A RESULT FOR SEMI-REGULAR CONTINUED FRACTIONS

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1. Introduction

If φ is a real number with $|\varphi| > 1$, then a semiregular continued fraction development of φ is denoted by

$$\varphi = [a_1, a_2, a_3, \cdots] = a_1 - \frac{1}{a_2 -} \frac{1}{a_3 -} \cdots,$$

where the a_i are integers such that $|a_i| \geqq 2$. The expansions arise geometrically by considering the sequence of divided cells of two-dimensional grids (see [1]), and are described by the following algorithm:

$$\varphi_n = a_{n+1} - \frac{1}{\varphi_{n+1}}, \quad |a_{n+1}| \geqq 2, \quad |\varphi_{n+1}| > 1,$$

for all $n \geqq 0$, taking $\varphi = \varphi_0$. Hence

$$a_{n+1} = [\varphi_n] \quad \text{or} \quad [\varphi_n + 1],$$

where in this case the square brackets are used to signify the integer-part function. It follows that each irrational φ has uncountably many such expansions, none of which has a_n constantly equal to 2 (or -2) for large n.

Hurwitz [4] has investigated one particular expansion from this uncountable set, namely the one for which

$$(1) \qquad -\tfrac{1}{2} < \frac{1}{\varphi_n} \leqq \tfrac{1}{2},$$

for all $n \geqq 0$. This expansion is a development 'to the nearest integer' in an obvious sense. Hurwitz proved that for any such expansion, if we put

$$\theta_n = [a_n, a_{n-1}, \cdots, a_1],$$

then

$$(2) \qquad \begin{aligned} \frac{1-\sqrt{5}}{2} < \frac{1}{\theta_n} < \frac{3-\sqrt{5}}{2} \quad &\text{when} \quad \varphi_n > 0, \\ \frac{\sqrt{5}-3}{2} < \frac{1}{\theta_n} < \frac{\sqrt{5}-1}{2} \quad &\text{when} \quad \varphi_n < 0. \end{aligned}$$

DEFINITION. *If $k \geqq 1$ is any real number, we will denote by $E(\varphi, 1/k)$ that unique semi-regular expansion of $\varphi \geqq k$, for which*

$$\frac{1-k}{k} < \frac{1}{\varphi_n} \leqq \frac{1}{k}, \qquad (n \geqq 0) \tag{3}$$

and by $E(1/k)$ the set

$$E\left(\frac{1}{k}\right) = \left\{E\left(\varphi, \frac{1}{k}\right) \middle| \varphi \geqq k, \varphi \text{ real}\right\}.$$

We can define a pair of minimal intervals $I^+(1/k)$ and $I^-(1/k)$, which are subintervals of $[-1, 1]$, and such that for all expansions of $E(1/k)$ we have

$$\frac{1}{\theta_n} \in \begin{cases} I^+\left(\dfrac{1}{k}\right) & \text{when} \quad \varphi_n > 0 \\[2ex] I^-\left(\dfrac{1}{k}\right) & \text{when} \quad \varphi_n < 0. \end{cases}$$

Then Hurwitz' result reduces to

$$I^+(\tfrac{1}{2}) = \left(\frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right)$$

$$I^-(\tfrac{1}{2}) = \left(\frac{\sqrt{5}-3}{2}, \frac{\sqrt{5}-1}{2}\right).$$

Davenport [3] and others have utilised this result to obtain results on the inhomogeneous minima of indefinite binary quadratic forms. In this case the $I^{\pm}(\frac{1}{2})$ are both of unit length, and so a different type of reduction for quadratic forms may be defined, by relaxing each of the open intervals to a half-open interval, and representing each reduced form by a doubly infinite chain of integers. This gives rise to the notion of a semi-regular expansion of the 'second kind' (see [4]).

It seemed to be of interest to investigate the intervals $I^{\pm}(1/k)$ for other values of $k \geqq 1$. If we put

$$L\left(\frac{1}{k}\right) = \max\left\{\text{length of } I^{\pm}\left(\frac{1}{k}\right)\right\},$$

then we may pose the following question. For what values of k is $L(1/k) = 1$, and hence after fixing a suitable convention, define a unique semi-regular expansion of the second kind?

In section **3** we will establish the following results. Define the sequences $\{\alpha_r\}$, $\{\beta_r\}$, $\{\delta_r\}$ by

$$\alpha_r = \frac{r}{r+1}, \quad \beta_r = \frac{3r-\sqrt{(r^2+4r)}}{2r}, \quad \delta_r = \frac{\sqrt{(r^2+4r)}-r}{2}.$$

THEOREM 1. *For all* $r \geqq 1$,

$$I^{+}(\alpha_r) = (\beta_r - 1, \beta_r)$$
$$I^{-}(\alpha_r) = (\delta_r - 1, \delta_r)$$
$$I^{\pm}(\beta_r) = (\beta_r - 1, \beta_r)$$
$$I^{\pm}(\delta_r) = (\delta_r - 1, \delta_r).$$

Similar results hold for $I^{\pm}(1-\alpha_r)$, $I^{\pm}(1-\beta_r)$, $I^{\pm}(1-\delta_r)$, and these are readily deduced from the theorem.

In section 4 the intervals $I^{\pm}(1/k)$ will be discussed for more general $k \geqq 1$. It will be shown that there are uncountably many 'bad' k, that is k for which $L(1/k) > 1$. The whole paper could have been presented in the context of binary quadratic forms, the proofs needing only minor modification. I would like to thank the referee for his helpful comments and suggestions on this paper, and to acknowledge the financial support of an 1851 Overseas Scholarship.

2. Preliminary results

In this section we will quote a few results for semi-regular continued fractions which we will need in what follows. If p_n/q_n is the n^{th} convergent for φ, that is

$$p_n/q_n = [a_1, a_2, \cdots, a_n],$$

then (see [1])

$$p_0 = 1,\ q_0 = 0,\ p_1 = a_1,\ q_1 = 1,$$
$$p_{n+1} = a_{n+1}p_n - p_{n-1},$$
$$q_{n+1} = a_{n+1}q_n - q_{n-1},$$

and

$$p_{n-1}q_n - q_{n-1}p_n = 1 \qquad (n \geqq 1).$$

It follows that

$$\varphi_0 = [a_1, a_2, \cdots, a_n, \varphi_n] = \frac{\varphi_n p_n - p_{n-1}}{\varphi_n q_n - q_{n-1}}. \tag{4}$$

LEMMA 1. *If* $\varphi = [a_1, a_2, \cdots]$ *where* $a_j > 0$ *for all* $j \geqq 1$, *then whenever* $a_1, a_2, \cdots, a_{n-1}$ *remain constant and* a_n *is increased, the value of* φ *is increased, whatever the values of* $a_{n+1}, a_{n+2}, \cdots$. *It follows that if* φ_n *and* λ_n *are also both positive, then*

$$[a_1, a_2, \cdots, a_n, \varphi_n] < [a_1, a_2, \cdots, a_n, -\lambda_n].$$

The proof may be found in [1].

LEMMA 2. *We have, if* $a_1 > 0$,

$$\varphi < [a_1, a_2, \cdots, a_n, \varphi]$$

if and only if

$$\varphi < [\overline{a_1, \cdots, a_n}],$$

where, as usual, the upper bar is used to denote a periodic expansion.

This result follows from (4).

The repetition of a finite sequence of partial quotients will be denoted by a suitable subscript, and we will incorporate the convention that a zero subscript means that the corresponding segment be deleted from the chain. We will use round external brackets to denote an ordinary continued fraction development.

LEMMA 3.

$$[2_r, \varphi'] = \frac{(r+1)\varphi' - r}{r\varphi' - (r-1)}, \quad \text{for} \quad r \geqq 0.$$

If in ordinary continued fractions

$$\varphi = (a, r+1, \varphi'), \ a \geqq 0, \ \varphi' > 1,$$

then in semi-regular continued fractions

$$\varphi + 1 = [a+2, 2_r, \varphi'+1].$$

The proof is straight forward and may be found in [2]. This relationship enables any expansion from the set $E(1)$ to be transformed into the corresponding ordinary continued fraction expansion. Using the above conventions, and inserting an appropriate 2_0 into the semi-regular expansion, if necessary, we have for $a_i > 0$,

$$\text{(5)} \qquad \begin{aligned} \varphi &= (a_1, a_2, a_3, a_4, \cdots) \text{ if and only if} \\ \varphi + 1 &= [a_1+2, 2_{a_2-1}, a_3+2, 2_{a_4-1}, \cdots]. \end{aligned}$$

It is clear that for $k > 1$, $E(1/k)$ contains no expansion with a partial quotient equal to ± 1, for if say

$$\varphi_n = 1 - \frac{1}{\varphi_{n+1}}, \quad \text{with} \quad \frac{1-k}{k} < \frac{1}{\varphi_{n+1}} < 0,$$

then $\varphi_n \geqq k$ implies that

$$k \leqq \varphi_n < 1 - \frac{1-k}{k} = 2 - \frac{1}{k},$$

a contradiction.

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3. Proof of Theorem 1

We first note that after using (4)

$$(6)\qquad \begin{cases} 1/\alpha_r = [2_r], \\ 1/\beta_r = [\overline{2_{r-1}, 3}], \\ 1/\delta_r = [2, \overline{2_{r-1}, 3}]. \end{cases}$$

It then follows from Lemma 2 that

$$(7)\qquad \cdots < \beta_{r-1} < \alpha_{r-1} < \delta_{r-1} < \beta_r < \alpha_r < \delta_r < \cdots.$$

We also have

$$(8)\qquad \begin{cases} 1/(1-\alpha_r) = r+1, \\ 1/(1-\beta_r) = [r+1, \overline{r+2}], \\ 1/(1-\delta_r) = [\overline{r+2}]. \end{cases}$$

These expansions may be obtained by an application of the transformation (5), as the following example demonstrates.

$$\frac{1}{\beta_r} = [2, 2_{r-2}, \overline{3, 2_{r-1}}],$$

hence by (5)

$$\frac{1}{\beta_r} - 1 = (0, r-1, \overline{1, r}),$$

and

$$\frac{\beta_r}{1-\beta_r} = (r-1, \overline{1, r}).$$

Thus, by (5) again, we obtain

$$\frac{\beta_r}{1-\beta_r} + 1 = \frac{1}{1-\beta_r} = [r+1, \overline{2_0, r+2}] = [r+1, \overline{r+2}].$$

Consider the set of expansions $E(\alpha_r)$. We cannot have for any n,

$$\varphi_n = [2_r, \varphi_{n+r}], \quad \text{where} \quad \varphi_{n+r} > 0,$$

else

$$\varphi_n = \frac{(r+1)\varphi_{n+r} - r}{r\varphi_{n+r} - (r-1)} < \frac{r+1}{r},$$

which contradicts that the expansion belongs to $E(\alpha_r)$. However, if

$$\varphi_n = [2_{r-1}, \varphi_{n+r-1}],$$

where $\varphi_{n+r-1} > 2$, then $\varphi_n > (r+1)/r$, as required, Hence, whenever $a_{m+1} > 0$, $a_m > 0$, we have by (6) and Lemma 1,

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$$\theta_m > [\overline{2_{r-1}, 3}] = 1/\beta_r.$$

Now if for some n

$$\varphi_n = [2_{r+1}, \varphi_{n+r+1}],$$

where $\varphi_{n+r+1} < -2$, then $\varphi_n < (r+1)/r$, but if

$$\varphi_n = [2_r, \varphi_{n+r}],$$

where $\varphi_{n+r} < 0$, then $\varphi_n > (r+1)/r$. Hence it follows from (6) and Lemma 1, that whenever $a_{m+1} < 0$, $a_m > 0$, that

$$\theta_m > [2, \overline{2_{r-1}, 3}] = 1/\delta_r.$$

If for some expansion from the set $E(\alpha_r)$ we have

$$\varphi_n = [-r-1, \varphi_{n+1}],$$

then we conclude that $\varphi_{n+1} > 0$, and clearly $r+1$ is the smallest absolute value permissible for negative partial quotients. Consequently, by (8) and Lemma 1, we have for all m for which $\varphi_m > 0$, and $a_m < 0$,

$$|\theta_m| > [r+1, \overline{r+2}] = 1/(1-\beta_r),$$

and for all m for which $\varphi_m < 0$, and $a_m < 0$,

$$|\theta_m| > [\overline{r+2}] = 1/(1-\delta_r).$$

Thus we have shown that

$$I^+(\alpha_r) = (\beta_r-1, \beta_r),$$
$$I^-(\alpha_r) = (\delta_r-1, \delta_r).$$

Let us now consider the expansions from the set $E(\beta_r)$. It is again clear that we cannot have for any n

$$\varphi_n = [2_r, \varphi_{n+r}], \quad \text{where} \quad \varphi_{n+r} > 0,$$

but, by Lemma 1, we may have

$$\varphi_n = [2_{r-1}, \varphi_{n+r-1}],$$

provided $\varphi_{n+r-1} \geqq [3, 1/\beta_r]$. In the case when

$$\varphi_n = [2_r, \varphi_{n+r}], \quad \text{where} \quad \varphi_{n+r} < 0,$$

then by hypothesis

$$1/|\varphi_{n+r}| < 1-\beta_r,$$

and so by (6) and Lemma 1,

$$\varphi_n = \left[2_{r-1}, 2+\frac{1}{|\varphi_{n+r}|}\right] < [2_{r-1}, 3-\beta_r] = 1/\beta_r.$$

It therefore follows that whatever the sign of φ_m, we have for $a_m > 0$,

$$\theta_m > [\overline{2_{r-1}, 3}] = 1/\beta_r.$$

Now suppose that for some n we have

$$\varphi_n = [-r, \varphi_{n+1}],$$

then clearly $\varphi_{n+1} > 0$, and

$$\begin{aligned} |\varphi_n| &= r + \frac{1}{|\varphi_{n+1}|} \\ &\leqq r + \beta_r \\ &= [r+1, (1-\beta_r)^{-1}] \\ &< 1/(1-\beta_r), \end{aligned}$$

if and only if $1/(1-\beta_r) > [\overline{r+1}]$, by Lemma 2, and by (8) this condition is clearly satisfied, leading to a contradiction. Now

$$\varphi_n = [-r-1, \varphi_{n+1}]$$

is a permissible expansion, whatever the sign of φ_{n+1}, provided it is large enough. However we cannot have

$$\varphi_n = [-r-1, -r-1, \varphi_{n+2}],$$

when $\varphi_{n+2} < 0$, since this implies

$$|\varphi_n| < 1/(1-\beta_r).$$

If $\varphi_{n+2} > 0$, then $\varphi_{n+2} \geqq 1/\beta_r$, and

$$\begin{aligned} |\varphi_n| &= \left[r+1, r+1+\frac{1}{\varphi_{n+2}} \right] \\ &\leqq [r+1, r+2, (1-\beta_r)^{-1}] \\ &< 1/(1-\beta_r), \end{aligned}$$

by (8) and Lemma 2, since $1/(1-\beta_r) > [\overline{r+1, r+2}]$.

Hence it follows readily that if $a_m < 0$,

$$|\theta_m| > [r+1, \overline{r+2}] = 1/1-\beta_r.$$

Consequently

$$I^{\pm}(\beta_r) = (\beta_r - 1, \beta_r).$$

The result for $I^{\pm}(\delta_r)$ follows analogously.

On examining the proofs for the preceding results, we notice that they remain valid if we replace (3) by

$$\frac{1-k}{k} \leqq \frac{1}{\varphi_n} < \frac{1}{k} \qquad (n \geqq 0),$$

the only modification required being the occasional interchange of $<$ and $\leqq$. Denote this set of expansions by $E'(1/k)$. Then the set obtained by negating all the expansions of $E'(1/k)$ is just the set $E(1-1/k)$. It follows from Theorem 1 that

$$\begin{aligned} I^+(1-\alpha_r) &= (-\delta_r,\ 1-\delta_r) \\ I^-(1-\alpha_r) &= (-\beta_r,\ 1-\beta_r) \\ I^\pm(1-\beta_r) &= (-\beta_r,\ 1-\beta_r) \\ I^\pm(1-\delta_r) &= (-\delta_r,\ 1-\delta_r). \end{aligned} \tag{9}$$

4. Further results

In this section we prove the following theorem.

THEOREM 2. *For any k satisfying either*

$$\alpha_{r-1} < \frac{1}{k} < \delta_{r-1}, \quad \textit{or} \quad \beta_r < \frac{1}{k} < \alpha_r,$$

we have

$$L\left(\frac{1}{k}\right) > 1.$$

PROOF. In the first case (6) and Lemma 1 imply that

$$k = [2_{r-1}, a, \cdots], \qquad \text{where} \qquad a \geqq 3;$$

hence

$$\varphi_n = [2_{r-1}, a_{n+r}, \cdots], \quad \text{where} \quad a_{n+r} > 0$$

satisfies $\varphi_n > k$ provided a_{n+r} is large enough. Now put

$$\varphi = r + \frac{1}{k} = \left[r+1, \frac{k}{k-1}\right],$$

then by Lemma 2, $\varphi > k/(k-1)$ provided

$$\frac{k}{k-1} < [\overline{r+1}] = 1/(1-\delta_{r-1}),$$

which is satisfied, by the hypothesis.

Now let

$$\gamma = \varphi - \frac{k}{k-1} > 0.$$

If we have for some $k^* > k$, $\varphi_n = [-r, k^*]$, then

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$$|\varphi_n| = r + \frac{1}{k^*} = \varphi + \frac{1}{k^*} - \frac{1}{k}$$
$$= \frac{k}{k-1} + \gamma + \frac{1}{k^*} - \frac{1}{k}$$
$$> \frac{k}{k-1},$$

provided k^* is sufficiently close to k. Choose k^* to be rational (and if k is rational, take $k^* = k$), and close enough to k. Expanding k^* by (3), we obtain the finite expansion $E(k^*, 1/k)$, say,

$$k^* = [2_{r-1}, b_1, \cdots, b_n], \quad \text{where} \quad b_1 > 0.$$

Consider the chain

$$[a_1, \cdots, a_t, -r, 2_{r-1}, b_1, \cdots, b_n, b, c, 2_{r-1}, b, \cdots],$$

with $a_t < 0$ and $c > 0$. It is clear that whether k is rational or irrational, $[b_j, \cdots, b_n]$ cannot equal either k or $k/(1-k)$ for any j $(1 \leqq j \leqq n)$, and so, provided b is sufficiently large and of the correct sign, the chain belongs to the set of expansions $E(1/k)$, for all large $a_1, \cdots, a_{t-1}$. Hence we have for the corresponding subscripts p and m,

$$\text{(10)} \qquad \begin{aligned} |\theta_p| &= [r, |a_t|, \cdots, -a_1] < r, \ \varphi_p > 0, \\ \theta_m &= [2_{r-1}, c, \cdots, a_1] < [2_{r-1}] = r/(r-1), \ \varphi_m > 0. \end{aligned}$$

Thus the length of $I^+(1/k)$ exceeds unity. Since the $a_1, \cdots, a_{t-1}$, are arbitrary, then there are chains in which there are infinitely many pairs p, m, for which (10) holds.

In the second case, it may be checked readily by methods similar to the above that

$$k = [2_r, x], \quad \text{where} \quad x < 0.$$

It can be verified that the following expansion belongs to the set $E(1/k)$, provided that b is large enough, and the $a_1, \cdots, a_p$ are chosen judiciously.

$$[a_1, \cdots, a_p, -r-1, b, c, 2_r, x],$$

where $a_p < 0$, $b < 0$, and $c > 0$. Thus for some m and t we have

$$|\theta_m| = [r+1, |a_p|, \cdots, -a_1] < r+1, \qquad \varphi_m < 0,$$
$$\theta_t = [2_r, c, \cdots, a_1] < [2_r] = (r+1)/r, \qquad \varphi_t < 0.$$

Hence the length of $I^-(1/k)$ exceeds unity.

By methods similar to those used to obtain (9), we may prove analogous results for sequences of intervals in $[0, \frac{1}{2}]$.

The question of the values taken by $L(1/k)$ when $\delta_{r-1} < 1/k < \beta_r$ has a more complicated answer, and we will not investigate it here. There are sequences of values of k in this range for which $L(1/k) = 1$. For example

$$k_{r,m} = [2_{r-1}, 3, \overline{(2_{r-1}, 3)_m, 2_{r-2}, 3}]$$
$$k_{r,m,p} = [(2_{r-1}, 3)_m, \overline{2_{r-1}, 3, (2_{r-2}, 3, (2_{r-1}, 3)_m)_p}].$$

There are also sequences of intervals for which $L(1/k) > 1$. It seems reasonable to conjecture that the set of k for which $L(1/k) = 1$, has measure zero.

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