AN IDENTITY FOR THE FIBONACCI AND LUCAS NUMBERS

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In this paper we prove an identity between sums of reciprocals of Fibonacci and Lucas numbers. The Fibonacci numbers are defined for all $n \geq 0$ by the recurrence relation $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, where $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n are defined for all $n \geq 0$ by the same recurrence relation, where $L_0 = 2$ and $L_1 = 1$. We prove the following identity.

THEOREM 1. *For the Fibonacci and Lucas numbers we have*

$$\sum_{n=1}^{\infty} \frac{1}{(F_{2n-1})^3} = \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} \left\{ \sum_{n=1}^{\infty} \frac{1}{(F_{2n-1})^2} - 5 \sum_{n=1}^{\infty} \frac{1}{(L_{2n})^2} \right\}.$$

The above theorem is an immediate corollary of the following result.

THEOREM 2. *For real α and β such that $\alpha\beta = -1$ and $-1 < \beta < 0$ we have*

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} - \beta^{2n-1})^3} = \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} - \beta^{2n-1})} \left\{ \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} - \beta^{2n-1})^2} + \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n} + \beta^{2n})^2} \right\}.$$

Theorem 1 is proved by noting that $F_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $L_n = \alpha^n + \beta^n$ where $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \frac{1}{2}(1 - \sqrt{5})$, so that α and β satisfy the conditions of Theorem 2. Before we prove Theorem 2 we require two elementary lemmas.

LEMMA 1. *For $|q| < 1$ we have*

$$\sum_{n=1}^{\infty} \frac{nq^n}{1 + q^n} = \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1 - q^{2n-1}}.$$

Proof. Logarithmically differentiate Euler's formula $\prod_{n=1}^{\infty} (1 + q^n)(1 - q^{2n-1}) = 1$.

LEMMA 2. *For $|q| < 1$ we have*

$$\sum_{n=1}^{\infty} \frac{q^n}{(1 + q^n)^2} = \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} - 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}}.$$

Proof.

$$\sum_{n=1}^{\infty} \frac{q^n}{(1 + q^n)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} m q^{mn} = \sum_{n=1}^{\infty} \{\sigma^o(n) - \sigma^e(n)\} q^n,$$

where

$$\sigma^o(n) = \sum_{d \mid n, d \text{ odd}} \mathrm{d} \quad \text{and} \quad \sigma^e(n) = \sum_{d \mid n, d \text{ even}} d.$$

Now $\sigma^o(n) - \sigma^e(n) = \sigma(n) - 4\sigma(n/2)$ where $\sigma(n) = \sum_{d \mid n} d$ and $\sigma(x) = 0$ for non-integral x. Therefore

$$\sum_{n=1}^{\infty} \frac{q^n}{(1 + q^n)^2} = \sum_{n=1}^{\infty} \left\{ \sigma(n) - 4\sigma\left(\frac{n}{2}\right) \right\} q^n = \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} - 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}}.$$

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To prove Theorem 2 we just note that it is an immediate corollary of the following result, with $q=\beta$, when α and β satisfy the conditions of Theorem 2.

THEOREM 3. *For* $|q|<1$ *we have*

$$\sum_{n=1}^{\infty}\frac{q^{6n-3}}{(1+q^{4n-2})^3}=\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\left\{\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1+q^{4n-2})^2}-\sum_{n=1}^{\infty}\frac{q^{4n}}{(1+q^{4n})^2}\right\}.$$

Proof. We use Jacobi's triple product identity [**1**], which states that for complex q and z such that $|q|<1$ and $z\neq 0$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1})=\sum_{n=-\infty}^{\infty}q^{n^2}z^n. \tag{1}$$

We now transform (1) by applying the following identities, which are effectively Chebyshev polynomials. For $n\geq 1$ and $z\neq 0$ we have

$$z^{2n}+\frac{1}{z^{2n}}=\sum_{j=0}^{n}(-1)^{n+j}\frac{2n}{n+j}\binom{n+j}{2j}(z+z^{-1})^{2j},$$

and for $n\geq 0$, $z\neq 0$ we have

$$z^{2n+1}+\frac{1}{z^{2n+1}}=\sum_{j=0}^{n}(-1)^{n+j}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(z+z^{-1})^{2j+1}.$$

From (1), with $x=z+z^{-1}$ and using the Chebshev polynomials to substitute for z^n+z^{-n}, then interchanging the order of summation we have

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^{2n})(1+xq^{2n-1}+q^{4n-2})&=1+\sum_{n=1}^{\infty}\left\{z^n+\frac{1}{z^n}\right\}q^{n^2}\\&=\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}(-1)^{n+j}\frac{2n}{n+j}\binom{n+j}{2j}x^{2j}q^{(2n)^2}+\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}(-1)^{n+j}\\&\quad\times\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}x^{2j+1}q^{(2n+1)^2},\end{aligned} \tag{2}$$

where $(2n)/(n+j)$ is taken to be 1 when $n=j=0$. Equating the coefficients of x^3 in (2) gives

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})&\left\{\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\right)^2-3\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\sum_{n=1}^{\infty}\right.\\&\left.\times\frac{q^{4n-2}}{(1+q^{4n-2})^2}+2\sum_{n=1}^{\infty}\frac{q^{6n-3}}{(1+q^{6n-3})^3}\right\}=\sum_{n=1}^{\infty}(-1)^{n+1}n(n+1)(2n+1)q^{(2n+1)^2}.\end{aligned} \tag{3}$$

To evaluate the term on the right hand side of (3) we logarithmically differentiate the following famous theorem of Jacobi's [**1**]. For $|q|<1$ we have

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{4}$$

Then let $q:=q^8$ and multiply through by q to give

$$\sum_{n=1}^{\infty}(-1)^{n+1}n(n+1)(2n+1)q^{(2n+1)^2}=6q\prod_{n=1}^{\infty}(1-q^{8n})^3\sum_{n=1}^{\infty}\frac{nq^{8n}}{1-q^{8n}}. \tag{5}$$

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We need two further results, (6) and (7), which are obtained by equating the coefficients of x and x^2 respectively in (2). For the last equality in (6), we again use (4).

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}=q\prod_{n=1}^{\infty}(1-q^{8n})^3. \quad (6)$$

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})\frac{1}{2}\left\{\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\right)^2-\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1+q^{4n-2})^2}\right\}=\sum_{n=1}^{\infty}(-1)^{n+1}n^2q^{4n^2}. \quad (7)$$

Now let $z=-1$ in (1) and logarithmically differentiate. Then multiply through by q to give

$$\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n-1})\left\{\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}+\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}\right\}=2\sum_{n=1}^{\infty}(-1)^{n+1}n^2q^{n^2}. \quad (8)$$

Then substitute for $\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}$ in (8), using Lemma 1, and let $q:=q^4$ to give

$$\prod_{n=1}^{\infty}(1-q^{4n})(1-q^{8n-4})\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{8n}}=\sum_{n=1}^{\infty}(-1)^{n+1}n^2q^{4n^2}. \quad (9)$$

From Lemma 2 with $q:=q^4$ we have,

$$\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{8n}}-3\sum_{n=1}^{\infty}\frac{nq^{8n}}{1-q^{8n}}=\sum_{n=1}^{\infty}\frac{q^{4n}}{(1+q^{4n})^2}. \quad (10)$$

With the help of Euler's identity, $\prod_{n=1}^{\infty}(1+q^n)(1-q^{2n-1})=1$, we can combine (7), (9) and (10) to give,

$$6\sum_{n=1}^{\infty}\frac{nq^{8n}}{1-q^{8n}}=\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\right)^2-\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1+q^{4n-2})^2}-2\sum_{n=1}^{\infty}\frac{q^{4n}}{(1+q^{4n})^2}. \quad (11)$$

Multiply (3) by $\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}$. Substitute for $\sum_{n=1}^{\infty}(-1)^{n+1}n(n+1)(2n+1)q^{(2n+1)^2}$ from (5) and $\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}$ from (6). The term $q\prod_{n=1}^{\infty}(1-q^{8n})^3$ cancels. We then use (11) to substitute for $6\sum_{n=1}^{\infty}\frac{nq^{8n}}{1-q^{8n}}$ in our new expression, and after the term in $\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\right)^3$ conveniently cancels out we are left with Theorem 3. So this completes the proof of Theorem 3 and hence that of Theorems 1 and 2.

Conclusion. It was first shown by Landau [2] that

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}=\frac{\sqrt{5}}{4}\theta_2^2\left(\frac{3-\sqrt{5}}{2}\right), \quad (12)$$

where for $|q| < 1$ we have,

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}.$$

Also it is known that

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^3} = \frac{5\sqrt{5}}{32}\,\theta_2^2\left(\frac{3-\sqrt{5}}{2}\right)\left\{1-\theta_4^4\left(\frac{3-\sqrt{5}}{2}\right)\right\}, \tag{13}$$

where for $|q| < 1$ we have,

$$\theta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

There are many other expressions for sums of reciprocals of Fibonacci and Lucas numbers in terms of the theta functions. See [**3**] for more examples. Also there are other known polynomial identities between these sums. For example

$$\sum_{n=1}^{\infty} \frac{1}{L_n^2} = 2\left(\sum_{n=1}^{\infty} \frac{1}{L_{2n}}\right)^2 + \sum_{n=1}^{\infty} \frac{1}{L_{2n}},$$

and

$$4\left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2 = 5\sum_{n=1}^{\infty} \frac{1}{L_n^2} + 3\sum_{n=1}^{\infty} \frac{1}{F_n^2}.$$

Of course we could prove Theorem 1 by showing that,

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} - 5\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} = \frac{5}{8}\left\{1-\theta_4^4\left(\frac{3-\sqrt{5}}{2}\right)\right\},$$

and then use results (12) and (13). However, such a proof would be less direct than the proof given.

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