (1.15) is a Cantor-type set. We will assume that m is odd. Let ν_m be the corresponding self-similar measure with probability weights $p_0 = p_1 = 1/2$. We first show that the m-fold convolution of ν_m is the self-similar measure defined by the IFS in (1.17).

Proposition 6.1 *Let $m \ge 3$ be an odd integer. Then $\mu_m := \nu_m^{*m}$ is the self-similar measure defined by the IFS* (1.16) *together with probability weights*

$$w_i := \frac{1}{2^m}\binom{m}{i}, \quad i = 0, 1, \dots, m.$$

That is, $\mu_m = \sum_{i=0}^m w_i \mu_m \circ S_i^{-1}$.

Proof We prove by induction the more general result that ν_m^{*k} is the self-similar measure defined by the IFS

$$S_i(x) = \frac{1}{m}x + \frac{m-1}{m}i, \quad i = 0, 1, \dots, k,$$

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with probability weights $\binom{k}{i}/2^k, i = 0, 1, \ldots, k$. This is clearly true if $k = 1$. Assume that it is true for some $k \geq 1$. Then for any Borel subset $A \subseteq \mathbb{R}$, by using the self-similar identity for ν_m and induction hypothesis, we have

$$\begin{aligned}
\nu_m^{*(k+1)}(A) &= \int_{\mathbb{R}} \nu_m^{*k}(A - x)\, d\nu_m \\
&= \frac{1}{2}\int_{\mathbb{R}} \nu_m^{*k}(A - S_0 x)\, d\nu_m + \frac{1}{2}\int_{\mathbb{R}} \nu_m^{*k}(A - S_1 x)\, d\nu_m \\
&= \sum_{j=0}^{k} \frac{1}{2^{k+1}} \binom{k}{j} \int_{\mathbb{R}} \nu_m^{*k}\big(S_j^{-1}(A - S_0 x)\big)\, d\nu_m \\
&\quad + \sum_{j=0}^{k} \frac{1}{2^{k+1}} \binom{k}{j} \int_{\mathbb{R}} \nu_m^{*k}\big(S_j^{-1}(A - S_1 x)\big)\, d\nu_m.
\end{aligned}$$

Rewrite the first and second summations above, respectively, as

$$\frac{1}{2^{k+1}}\binom{k+1}{0}\int_{\mathbb{R}} \nu_m^{*k}(S_0^{-1}A - x)\, d\nu_m + \frac{1}{2^{k+1}}\sum_{j=1}^{k}\binom{k}{j}\int_{\mathbb{R}} \nu_m^{*k}(S_j^{-1}A - x)\, d\nu_m,$$

$$\frac{1}{2^{k+1}}\sum_{j=1}^{k}\binom{k}{j-1}\int_{\mathbb{R}} \nu_m^{*k}(S_j^{-1}A - x)\, d\nu_m + \frac{1}{2^{k+1}}\binom{k+1}{k+1}\int_{\mathbb{R}} \nu_m^{*k}(S_{k+1}^{-1}A - x)\, d\nu_m,$$

and using the identity $\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$, we get

$$\nu_m^{*(k+1)}(A) = \sum_{j=0}^{k+1} \frac{1}{2^{k+1}} \binom{k+1}{j} \nu_m^{*(k+1)}(S_j^{-1}A).$$

This completes the induction, and the proposition follows. ■

Note that $\operatorname{supp}(\mu_m) = [0, m]$ and that the IFS (1.16) is not PCF and does not satisfy the open set condition, since

$$S_i[0, m] \cap S_{i+1}[0, m] = \left[\frac{m-1}{m} + \frac{(m-1)i}{m},\ 1 + \frac{(m-1)i}{m}\right] \quad \text{for} \quad i = 0, 1, \ldots, m.$$

For the rest of this section we will fix an odd integer $m \geq 3$, and let $\mu := \mu_m$ for convenience.

For the IFS in (1.16), it is shown in [LN3] that μ satisfies a family of second-order identities with respect to the IFS

$$T_i(x) = \frac{1}{m}x + i, \quad i = 0, 1, \ldots, m - 1.$$

In fact, for $i, j, k \in \{0, 1, \dots, m-1\}$, define

$$m^{(i)}_{j,k} = \begin{cases} w_\ell, & \text{if } 0 \le \ell \le m \text{ and } i + mj - (m-1)\ell = k, \\ 0, & \text{otherwise,} \end{cases}$$

and let M_i be the matrix

$$M_i = \left[m^{(i)}_{j+1,k+1} \right]^{m-1}_{j,k=0}. \tag{6.1}$$

Then μ satisfies

$$\begin{bmatrix} \mu(T_{0i}A) \\ \vdots \\ \mu(T_{(m-1)i}A) \end{bmatrix} = M_i \begin{bmatrix} \mu(T_0 A) \\ \vdots \\ \mu(T_{m-1}A) \end{bmatrix}, \quad i = 0, 1, \dots, m-1. \tag{6.2}$$

For the IFS's $\{S_i\}^m_{i=0}$ and $\{T_i\}^{m-1}_{i=0}$ we consider here, $L = m-1$, $\widetilde{n} = \overline{n} = 1$. Moreover, it is shown in [LN3] that conditions (C1), (C2), and (C3) hold with $\omega = 2$, $\mathcal{J}_0 = \{0, m-1\}$, $\mathcal{J}_1 = \{1, \dots, m-2\}$, and $\mathbf{I} = \{(i, i') : i = 1, \dots, m-2\}$, where $i' = m-1-i$. Furthermore, we have

$$\begin{aligned} \mathbf{I}^0_2 &= \{(i,k) : i = 1, \dots, m-2, k \ne i'\}, \\ \mathbf{I}^0_2(i) &= \{(i,k) : k \ne i'\}, \quad i = 1, \dots, m-2, \\ \mathbf{I}(i) &= \{(i,i')\}, \quad i = 1, \dots, m-2. \end{aligned}$$

Let $i = 1, \dots, m-2$, $I = (i, i')$, and $n_t > \omega = 2$. The renewal equation (3.6) can be written as

$$\begin{aligned} f_i(t) = &\sum_{J \in \mathbf{I}^0_2(i)} \left(\frac{c(J, j(J))}{m} \right)^\alpha f_{j(J)} \left(t + \ln\left(\frac{c(J, j(J))}{m} \right) \right) \\ &+ \sum_{k=0}^\infty \sum_{J \in \mathcal{J}^k_0} \sum_{\ell=1}^{m-2} \left(\frac{c(I, J, \ell, j)}{m^{k+2}} \right)^\alpha f_{j(I,J,\ell)} \left(t + \ln\left(\frac{c(I, J, \ell, j)}{m^{k+2}} \right) \right) + z^{(\alpha)}_i(t), \end{aligned}$$

where

$$z^{(\alpha)}_i(t) = \sum_{J \in \mathcal{J}^{n_t - 2}_0} e^{-\alpha t} N\left(e^t, -\Delta_{\mu|_{T_{IJ}[0,m]}}\right) + e^{-\alpha t} \varepsilon_{n_t}(i) \tag{6.3}$$

and $0 \le \varepsilon_{n_t}(i) \le \#\mathcal{P}_{n_t}(i)$.

Proposition 6.2 *The spectral radius of* $\mathbf{M}_\alpha(\infty)$ *is*

$$\frac{1}{m^\alpha} \sum_{i=1}^{m-1} w_i^\alpha + \sum_{k=0}^\infty \frac{1}{m^{(k+2)\alpha}} \sum_{i=1}^{m-2} \sum_{J \in \mathcal{J}^k_0} c^\alpha_{i,J},$$

where $c_{i,J}$ *is defined as in* (1.19) *and* (1.20).

Proof It follows from the renewal equations above that the (j,i)-entry of $\mathbf{M}_\alpha(\infty)$, *i.e.*, $\mu_{ji}^{(\alpha)}(\infty)$, is equal to

$$\sum_{J\in \mathbf{I}_2^0(i),\, j(J)=j} \left(\frac{c(J,j(J))}{m}\right)^{\alpha} + \sum_{k=0}^{\infty}\sum_{J\in \mathcal{J}_0^k}\sum_{\ell=1}^{m-2}\left(\frac{c(I,J,\ell,j)}{m^{k+2}}\right)^{\alpha} \#\big\{\, j(I,J,\ell) : j(I,J,\ell)=j\big\},$$

where $I=(i,i')$.

We will show that the row sums of $\mathbf{M}_\alpha(\infty)$ are all equal. In fact, it follows from [LN3, Propositions 4.3(a)] that for each $J=(i,k)\in \mathbf{I}_2^0(i)$,

$$c(J,j)=\begin{cases} w_i\delta_{j,k+i}, & \text{if } 0\le k<i',\\ w_{i+1}\delta_{j,k-i'}, & \text{if } i'<k\le m-1.\end{cases}$$

Hence,

$$\sum_{J\in \mathbf{I}_2^0(i),\, j(J)=j}\left(\frac{c(J,j(J))}{m}\right)^{\alpha}=\begin{cases} w_{i+1}^{\alpha}/m^{\alpha}, & \text{if } j<i,\\ (w_i^{\alpha}+w_{i+1}^{\alpha})/m^{\alpha}, & \text{if } j=i,\\ w_i^{\alpha}/m^{\alpha}, & \text{if } j>i,\end{cases}$$

and therefore

$$\sum_{i=1}^{m-2}\sum_{J\in \mathbf{I}_2^0(i),\, j(J)=j}\left(\frac{c(J,j(J))}{m}\right)^{\alpha}=\frac{1}{m^{\alpha}}\sum_{i=1}^{m-1} w_i^{\alpha}.$$

Also, it follows from [LN3, Propositions 4.4] that $c(I,J,\ell,j)=c_{i,J}\delta_{\ell,j}$. Thus,

$$\sum_{i=1}^{m-2}\sum_{k=0}^{\infty}\sum_{J\in \mathcal{J}_0^k}\sum_{\ell=1}^{m-2}\left(\frac{c(I,J,\ell,j)}{m^{k+2}}\right)^{\alpha}\#\big\{\, j(I,J,\ell) : j(I,J,\ell)=j\big\} = \sum_{k=0}^{\infty}\frac{1}{m^{(k+2)\alpha}}\sum_{i=1}^{m-2}\sum_{J\in \mathcal{J}_0^k} c_{i,J}^{\alpha},$$

which completes the proof. ∎

We now turn to estimating the error term $\mathbf{z}^{(\alpha)}(t)$. For each $t>0$, n_t is the smallest integer such that

$$t+\max\Big\{\ln\Big(\frac{c_{i,J}}{m^{n_t}}\Big) : i=1,\dots,m-2,\, J\in \mathcal{J}_0^{n_t-2}\Big\}<t_o,$$

where t_o is defined as in (3.5). We need to show that for each $i=1,\dots,m-2$, there exists some $\sigma>0$ such that $z_i^{(\alpha)}(t)$ is of order $o(e^{-\sigma t})$ as $t\to\infty$ (Lemma 6.14).

We first compute the lower L^∞-dimension of μ. It is proved in [LN3, Theorem 1.2] that for the self-similar measure in (1.17), the L^q-spectrum $\tau(q)$, $q > 0$, is the unique real number α satisfying

$$m^\alpha \sum_{i=1}^{m-1} w_i^q + \sum_{k=0}^{\infty} m^{(k+2)\alpha} \Big(\sum_{i=1}^{m-2} \sum_{J \in \mathcal{I}_0^k} c_{i,J}^q \Big) = 1. \tag{6.4}$$

Let

$$\widetilde{P}_0 = \begin{bmatrix} 1 & 0 \\ 1 & m \end{bmatrix} \qquad \text{and} \qquad \widetilde{P}_{m-1} = \begin{bmatrix} m & 1 \\ 0 & 1 \end{bmatrix}.$$

Then for $J \in \mathcal{I}_0^k$,

$$c_{i,J} = \frac{1}{2^{2m}} \left[\tbinom{m}{i+1}, \tbinom{m}{i} \right] P_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2^{2m+mk}} \left[\tbinom{m}{i+1}, \tbinom{m}{i} \right] \widetilde{P}_J \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{6.5}$$

Proposition 6.3 *Let $k \geq 1$. Then*

(a) $\widetilde{P}_0^k = \begin{bmatrix} 1 & 0 \\ \frac{m^k-1}{m-1} & m^k \end{bmatrix}$ *and* $\widetilde{P}_{m-1}^k = \begin{bmatrix} m^k & \frac{m^k-1}{m-1} \\ 0 & 1 \end{bmatrix}$;

(b) $(m-1, \dots, m-1), (0, \dots, 0) \in \mathcal{I}_0^k$ *maximize, respectively, the first and second column sums over all $\widetilde{P}_J$, $J \in \mathcal{I}_0^k$.*

Proof Part (a) follows directly by induction.

(b) We use induction again. The assertion clearly holds for $k = 1$. Assume that it holds for some $k \geq 1$. Write $\widetilde{P}_J$, $J \in \mathcal{I}_0^k$, as

$$\widetilde{P}_J = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{6.6}$$

Then

$$\widetilde{P}_J \widetilde{P}_0 = \begin{bmatrix} a+b & mb \\ c+d & md \end{bmatrix} \quad \text{and} \quad \widetilde{P}_J \widetilde{P}_{m-1} = \begin{bmatrix} ma & a+b \\ mc & c+d \end{bmatrix}.$$

By induction hypothesis, $a + c \leq m^k$ and $b + d \leq m^k$. Thus,

$$a+b+c+d \leq 2m^k \leq m^{k+1}, \quad m(a+c) \leq m^{k+1}, \quad \text{and} \quad m(b+d) \leq m^{k+1}.$$

By using (a), we see that the assertion holds for $J \in \mathcal{I}_0^{k+1}$. ∎

Since m is odd, the maximum of the binomial coefficients $\binom{m}{i}$, $0 \leq i \leq m$, is attained when $i = [(m-1)/2]$ or $i = (m+1)/2$. Let

$$c_m := \binom{m}{(m-1)/2} = \binom{m}{(m+1)/2}.$$

Proposition 6.4 *Let $k \geq 1$. Then*

$$\frac{c_m}{4^m} \Big(\frac{m}{2^m} \Big)^k \leq \max\{c_{i,J} : i = 1, \dots, m-2, J \in \mathcal{I}_0^k\} \leq \frac{2c_m}{4^m} \Big(\frac{m}{2^m} \Big)^k.$$

Proof Let $J \in \mathcal{J}_0^k$ and write $\widetilde{P}_J$ as in (6.6). Then for $i = 1, \dots, m-2$,

$$\begin{aligned}
c_{i,J} &= \frac{1}{2^{2m+mk}} \left[\binom{m}{i+1}, \binom{m}{i}\right] \widetilde{P}_J \begin{bmatrix} 1 \\ 1 \end{bmatrix} && \text{(by (6.5))} \\
&\le \frac{1}{2^{2m+mk}} \left[c_m, c_m\right] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{c_m(a+b+c+d)}{2^{2m+mk}} \\
&\le \frac{2c_m m^k}{2^{2m+mk}}. && \text{(Proposition 6.3)}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\max\{c_{i,J} : i = 1, \dots, m-2,\, J \in \mathcal{J}_0^k\} &\ge \frac{1}{2^{2m+mk}} \left[c_m, c_m\right] \widetilde{P}_0^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{2^{2m+mk}} \left[c_m, c_m\right] \begin{bmatrix} 1 & 0 \\ \frac{m^k-1}{m-1} & m^k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&\ge \frac{c_m m^k}{2^{2m+mk}}.
\end{aligned}$$

This completes the proof. ■

Theorem 6.5 *Let $\mu = \mu_m$ be the self-similar measure in* (1.17)*. Then*

$$\underline{\dim}_\infty(\mu) = \frac{\ln(2^m/c_m)}{\ln m}.$$

Proof The proof is obtained by modifying that of [LN3, Theorem 6.2]; we include it for completeness.

For $q > 0$, (6.4) implies that $m^{\tau(q)}(c_m/2^m)^q < 1$. Thus,

$$\lim_{q\to\infty} \frac{\tau(q)}{q} \le \frac{\ln(2^m/c_m)}{\ln m} =: \ell.$$

Suppose $\lim_{q\to\infty} \tau(q)/q < \ell - \epsilon$ for some $\epsilon > 0$. We first notice that this would imply

$$\lim_{q\to\infty} m^{\tau(q)} \sum_{i=1}^{m-1} w_i^q = 0. \tag{6.7}$$

In fact, for all $q > 0$ sufficiently large, we would get

$$m^{\tau(q)} \left(\frac{c_m}{2^m}\right)^q = \left(m^{\tau(q)/q} \frac{c_m}{2^m}\right)^q \le \left(m^{\ln(2^m/c_m)/\ln m - \epsilon} \frac{c_m}{2^m}\right)^q = \left(\frac{1}{m^\epsilon}\right)^q,$$

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which implies (6.7).

Next, by putting $q = 1$ into (6.4) and using the fact that $\tau(1) = 0$, we get

$$\sum_{i=1}^{m-1} w_i + \sum_{k=0}^{\infty}\Big(\sum_{i=1}^{m-2}\sum_{J\in\mathcal{I}_0^k} c_{i,J}\Big) = 1.$$

Thus,

$$\sum_{k=0}^{\infty}\sum_{i=1}^{m-2}\sum_{J\in\mathcal{I}_0^k} c_{i,J} = 1 - \sum_{i=1}^{m-1} w_i = w_0 + w_m > 0.$$

Now let q_0 be sufficiently large so that for all $q \geq q_o$,

$$m^{\tau(q)}\sum_{i=1}^{m-1} w_i^q < 1 - (w_0 + w_m).$$

Then for all $q \geq q_o$,

$$\begin{aligned}
w_0 + w_m &< \sum_{k=0}^{\infty} m^{(k+2)\tau(q)}\Big(\sum_{i=1}^{m-2}\sum_{J\in\mathcal{I}_0^k} c_{i,J}^q\Big)\\
&\leq \sum_{k=0}^{\infty} m^{(k+2)\tau(q)}\big(\max\{c_{i,J} : i = 1,\dots,m-2,\, J \in \mathcal{I}_0^k\}\big)^{q-1}\Big(\sum_{i=1}^{m-2}\sum_{J\in\mathcal{I}_0^k} c_{i,J}\Big)\\
&\leq \sum_{k=0}^{\infty} m^{(k+2)\tau(q)}\Big(\frac{2c_m}{4^m}\Big(\frac{m}{2^m}\Big)^k\Big)^{q-1}\Big(\sum_{i=1}^{m-2}\sum_{J\in\mathcal{I}_0^k} c_{i,J}\Big)\\
&\leq (w_0 + w_m)\sup_{k\geq 0}\Big(m^{\tau(q)}\Big(\frac{c_m}{2^m}\Big)^{q-1}\Big)^{k+2},
\end{aligned}$$

where in the last inequality we have used the fact that $2c_m m^k \leq c_m^{k+2}$. It follows that $m^{\tau(q)}(c_m/2^m)^{q-1} \geq 1$ for all $q \geq q_o$, and thus $\lim_{q\to\infty} \tau(q)/(q-1) \geq \ell$. This contradiction completes the proof. ■

We now find an upper estimate for

$$\sum_{J\in\mathcal{I}_0^{n_t-2}} e^{-\alpha t} N(e^t, -\Delta_{\mu|_{T_{IJ}[0,m]}}),$$

where $I = (i, i')$. Again, we need a few lemmas; the estimate is given in Lemma 5.6.

By symmetry it suffices to consider those $J = (j_1,\dots,j_{n_t-2}) \in \mathcal{I}_0^{n_t-2}$ with $j_{n_t-2} = 0$. Let $J = (J', 0)$, where $J' \in \mathcal{I}_0^{n_t-2}$. We divide $T_{IJ}[0,m]$ into the following m subintervals:

$$T_{IJ0}[0,m], \quad T_{IJj}[0,m], \quad j = 1,\dots,m-2, \quad \text{and} \quad T_{IJ(m-1)}[0,m].$$

We consider the Laplacian defined by the restriction of μ to these subintervals.

Case 1. $T_{IJj}[0,m]$, $j = 1,\dots,m-2$.

Lemma 6.6 *For* $I=(i,i')$, $J\in\mathcal{I}_0^{n_t-2}$ *and* $j=1,\dots,m-2$,

$$-\Delta_{\mu|_{T_{IJj}[0,m]}}\approx\frac{1}{\rho_{IJ}c_{i,J}}\big(-\Delta_{\mu|_{T_j[0,m]}}\big).$$

Proof Let $A\subseteq T_{IJj}[0,m]$. Then there exists $B\subseteq[0,m]$ such that $A=T_{IJj}B$. Hence by [LN3, Proposition 4.4(c)(i)],

$$\mu(A)=\mu(T_{IJj}B)=c_{i,J}\mu(T_jB)=c_{i,J}\mu\circ T_{IJ}^{-1}(A).$$

The lemma now follows from Proposition 2.2. ∎

We now consider the following.

Case 2. $T_{IJ(m-1)}[0,m]=T_{IJ'0(m-1)}[0,m]$.

Lemma 6.7 *Let* $I=(i,i')$ *and* $J'\in\mathcal{I}_0^{n_t-3}$. *There exists a constant* $C>0$ *such that for all* $A\subseteq T_{IJ'0(m-1)}[0,m]$,

$$c_{i,J'}\mu\circ T_{IJ'}^{-1}(A)\le\mu(A)\le Cc_{i,J'}\mu\circ T_{IJ'}^{-1}(A).$$

Proof Let $A\subseteq T_{IJ'0(m-1)}[0,m]$. Then there exists $D\subseteq[0,m]$ such that

$$A=T_{IJ'0(m-1)}D=T_{IJ'}T_1T_1^{-1}T_{0(m-1)}D.$$

Note that $T_{0(m-1)}[0,m]\subseteq[(m-1)/m,1]$. Thus

$$T_1^{-1}T_{0(m-1)}[0,m]\subseteq T_1^{-1}\Big[\frac{m-1}{m},1\Big]\subseteq[-1,0].$$

We now apply [LN3, Proposition A.2] with $B:=T_1^{-1}T_{0(m-1)}D$ to obtain a constant $C>0$ such that

$$c_{i,J'}\mu(T_1B)\le\mu(A)=\mu(T_{IJ'1}B)\le Cc_{i,J'}\mu(T_1B),$$

from which the asserted inequalities follow. ∎

Lemma 6.8 *Let* $I=(i,i')\in\mathbf{I}$ *and* $J=(J',0)\in\mathcal{I}_0^{n_t-2}$.

(a) $-\Delta_{\mu\circ T_{IJ'}^{-1}|_{T_{IJ'0(m-1)}[0,m]}}\approx\frac{1}{\rho_{IJ'}}(-\Delta_{\mu|_{T_{0(m-1)}[0,m]}})$.

(b) *There exists a constant* $C>0$ *such that*

$$\lambda_n\big(-\Delta_{\mu|_{T_{IJ'0(m-1)}[0,m]}}\big)\ge\frac{1}{Cc_{i,J'}}\lambda_n\big(-\Delta_{\mu\circ T_{IJ'}^{-1}|_{T_{IJ'0(m-1)}[0,m]}}\big).$$

Proof (a) follows from Proposition 2.2.

(b) follows from Proposition 2.3 and Lemma 6.7. ∎

Case 3. $T_{IJ0}[0,m] = T_{IJ}T_{m-1}^{-1}T_{m-1}T_0[0,m] = T_{IJ}T_{m-1}^{-1}[m-1, m-1+\frac{1}{m}]$.

We recall a result and some notation from [LN3] before proving the next lemma. For each $i = 1,\dots,m-2$, let $\widetilde{M}_i$ be the matrix formed from the M_i in (6.1) by keeping its $(m-i)$-th row and assigning 0 to all other entries. For $i = 0$ or $m-1$, let $\widetilde{M}_i$ be the matrix formed from M_i by keeping its first and last rows and assigning 0 to all other entries. Then for $i = 1,\dots,m-2$,

$$c_{i,J} = [w_{i+1}, \mathbf{0}, w_i]\widetilde{M}_J \begin{bmatrix} w_0 \\ \mathbf{0} \\ w_m \end{bmatrix} \quad \text{and} \quad [w_{i+1}, \mathbf{0}, w_i] = \mathbf{e}_i M_{i'} = \mathbf{e}_i \widetilde{M}_{i'},$$

where $\mathbf{e}_i$ denotes the unit vector in $\mathbb{R}^m$ whose $(i+1)$-st coordinate is 1.

Lemma 6.9 *Let $I = (i,i')$ and $J \in \mathcal{I}_0^{n_t-2}$. Then for all μ-measurable subsets $A \subseteq T_{IJ0}[0,m]$,*

$$\mu(A) \le \frac{c_{i,J}}{w_0}\mu \circ (T_{IJ}T_{m-1}^{-1})^{-1}(A).$$

Proof Let $A \subseteq T_{IJ0}[0,m]$. Then there exists $B \subseteq T_0[0,m]$ such that $A = T_{IJ}B$. Thus by the proofs of [LN3, Proposition 4.4(c)(ii)] and [LN3, Proposition 4.4(a)],

$$\begin{aligned}
\mu(A) &= \mu(T_{IJ}B) = \mu(T_{IJ'0}B)\\
&= \mathbf{e}_i \widetilde{M}_{i'}\widetilde{M}_{J'} \begin{bmatrix} \mu(T_0T_0B) \\ \mathbf{0} \\ \mu(T_{m-1}T_0B) \end{bmatrix} \qquad \text{(proof of [LN3, Proposition 4.4(c)(ii)])}\\
&= \mathbf{e}_i \widetilde{M}_{i'}\widetilde{M}_{J'}\widetilde{M}_0 \begin{bmatrix} \mu(T_0B) \\ \mathbf{0} \\ \mu(T_{m-1}B) \end{bmatrix} \qquad \text{(proof of [LN3, Proposition 4.4(a)])}\\
&= [w_{i+1}, \mathbf{0}, w_i]\widetilde{M}_J \begin{bmatrix} \mu(T_0B) \\ \mathbf{0} \\ \mu(T_{m-1}B) \end{bmatrix}.
\end{aligned}$$

By using the self-similar identity (1.17) we get

$$\mu(T_0B) = w_0\mu(B) \quad \text{and} \quad \mu(T_{m-1}B) = w_{m-1}\mu(S_{m-1}^{-1}T_{m-1}B) + w_m\mu(B).$$

Since $w_0 = w_m = 1/2^m$, we have $\mu(T_0B) \le \mu(T_{m-1}B)$. Consequently,

$$\mu(A) \le \frac{1}{w_0}[w_{i+1}, \mathbf{0}, w_i]\widetilde{M}_J \begin{bmatrix} w_0 \\ \mathbf{0} \\ w_m \end{bmatrix} \mu(T_{m-1}B) = \frac{c_{i,J}}{w_0}\mu\big((T_{IJ}T_{m-1}^{-1})^{-1}A\big). \qquad \blacksquare$$

Lemma 6.10 *Let $I = (i,i')$ and $J \in \mathcal{I}_0^{n_t-2}$.*

(a) $-\Delta_{\mu\circ(T_{IJ}T_{m-1}^{-1})^{-1}|_{T_{IJ0}[0,m]}} \approx \frac{1}{\rho_{IJ}\rho_{m-1}^{-1}}(-\Delta_{\mu|_{T_{m-1}T_0[0,m]}})$.

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(b) $\lambda_n(-\Delta_{\mu|_{T_{IJ0}[0,m]}}) \geq \frac{w_0}{c_{i,J}} \lambda_n(-\Delta_{\mu\circ(T_{IJ}T_{m-1}^{-1})^{-1}}|_{T_{IJ0}[0,m]})$.

Proof (a) follows from Proposition 2.2. (b) follows from Proposition 2.3 and Lemma 6.9. ∎

Lemma 6.11 *Let $I = (i, i')$. Then there exists some $C_o > 0$ such that for all $t \in \mathbb{R}$,*

$$\sum_{J \in \mathcal{I}_0^{n_t - 2}} e^{-\alpha t} N\big(e^t, -\Delta_{\mu|_{T_{IJ}[0,m]}}\big) \leq C_o e^{-\alpha t} 2^{n_t}.$$

Proof For $I = (i, i')$, $J = (j_1, \dots, j_{n_t}, 0) = (J', 0) \in \mathcal{I}_0^{n_t-2}$, and C as in Lemma 6.8(b), by using the lemmas above, we have

$$\begin{aligned}
N\big(e^t, -\Delta_{\mu|_{T_{IJ}[0,m]}}\big) &\leq \sum_{j=0}^{m-1} N(e^t, -\Delta_{\mu|_{T_{IJj}[0,m]}}) + m - 1 && \text{((2.3) and (2.4))}\\
&\leq N\Big(\frac{c_{i,J}}{w_0} e^t, -\Delta_{\mu\circ(T_{IJ}T_{m-1}^{-1})^{-1}|_{T_{IJ0}[0,m]}}\Big) && \text{(Lemma 6.10(b))}\\
&\quad + \sum_{j=1}^{m-2} N\big(\rho_{IJ} c_{i,J} e^t, -\Delta_{\mu|_{T_j[0,m]}}\big) && \text{(Lemma 6.6)}\\
&\quad + N\big(C c_{i,J'} e^t, -\Delta_{\mu\circ T_{IJ'}^{-1}|_{T_{IJ'0(m-1)}[0,m]}}\big) && \text{(Lemma 6.8(b))}\\
&\quad + m - 1
\end{aligned}$$

By applying Lemma 6.10(a) and Lemma 6.8(a) to the first and third terms in the last expression, respectively, and using the fact that $\rho_i = 1/m$ for $i = 0, 1, \dots, m-1$, we get

$$\begin{aligned}
N\big(e^t, -\Delta_{\mu|_{T_{IJ}[0,m]}}\big) &\leq N\Big(\frac{\rho_{IJ} c_{i,J}}{\rho_{m-1} w_0} e^t, -\Delta_{\mu|_{T_{m-1}T_0[0,m]}}\Big) + \sum_{j=1}^{m-2} N\big(\rho_{IJ} c_{i,J} e^t, -\Delta_{\mu|_{T_j[0,m]}}\big)\\
&\quad + N\big(C \rho_{IJ'} c_{i,J'} e^t, -\Delta_{\mu|_{T_0 T_{m-1}[0,m]}}\big) + m - 1\\
&= N\Big(\frac{c_{i,J}}{m^{n_t - 1} w_0} e^t, -\Delta_{\mu|_{T_{m-1}T_0[0,m]}}\Big) + \sum_{j=1}^{m-2} N\Big(\frac{c_{i,J}}{m^{n_t}} e^t, -\Delta_{\mu|_{T_j[0,m]}}\Big)\\
&\quad + N\Big(\frac{C c_{i,J'}}{m^{n_t - 1}} e^t, -\Delta_{\mu|_{T_0 T_{m-1}[0,m]}}\Big) + m - 1.
\end{aligned}$$

Now, it follows from the choice of n_t that there exists a constant $C_1 > 0$ such that

$$N\big(e^t, -\Delta_{\mu|_{T_{IJ}[0,m]}}\big) \leq C_1.$$

By symmetry, the same bound holds if $J = (j_1, \dots, j_{n_t-2}) \in \mathcal{I}_0^{n_t-2}$ and $j_{n_t-2} = m - 1$. Hence, the lemma follows. ∎

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Since all column sums of $\mathbf{M}_\alpha(\infty)$ are equal, (1.9) becomes

$$F(\alpha) = \frac{1}{m^\alpha}\sum_{i=1}^{m-1} w_i^\alpha + \sum_{k=0}^{\infty}\frac{1}{m^{(k+2)\alpha}}\sum_{i=1}^{m-2}\sum_{J\in\mathcal{J}_0^k} c_{i,J}^\alpha \tag{6.8}$$

$$D = \{\alpha\in\mathbb{R} : F(\alpha)<\infty\},\qquad \tilde{\alpha} = \inf D.$$

Lemma 6.12 *Let $F(\alpha)$ and $\tilde{\alpha}$ be defined as in* (6.8). *Then* $\lim_{\alpha\to\tilde{\alpha}^+} F(\alpha)=\infty$ *and* $F(\tilde{\alpha})=\infty$.

Proof Define $d_0 := \min\{w_0, w_m\}$ and $d_i := \min\{w_i, w_{i+1}\}$ for $i = 1,\dots,m-2$. Then

$$F(\alpha) = \frac{1}{m^\alpha}\sum_{i=1}^{m-1} w_i^\alpha + \sum_{k=0}^{\infty}\frac{1}{m^{(k+2)\alpha}}\sum_{i=1}^{m-2} d_0^\alpha d_i^\alpha \sum_{J\in\mathcal{J}_0^k}\left([\widetilde{w}_{i+1},\widetilde{w}_i]P_J\begin{bmatrix}\widetilde{w}_0\\ \widetilde{w}_m\end{bmatrix}\right)^\alpha,$$

where $\widetilde{w}_i := w_i/d_i$, $\widetilde{w}_{i+1} =: w_{i+1}/d_i$, and $\widetilde{w}_0, \widetilde{w}_m$ are similarly defined. Let

$$s_{i,k}(\alpha) := \sum_{J\in\mathcal{J}_0^k}\left([\widetilde{w}_{i+1},\widetilde{w}_i]P_J\begin{bmatrix}\widetilde{w}_0\\ \widetilde{w}_m\end{bmatrix}\right)^\alpha.$$

Since $\widetilde{w}_i \geq 1$ for $i = 0,\dots,m$, it is straightforward to check that $\{s_{i,k}(\alpha)\}_k$ is submultiplicative. Moreover, it is shown in [LN3, Proposition A.1] that for $i = 1,\dots,m-2$,

$$\lim_{k\to\infty}\Bigl(\sum_{J\in\mathcal{J}_0^k} c_{i,J}^\alpha\Bigr)^{1/k}$$

is independent of i. It follows that $R(\alpha) := \lim_{k\to\infty}(s_{i,k}(\alpha))^{1/k}$ is also independent of i. The result now follows by using a similar argument as in the proof of Lemma 5.7. ∎

Before proving the final error estimate, we first prove the following.

Lemma 6.13 *Let $i = 1,\dots,m-1$. Then for each integer $n \geq 2$,*

$$\#\mathcal{P}_n = \#\mathcal{P}_n(i) = 2 + (m-1)2^{n-2}.$$

Proof Clearly, $\#\mathcal{P}_1 = \#\{T_i(0), T_i(m)\} = 2$. $\mathcal{P}_2$ contains the end-points of the intervals $T_iT_j[0,m]$, $j = 0,1,\dots,m-1$. Thus, $\#\mathcal{P}_2 = \#\mathcal{P}_1 + (m-1)$. $\mathcal{P}_3$ contains $\mathcal{P}_2$ together with all end-points of the intervals $T_{ii'j}[0,m]$, $j = 0,1,\dots,m-1$. Thus $\#\mathcal{P}_3 = \#\mathcal{P}_2 + (m-1)$. $\mathcal{P}_4$ contains $\mathcal{P}_3$ together with the end-points of the intervals $T_{ii'Jj}$, $J\in\mathcal{J}_0$, $j = 0,1,\dots,m-1$. Thus $\#\mathcal{P}_4 = \#\mathcal{P}_3 + 2(m-1)$. Inductively, for $n > 2$,

$$\#\mathcal{P}_n = \#\mathcal{P}_{n-1} + 2^{n-3}(m-1) = 2 + (m-1) + (m-1)\sum_{k=0}^{n-3} 2^k = 2 + (m-1)2^{n-2}. \quad ∎$$

Lemma 6.14 *Let α be the unique number satisfying $F(\alpha) = 1$. Then there exists $\sigma > 0$ such that $e^{-\alpha t}2^{n_t} = o(e^{-\sigma t})$ as $t \to \infty$. Consequently, for all $i \in \mathcal{I}_1$, $z_i^{(\alpha)}(t) = o(e^{-\sigma t})$ as $t \to \infty$.*

Proof We first get an upper bound for n_t in terms of t. Fix t. Since n_t is the smallest integer such that $t + \max\{\ln(c_{i,J}/m^{n_t}) : i = 1, \dots, m-2,\ J \in \mathcal{I}_0^{n_t-2}\} < t_o$, there exist $J_o \in \mathcal{I}_0^{n_t-3}$ and $1 \le i_o \le m-2$ such that

$$t + \ln\left(\frac{c_{i_o,J_o}}{m^{n_t-1}}\right) \ge t_o.$$

Thus, by Proposition 6.4, we have

$$m^{n_t} \le mc_{i_o,J_o}e^{t-t_o} \le m\left(\frac{2c_m}{4^m}\right)\left(\frac{m}{2^m}\right)^{n_t-3}e^{t-t_o}.$$

From this we get

$$n_t < \frac{K_m + t}{m\ln 2}, \tag{6.9}$$

where $K_m := \ln(2^{m+1}c_m/m^2) - t_o$ is a constant independent of t.

Next, it follows from the same argument as in Section 5 that

$$\alpha > \frac{\underline{\dim}_\infty(\mu)}{\underline{\dim}_\infty(\mu) + 1}. \tag{6.10}$$

By combining (6.9), (6.10), and Theorem 6.5, we have

$$\begin{aligned} e^{-\alpha t}2^{n_t} &\le \exp\left(\frac{-\underline{\dim}_\infty(\mu)t}{\underline{\dim}_\infty(\mu)+1}\right)\exp\left(\frac{K_m+t}{m}\right) \\ &= \exp\left(\frac{K_m}{m}\right)\exp\left(-\frac{(m-1)\ln(2^m/c_m) - \ln m}{m(\ln(2^m/c_m)+\ln m)}\,t\right). \end{aligned}$$

Thus, it suffices to show that $(m-1)\ln(2^m/c_m) - \ln m > 0$. It is clear that $2^m/c_m \ge 2$. Therefore we need only show that $(m-1)\ln 2 - \ln m > 0$ for all $m \ge 3$. But this is clear by using calculus. Thus the first part of the lemma holds. Using the definition of $z_i^{(\alpha)}(t)$ in (6.3), Lemma 6.11, and Lemma 6.13, we have $z_i^{(\alpha)}(t) \le Ce^{-\alpha t}2^{n_t}$ for some constant C. Hence the second part of the lemma also follows. ■

We can now complete the proofs of the remaining main results.

Proof of Theorem 1.3 Combine Proposition 6.2, Lemmas 6.12 and 6.14, and Theorem 1.1. ■

Proof of Corollary 1.4 Recall that $\dim_s(\mu)/2$ is the unique q coordinate of the intersection of the curve $y = \tau(q)$ and the line $y = -q$ in the (q, y)-plane. Since $\tau(q)$ is strictly concave with $\tau(0) = -1$ and $\tau(1) = 0$, we have $\dim_s(\mu)/2 < 1/2$, and the result follows. ■

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College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, China

and

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460-8093, U.S.A.
e-mail: smngai@georgiasouthern.edu