TWO-VARIABLE LAWS FOR $PSL(2, 5)$

R. M. BRYANT [1] and M. B. POWELL

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

In [2], John Cossey and Sheila Oates Macdonald give a basis for the set of laws of $PSL(2, 5)$ — the simple group of order 60 — and with one extreme exception the laws of their basis involve at most two variables. They raise the problem of finding a basis in which all of the laws involve only a small number of variables, and remark that they have shown that five variables will suffice. Here we give a basis consisting entirely of two-variable laws.

In notation and terminology we follow [2].

We use a set of two-variable laws including (1)—(6) of [2], and the fact proved there that this set will be a basis for the laws of $PSL(2, 5)$ so long as $PSL(2,5)$ is the only non-abelian simple group in the variety $\mathfrak{V}$ that the set defines. Consequently we prove

THEOREM. *The only non-abelian simple group in* $\mathfrak{V}$ *is* $PSL(2,5)$.

The method used to establish the structure of the simple group is based upon an idea introduced in [1]. If $\{w_i\}_{i=1}^{n}$ is a set of words of the free group F, we shall say that a group G satisfies the *disjunction* $w_1 \vee w_2 \vee \cdots \vee w_n$ if for every homomorphism $\phi : F \to G$, $w_i\phi = 1$ for at least one value of i. From a disjunction we may construct words which are laws of every group satisfying the disjunction, and, what is more important, which allow us to recover the disjunction in certain groups — for example non-abelian simple groups — having these words as laws.[2]

For example, in $PSL(2,5)$ every element has order 1, 2, 3 or 5: therefore $PSL(2, 5)$ satisfies the disjunctions

(7′) $x^3 \vee x^{10}$, and
(8′) $x^5 \vee x^6$,

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[2] A particular case is that of the *chief centralizer law* and its application in [4] of L. G. Kovács and M. F. Newman, which in fact suggested the more general idea.

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giving rise to the two-variable laws

(7) $[x^3, (x^{10})^y]$, and

(8) $[x^5, (x^6)^y]$.

Now if G is a non-abelian simple group, no non-identity element of G can commute with a complete set of conjugates of any other non-identity element. Thus if G satisfies (7) and (8), any element of G is of order dividing 3 or 10, and again of order dividing 5 or 6. Consequently we retrieve in G the fact that every element has order 1, 2, 3 or 5.

The other disjunctions we use are

(9′) $[x^6, y^6] \vee (xy)^3 \vee (xy^{-1})^3$, and

(10′) $[x^6, y^6] \vee (xy)^2 \vee (xy^2)^2 \vee (xy^3)^2 \vee (xy^4)^2$,

which are easily checked to be satisfied by $PSL(2, 5)$. Essentially they express the fact that two non-commuting elements of order 5 generate the whole group. From these disjunctions we construct the sets of laws under (9) and (10) below. The choice of the conjugating words u_1, u_2, u_3 and u_4 is determined by the need to restrict to two-variable laws, but as a consequence the recovery of the corresponding disjunctions is more difficult. (See Lemma 1.)

2. The basis

$\mathfrak{B}$ is the variety defined by the following set of two-variable laws.

(1) x^{30}

(2) $\{(x^{10}y^{10})^6[x^{10}, y^{10}]^2\}^5$

(3) $\{((x^6y^{12})^5(x^6y^{18})^5)^3[x^6, y^6]^6\}^6$

(4) $[x^3, y^3]^{15}$

(5) $\{[x^6y^{10}x^{-6}, y^{-10}][y^{10}, x^6]\}^{10}$

(6) $\{[y^{10}x^6y^{-10}, x^{-6}][y^{10}, x^6]^2\}^6$

(7) $[x^3, (x^{10})^y]$

(8) $[x^5, (x^6)^y]$

(9) The set of words

$[x^6, y^6, (xy)^{3u_1}, (xy^{-1})^{3u_2}]$

with u_1, u_2, taken from the set $\{1, x, y\}$.

(10) The set of words

$[x^6, y^6, (xy)^{2u_1}, (xy^2)^{2u_2}, (xy^3)^{2u_3}, (xy^4)^{2u_4}]$

with u_1, u_2, u_3, u_4 from $\{1, x, y\}$.

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3. Proof of the theorem

We have included as (1)—(6) the laws (1)—(6) of [2], and so we have all the results of § 3 of that paper. Thus

(i) *p-groups in $\mathfrak{V}$ are elementary abelian*, and

(ii) *an element of order p which belongs to the normalizer of a q-subgroup of a group in $\mathfrak{V}$ belongs to its centralizer if p and q take the values* 5 *and* 2, 5 *and* 3, *or* 3 *and* 5.

We let G be a non-abelian simple group in $\mathfrak{V}$. Then as remarked above, (7) and (8) give

(iii) *every element of G has order* 1, 2, 3 *or* 5.

(i) and (iii) put us in a familiar situation where we can deduce

(iv) *the Sylow (i.e. maximal) p-subgroups of G are the centralizers of each of their non-identity elements,*

(v) *each Sylow subgroup has trivial intersection with any distinct conjugate*, and

(vi) *two elements of a Sylow p-subgroup P of G are conjugate in G if and only if they are conjugate in the normalizer $N_G(P)$.*

If P is a non-trivial Sylow p-subgroup we see from (iv) that $N_G(P)/P$ acts fixed point freely on P by conjugation, and from (ii) that it is a q-group, where $q = 3$, 2 or 2 according as $p = 2$, 3 or 5. By a well-known theorem of Frobenius, any subgroup of $N_G(P)/P$ of order q^2 is cyclic: and using (i) we have

(vii) *$[N_G(P) : P]$ divides* 3, 2 *or* 2 *according as $p = 2$,* 3 *or* 5.

We are now able to make our deductions from (9) and (10).

LEMMA 1. *G satisfies* (9′) *and* (10′).

PROOF. Suppose otherwise that, say, (9′) is not satisfied for $x = a$ and $y = b$. Let $c_0 = [a^6, b^6]$, $c_1 = (ab)^3$ and $c_2 = (ab^{-1})^3$, so that c_0, c_1 and c_2 are all non-trivial. Let $H = \langle a, b\rangle$, the group generated by a and b.

From (9) we have the relations $[c_0, c_1^{u_1}, c_2^{u_2}] = 1$ for $u_i = 1$, a or b, $i = 1, 2$. Suppose that $[c_0, c_1^{u_1}] = c \neq 1$ for some choice of u_1, and let P be the centralizer of c in G. Since $c_2^{u_2} \in P$ for $u_2 = 1$, a or b, it follows by (iv) and (v) that $H \leqq N_G(P)$. Thus $c_0 = [a^6, b^6] = 1$ by (vii), which is a contradiction. Consequently $[c_0, c_1^{u_1}] = 1$ for $u_1 = 1$, a or b, and arguing in the same way again we deduce that $c_0 = 1$, a final contradiction.

In a similar way we can prove that (10′) is satisfied.

LEMMA 2. *Any two non-commuting elements of G of order* 5 *generate a subgroup isomorphic to $PSL(2, 5)$.*

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Proof. Suppose that $[a^6, b^6] \neq 1$. Then from (10′) we have $(ab^j)^2 = 1$ for $j = 1, 2, 3$ or 4, and putting $x = a$, $y = b^j$ in (9′) gives $(ab^{-j})^3 = 1$ and so $(b^{-j}a)^3 = 1$. Thus if $H = \langle a, b\rangle$, H is also generated by $c = ab^j$ and $d = b^{-j}a$ with relations $c^2 = d^3 = (cd)^5 = 1$. Since these are defining relations for $PSL(2, 5)$, H is isomorphic to a factor group of $PSL(2, 5)$, and so actually to $PSL(2, 5)$; which proves the lemma.

G contains elements of order 5, for otherwise it would have exponent 6. But a group of exponent 6 is soluble: its finitely generated subgroups are finite and consequently soluble of bounded derived length by the results of [3]. Since G is simple it contains elements of order 5 which do not commute and so

(viii) *G contains a subgroup H isomorphic to* $PSL(2, 5)$.

Let a and b be elements of order 5 which are in distinct Sylow 5-subgroups of G. Then $\langle a, b\rangle \cong PSL(2, 5)$ by Lemma 2 and so a is conjugate to b or to b^2. Thus G contains at most two conjugacy classes of elements of order 5. Applying (vi) and (vii) we obtain

(ix) *the Sylow 5-subgroups of G have order* 5.

If a and b are elements of G of order 2 which are in distinct Sylow 2-subgroups of G, then $\langle a, b\rangle$ cannot be a four-group by (iv). Thus by (iii) $\langle a, b\rangle$ must have twice odd order and a and b are conjugate. Consequently G contains only one conjugacy class of elements of order 2 and so by (vi) and (vii) a Sylow 2-subgroup of G has order not greater than 4. Therefore by (viii)

(x) *the Sylow 2-subgroups of G have order* 4.

Now let a be any element of G of order 5. Then there is some element b of H of order 5 such that $[a, b] \neq 1$, and so $K = \langle a, b\rangle \cong PSL(2, 5)$ by Lemma 2. $N_H\langle b\rangle = N_K\langle b\rangle = N_G\langle b\rangle$ by (vii) and (ix). By (x), H and K contain complete Sylow 2-subgroups of G and each of these intersects $N_H\langle b\rangle = N_K\langle b\rangle$. Thus H and K contain the same elements of order 2 and, since they are generated by these elements, we have $H = K$ and so $a \in H$. Thus H contains every element of G of order 5. Since G is generated by these elements we have $G = H$ and the proof of the theorem is complete.

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