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The Erdős-Hajnal problem list

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Dedicated to Pál Erdős (1913–1996) and András Hajnal (1931–2016)

Abstract

We give an update on the problem list of Erdős and Hajnal.

In 1967, Pál Erdős and András Hajnal wrote up a paper consisting the most interesting open problems emerging from their research, 82 in all ([51]). This was intended for the UCLA set theory conference that year but the authors sent mimeographed copies to practically everybody working in the field. This had an immense effect, several people started to work on the problems, using various methods of set theory. Already the original paper (which appeared in 1971) contained extensive comments, but seeing the tremendous progress, the authors decided to write up a second paper on the problem list, which eventually came out in 1974 ([52]). In this second installment some new problems were also added.

Detailed descriptions of Erdős's (and consequently Hajnal's) set theory work are in Hajnal's [90] and Kanamori's [110]. The former also tells the story of the problem list.

In this paper I try to survey these problems with an effort to describe the progress on them in the last 50 years.

Notation. Definitions. We use the notions and definitions of axiomatic set theory. In particular, each ordinal is a von Neumann ordinal, each cardinal is identified with the least ordinal of that cardinality. $\mathrm{cf}(\kappa)$ is the cofinality of κ. If κ is an infinite cardinal, then κ^+ is its successor cardinal. If A, B are subsets of the same ordered set, then $A < B$ denotes that $x < y$ holds for any $x \in A$, $y \in B$. If A or B is a singleton, we write $a < B$ instead of

$\{a\} < B$, etc. If S is a set, κ a cardinal, then $[S]^\kappa = \{x \subseteq S : |x| = \kappa\}$, $[S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}$, $[S]^{\leq\kappa} = \{x \subseteq S : |x| \leq \kappa\}$.

$\mathfrak{c} = 2^{\aleph_0}$.

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1. Ordinary partition relations for cardinals

If λ, κ_ν $(\nu < \gamma)$ are cardinals, $1 \leq r < \omega$, then $\lambda \to (\kappa_\nu)^r_\gamma$ denotes the following statement: if $F : [\lambda]^r \to \gamma$, then there are $\nu < \gamma$, $A \in [\lambda]^{\kappa_\nu}$ such that F is homogeneously of color ν on $[A]^r$. If this fails (i.e., there is F for which it is not true), we cross the arrow: $\lambda \not\to (\kappa_\nu)^r_\gamma$. Erdős, Hajnal, and Rado worked toward a full disscussion of this relation. Problems 1–5 cover some of the unsolved cases.

Problem 1. (GCH) $\lambda \not\to (\kappa_n : n < \omega)^3$ *where* $\lambda = \kappa_0 = \aleph_{\omega_{\omega+1}+1}$, $\kappa_m = 4$ $(n > 0)$.

Problem 2. (Erdős, Hajnal, Rado) *Assume* $2 \leq r < \omega$, λ *is an infinite cardinal,* $\gamma \leq 2$, $\kappa_\nu > r$ *are cardinals* $(\nu < \gamma)$, *and* $\lambda \not\to (\kappa_\nu)^r_\gamma$ *holds. Does then* $2^\lambda \not\to (1 + \kappa_\nu)^{r+1}_\gamma$ *hold?*

This is the negative stepping-up lemma, cf section 24 in [55]. The curious piece of notation, $1+\kappa_\nu$, denotes that this is cardinal addition, i.e., $1+4 = 5$ and $1 + \aleph_0 = \aleph_0$. The reference just mentioned reports on the following special cases already proved:

(a) All κ_ν are finite.
(b) κ_0, κ_1 are infinite, κ_0 regular.
(c) $r \geq 3$, κ_0 is infinite, regular.
(d) $r \geq 3$, κ_0, κ_1 are infinite.
(e) $r \geq 4$, κ_0 is infinite.

Problem 3. (Erdős, Hajnal, Rado) *If* $\aleph_\omega < 2^{\aleph_{n_0}} < 2^{\aleph_{n_1}} < \cdots$ $(n_i < \omega)$, $\lambda = 2^{\aleph_{n_0}} + 2^{\aleph_{n_1}} + \cdots$, *then* $\lambda \to (\aleph_\omega)^2_2$.

This was proved by Shelah in [167]. Essentially this was the last remaining case of the discussion of the relation $\lambda \to (\kappa)^2_2$.

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Hajnal conjectured that under the same assumption the stronger $\lambda \to (\aleph_\omega, 4)^3$ holds. This conjecture has so far been unproven.

Problem 4. (Erdős, Hajnal, Rado) *Assume that* λ *is singular,* λ *and* $\mathrm{cf}(\lambda)$ *are both* $\aleph_0$*-inaccessible. Does then*

$$\lambda \to (\lambda, \aleph_1)^2$$

hold?

Here a cardinal κ is $\aleph_0$-inaccessible, if for $\mu < \kappa$ we have $\mu^{\aleph_0} < \kappa$. The conditions are necessary, as $\tau^{\aleph_0} \not\to (\tau^+, \aleph_1)^2$ and if $\mathrm{cf}(\kappa) \not\to (\mathrm{cf}(\kappa), \aleph_1)^2$, then $\kappa \not\to (\kappa, \aleph_1)^2$.

The simplest case is $\lambda = \aleph_{\mathfrak{c}^+}$. In this case the second condition implies the first one by a remarkable theorem of Shelah: if $\mu < \aleph_{\mathfrak{c}^+}$, then $\mu^{\aleph_0} < \aleph_{\mathfrak{c}^+}$, ([174], see also in [2]). An Erdős–Rado theorem (Theorem 35.4 of [55]) states that the positive relation holds if $\aleph_{\mathfrak{c}^+}$ is strong limit.

Shelah and Stanley in [186] constructed a historic forcing that preserves the value of 2^τ up to some predetermined cardinal $\mu < \aleph_{\mathfrak{c}^+}$ and adds a counterexample to $\aleph_{\mathfrak{c}^+} \to (\aleph_{\mathfrak{c}^+}, \aleph_1)^2$. In [187] they deduced from the consistency of the existence of $\mathfrak{c}^+$ measurable cardinals the consistency of $\aleph_{\mathfrak{c}^+}$ is not strong limit yet $\aleph_{\mathfrak{c}^+} \to (\aleph_{\mathfrak{c}^+}, \aleph_1)^2$ holds.

Problem 5. (Erdős, Hajnal, Rado) *Can one prove without assuming the GCH that*

$$\aleph_{\omega+1} \not\to (\aleph_{\omega+1}, (3)_{\aleph_0})^2$$

holds?

The proof that $2^\lambda = \lambda^+$ implies $\lambda^+ \not\to (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ for λ singular is in Section 20 of [55]. Notice that $2^{\mathrm{cf}(\lambda)} \geq \lambda^+$ also implies the negative relation as $2^\kappa \not\to (3)^2_\kappa$ holds in general.

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2. Ordinary partition relations for order types

Ordinary partition relations for countable ordinals are of the form $\alpha \to (\beta, n)^2$ with $\omega < \alpha, \beta < \omega_1$, $n < \omega$. The reason why only partitions of this

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form are considered is given by the following easy results: $\theta \not\to (\omega+1,4)^3$ for any countable order type θ and $\alpha \not\to (\omega+1,\omega)^2$ $(\alpha < \omega_1)$.

It is equally easy to see, that if $\alpha \to (\alpha,3)^2$ then α is AI (additively indecomposable, i.e., a power of ω).

Specker proved in [189] that

$$\begin{aligned} \omega^2 &\to (\omega^2,k)^2 \quad (k<\omega), \\ \omega^n &\not\to (\omega^n,3)^2 \quad (3 \le n < \omega). \end{aligned}$$

E. C. Milner proved ([141])

$$\begin{aligned} \omega^{\alpha 3} &\not\to (\omega^{\alpha 2+1},3)^2 \quad (\alpha<\omega_1) \\ \omega^4 &\to (\omega^3,3)^2 \\ \omega^3 &\to (\omega^2 l,k)^2 \quad (l,k<\omega). \end{aligned}$$

Erdős proved

$$\omega^{\alpha 2+1} \to (\omega^{\alpha+1},4)^2 \quad (\alpha<\omega_1).$$

The usual technique of proving these theorems is based on a coding of an arbitrary coloring of $[\omega^n]^2$ into a coloring of ${}^m\omega$, for some $m>n$.

Independently Galvin, Hajnal, Haddad and Sabbagh proved that if $F : [{}^k\omega]^2 \to l$, then there is an $A \in [\omega]^\omega$ such that $F|[{}^kA]^2$ is canonical, i.e., $F(\langle n_0,\ldots,n_{k-1}\rangle,\langle m_0,\ldots,m_{k-1}\rangle)$ only depends on the truth values of $n_i = m_j$, $n_i < m_j$ $(i,j<k)$ for $n_i, m_i \in A$ $(i<k)$ ([82]).

This suffices to show that for every $k<\omega$ there is $n<\omega$ such that $\omega^k \not\to (\omega^3,n)^2$ but is not strong enough to calculate the least such n.

Problem 6. $\omega^5 \to (\omega^3,5)^2$?

This was the simplest unsolved case at the moment of writing [51] but in reality the problem asked for a full characterization of $\omega^n \to (\omega^k,l)^2$.

Haddad and Sabbagh ([82]), Hajnal, Galvin, E. C. Milner (unpublished)proved that

$$\omega^4 \to (\omega^3,4)^2$$

but

$$\omega^4 \not\to (\omega^3,5)^2.$$

Eva Nosal in [152] proved $\omega^n \to (\omega^3, 2^{n-2})^2$ and

$$\omega^n \not\to (\omega^3, 2^{n-2}+1)^2 \quad (4 \le n < \omega).$$

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Another result of Eva Nosal ([153]) is $\omega^n \to (\omega^m, p)^2$ and $\omega^n \not\to (\omega^m, p+1)^2$ where

$$p = 2^{[\frac{n-1}{m-1}]}$$

$(5 \leq m \leq n)$.

For more colors, Darby and Larson proved in [22] that if $t = t_1 + \cdots + t_m$ then

$$\omega^{t+2} \not\to (\omega^3, 2^{t_1} + 1, \ldots, 2^{t_m} + 1)^2$$

and

$$\omega^{t+3} \to (\omega^3, 2^{t_1+1}, \ldots, 2^{t_m+1})^2.$$

Problem 7. $\omega^\omega \to (\omega^\omega, 3)^2$?

This was proved by C. C. Chang in [20]. The complicated (90 page long) original proof was somewhat simplified and generalized to $\omega^\omega \to (\omega^\omega, k)^2$ $(k < \omega)$ by Milner (unpublished). Then Larson gave a reasonably short proof in [129], [130].

Galvin noticed that if $\omega^\alpha \to (\omega^\alpha, 3)^2$ and $2 < \alpha < \omega_1$ then necessarily α is of the form ω^ξ. ([74])

For the formulation of the following results set $a(\gamma) = k$ where $\gamma = \gamma_1 + \cdots + \gamma_k$ is the decomposition of γ into nonzero AI ordinals with $\gamma_1 \geq \cdots \geq \gamma_k$.

Theorem 1. (Schipperus, Darby) *If $a(\gamma) = 1$, then $\omega^{\omega^\gamma} \to (\omega^{\omega^\gamma}, 3)^2$.*

Theorem 2. (Darby) *If $\gamma = \omega^{\delta+1}$ for some δ, then $\omega^{\omega^\gamma} \not\to (\omega^{\omega^\gamma}, m)^2$ for $m \to (4)^3_{2^{32}}$.*

Theorem 3. (Darby, Schipperus, Larson) *If $a(\gamma) = 2$, then $\omega^{\omega^\gamma} \to (\omega^{\omega^\gamma}, 3)^2$ and $\omega^{\omega^\gamma} \not\to (\omega^{\omega^\gamma}, 5)^2$. Further, $\omega^{\omega^2} \to (\omega^{\omega^2}, 4)^2$.*

Theorem 4. (Schipperus, [164]) *If $a(\gamma) > 3$, then $\omega^{\omega^\gamma} \not\to (\omega^{\omega^\gamma}, 3)^2$.*

Theorem 5. (Schipperus) *If $\alpha > \omega^\omega$ is not an epsilon number then $\alpha \not\to (\alpha, m)^2$ where $m \to (6)^3_{2^{32}}$.* (α is an epsilon number if $\omega^\alpha = \alpha$.)

Conjecture. (Darby) *If $\alpha < \omega_1$ is an epsilon number then $\alpha \to (\alpha, n)^2$ for every $n < \omega$.*

Erdős and Hajnal raised the possibility that if $\alpha < \omega_1$, $3 \leq m < n < \omega$ then $\alpha \to (\alpha, m)^2$ implies $\alpha \to (\alpha, n)^2$ (as in the case of $\alpha = \omega^2$ or ω^ω). This

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is false by Theorems 2 and 3, but Theorem 5 and the Conjecture imply that it holds for m sufficiently large.

Problem 8. $\omega_1 \not\to (\omega_1, \omega+2)^2$?

Already in [62] Erdős and Rado showed $\omega_1 \to (\omega_1, \omega+1)^2$. After that, this is a natural question.

Hajnal showed in [85] that under CH $\omega_1 \not\to (\omega_1, \omega+2)^2$ holds. This is not as easy as it seems. Todorcevic in [?] proved that if $\mathfrak{b} = \omega_1$, then $\omega_1 \not\to (\omega_1, \omega : 2)^2$.

Raghavan and Todorcevic proved in [159] that $\omega_1 \not\to (\omega_1, \omega+2)^2$ also follows from the existence of an ω_1-Suslin tree.

Todorcevic proved in [198] that it is consistent (in fact, follows from PFA) that $\omega_1 \to (\omega_1, \alpha)^2$ holds for every $\alpha < \omega_1$. Erdős and Hajnal asked in [52] if $\omega_1 \to (\omega_1, \alpha)^2$ follows from MA_{ω_1} for $\alpha < \omega_1$. In unpublished work, Todorcevic showed this for $\alpha = \omega^2$, but the general case is open.

Problem 9. (GCH) $\omega_{\omega+1} \to (\omega_{\omega+1}, \omega+2)^2$.

Erdős and Rado showed in [62] that if $\kappa > \omega$ is regular, then $\kappa \to (\kappa, \omega + 1)^2$. For singular cardinals, the corresponding statement is unsolved: if $\lambda > \mathrm{cf}(\lambda) > \omega$, then $\lambda \to (\lambda, \omega+1)^2$. Standard theory gives this if λ is strong limit. Shelah proved the statement if $2^{\mathrm{cf}(\lambda)} < \lambda$ ([184]).

Hajnal in [85] proved that CH implies $\omega_1 \not\to (\omega_1, (\omega : 2))^2$.

Laver proved that if $\mathfrak{c} = \aleph_2$ and MA_{ω_1} holds, then $\omega_2 \not\to (\omega_2, (\omega : 2))^2$. Interestingly, this cannot be lifted to ω_3: Todorcevic in [197] proved that it is consistent that $\mathfrak{c} = \aleph_3$, MA_{ω_2} holds, yet $\omega_3 \to (\omega_3, (\omega : 2))^2$.

Baumgartner proved with his thinning-out process that if GCH holds and κ is regular, then in a cardinal and cofinality forcing extension $\mathfrak{c} = \kappa^+$ and $\kappa^+ \not\to (\kappa^+, (\omega : 2))^2$ ([7]).

Extending Baumgartner's thinnig-out forcing, Laver proved in [133] that if GCH holds and κ is strongly Mahlo, then with some cardinal and cofinality preserving forcing a model is obtained where $\mathfrak{c} = \kappa$, weakly Mahlo, and $\kappa \not\to (\kappa, (\omega : 2))^2$.

Kunen proved in [125] that if κ is real valued measurable then $\kappa \to (\kappa, \alpha)^2$ for $\alpha < \omega_1$ (see also [72]).

Problem 9 was asked in this form, because $\omega_{\omega+1} \to (\omega_{\omega+1}, \omega+1)^2$ by the Erdős–Rado theorem and $\omega_{\omega+1} \not\to (\omega_{\omega+1}, \omega_1)^2$ by Sierpinski's example giving

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$\kappa^\mu \not\to (\kappa^+, \mu^+)^2$ where $\mu = \mathrm{cf}(\kappa)$. Raghavan and Todorcevic proved in [159] that if λ is an infinite cardinal and there is a λ^+-Suslin tree and $\lambda^\mu > \lambda$, then $\lambda^+ \not\to (\lambda^+, \mu+2)^2$ holds. In particular, if there is an $\omega_{\omega+1}$-Suslin tree, then $\omega_{\omega+1} \not\to (\omega_{\omega+1}, \omega+2)^2$.

Problem 10. (GCH) *Does* $\kappa^+ \to (\alpha)^2_2$ *hold for every* κ *and* $\alpha < \kappa^+$*?*

Problem 10/A. *Does*

$$\omega_1 \to (\omega 2, \omega^2)^2$$

or

$$\omega_1 \to (\omega 3)^2_2$$

or

$$\omega_1 \to (\omega + n)^2_3 \quad (n < \omega)$$

hold?

Various special cases of the general statement $\omega_1 \to (\alpha)^2_k$ ($\alpha < \omega_1$, $k < \omega$) were proved in [62], [85], clearly Erdős and his friends were fascinated by the question. In [157], Prikry proved $\omega_1 \to (\omega : \omega_1, \alpha)^2$ for $\alpha < \omega_1$.

The general case was eventually proved by Baumgartner and Hajnal in [10]. See the comments at Problem 11 on this proof using forcing and absoluteness. The results of Prikry and Baumgartner-Hajnal were extended by Todorcevic who in [**?**] proved that for $\alpha < \omega_1$ and $k < \omega$ one has $\omega_1 \to ((\alpha, \omega_1), (\alpha)_k)^2$, i.e., if $f : [\omega_1]^2 \to k$, then either there are $H_0 < H_1 \subseteq \omega_1$ with $\mathrm{tp}(H_0) = \alpha$, $\mathrm{tp}(H_1) = \omega_1$ such that $[H_0]^2 \cup (H_0 \times H_1)$ is homogeneous in color 0, or else there is $K \subseteq \omega_1$, $\mathrm{tp}(K) = \alpha$, that K is homogeneous in color j for some $j > 0$.

In all his life, Hajnal was very proud of this argument.

Soon after the Baumgartner–Hajnal proof, Galvin gave a purely combinatorial proof of the theorem ([73]).

The following conjecture, which implies the Baumgartner–Hajnal theorem, is open: $\omega_1 \to (\alpha, n)^3$ for $\alpha < \omega_1$, $n < \omega$. Erdős and Rado proved that if θ is a real order, then $\theta \to (\omega + n, 4)^3$ ($n < \omega$) ([62], see a short proof by Jones, [103]). Milner and Prikry showed $\omega_1 \to (\omega + n, 4)^3$ ([142]), then $\omega_1 \to (\omega 2 + 1, 4)^3$ ([143]. Recently Jones extended this to $\omega_1 \to (\omega 2 + 1, n)^3$ ($n < \omega$) ([106]). In unpublished work, Schipperus proved $\omega_1 \to (\omega^2 + 1, 4)^3$.

Problem 10/B. (GCH) *Does* $\omega_2 \to (\omega_1 + \omega)^2_2$ *hold?*

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In unpublished work, Hajnal showed that $\omega_2 \not\to (\omega_1+2)^2_\omega$ and $\omega_2 \not\to [\omega_1+\omega]^2_{\omega_1}$ are separately consistent with GCH. In [160], Rebholz deduced this from the existence of a gap-2 morass and $\diamondsuit$.

In the positive direction, Erdős and Hajnal proved that if CH holds, then $\omega_2 \to (\omega_1+n)^2_2$ holds for $n<\omega$ (unpublished).

Shelah proved that if GCH holds, κ is regular, $|\alpha|^+ < \kappa$, then $\kappa^+ \to (\kappa+\alpha)^2_2$ ([166]).

Baumgartner, Hajnal, and Todorcevic proved that if $\kappa > \omega$ is regular, $n<\omega$, $2^{|\alpha|} < \kappa$, then $(2^{<\kappa})^+ \to (\kappa+\alpha)^2_n$ ([13]).

Shelah also proved that if κ is strongly compact, $\lambda > \kappa$ is regular, $\alpha < \kappa$ is an ordinal, $\mu < \kappa$ is a cardinal, then $(2^{<\lambda})^+ \to (\lambda+\alpha)^2_\mu$ holds ([183]).

Laver showed that if CH holds and there is a normal $(\aleph_2,\aleph_2,\aleph_0)$-saturated ideal on ω_1, then $\omega_2 \to (\omega_1 2+1,\alpha)^2$ $(\alpha<\omega_2)$ ([134]). Foreman and Hajnal proved that if the saturation property is replaced with the existence of an $\aleph_1$-dense ideal, then this can be improved to $\omega_2 \to (\omega_1^2+1,\alpha)^2$ $(\alpha<\omega_2)$ ([66]).

Kanamori ([109]) proved that if κ is measurable, then $\kappa^+ \to (\kappa 2+1,\alpha)^2$ $(\alpha<\kappa^+)$.

Foreman and Hajnal in [66] prove that if κ is measurable, then $\kappa^+ \to (\alpha)^2_\kappa$ holds for $\alpha < \Omega(\kappa)$, where $\Omega(\kappa)$ is a technically defined, very large ordinal, but $\Omega(\kappa) < \kappa^+$.

Erdős and Hajnal in [52] raised the following weakening of Problem 10: if GCH holds then for every $\alpha<\omega_2$ there is $\beta<\omega_3$ with $\beta \to (\alpha)^2_2$. The following result of Shelah may be relevant here: it is consistent that CH holds, $2^{\aleph_1}$ is anything it can be, and $2^{\aleph_1} \not\to (\omega_1\omega)^2_2$ ([180]).

We notice that Problem 10 is false if the number of colors is infinite: if κ is an infinite cardinal, then $\kappa^+ \not\to (\kappa^\omega+1)^2_\omega$ holds by the Milner–Rado paradox (see [144]).

Problem 11. *Does $\lambda \to (\alpha)^2_k$ hold for $\alpha<\omega_1$, $k<\omega$?*

Problem 11/A. *Do $\lambda \to (\omega^2)^2_2$, $\lambda \to (\omega 2)^2_3$, $\lambda \to (\omega+n)^2_4$ $(n<\omega)$ hold?*

Here λ is the order type of $(\mathbb{R},<)$. These were the then simplest unknown cases of the conjecture $\lambda \to (\alpha)^2_k$ $(\alpha<\omega_1,\ k<\omega)$. The problem was eventually settled by the following result.

Theorem. (Baumgartner–Hajnal, [10]) *If φ is an order type satisfying $\varphi \to (\omega)^1_\omega$ then $\varphi \to (\alpha)^2_k$ holds for $\alpha<\omega_1$, $k<\omega$.*

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The proof used a metamatematical argument. It was first shown that $\mathrm{MA}_{|\varphi|}$ implies the result, then a rather complicated argument was given showing that if $\varphi \to (\omega)^1_\omega$, then it remains true in any ccc forcing extension, finally an argument of Silver was used to show that if $(S, <)$ is an ordered set, F is a coloring of $[S]^2$, and F has a homogeneous subset of type α in some forcing extension, for some countable α, then it has one in the ground model. As each model has a ccc forcing extension satisfying MA_κ, for any cardinal κ, the theorem follows.

Notice that the condition on φ is necessary, as $\varphi \not\to (\omega)^1_\omega$ implies $\varphi \not\to (\omega, \omega+1)^2$ as can readily be seen.

Todorcevic showed in [203] that if π is a partial order type with $\pi \to (\omega)^1_\omega$ then $\pi \to (\alpha)^2_k$ holds for $\alpha < \omega_1$, $k < \omega$. Further, if κ is regular, $\lambda^{<\lambda} < \kappa$, π is a partial order type, $\pi \to (\kappa)^1_{2^{<\kappa}}$, then $\pi \to (\kappa + \alpha)^2_2$ $(\alpha < \lambda)$.

Problem 12. *Is there a relation $\lambda \to (\theta_0, \theta_1)^2$ without having $\varphi \to (\theta_0, \theta_1)^2$ for every real type φ ?*

Here $\lambda = \mathrm{tp}(\mathbb{R}, <)$, φ is a real type, if $|\varphi| \geq \aleph_1$ and $\omega_1, \omega_1^* \not\leq \varphi$. The answer is No by Sierpiński's coloring and the Baumgartner–Hajnal theorem described at Problem 11.

In [52], Erdős and Hajnal extends the question to the following. Let $(R, <)$ be an ordered set of order type φ. What are the implications between the following statements?

(i) R is uncountable and contains a countable dense set.

(ii) $\omega_1, \omega_1^* \not\leq \varphi$, $|\varphi| \geq \aleph_1$.

(iii) $\varphi \to (\eta)^1_\omega$.

(iv) $\varphi \to (\omega)^1_\omega$.

(v) There is $\psi \leq \varphi$, $|\psi| \geq \aleph_1$, $\omega_1^* \not\leq \psi$.

Galvin showed that if $\omega_1 \not\leq \varphi$ then (iv) implies (iii).

Baumgartner proved the following.

Theorem. (Baumgartner, [8])

(a) *There is φ, $|\varphi| = \aleph_1$, satisfying (iv) but not (v).*

(b) (V=L) *There is φ with $|\varphi| = \aleph_2$, $\varphi \to (\omega)^1_\omega$ such that $\psi \not\to (\omega)^1_\omega$ holds for every $\psi \leq \varphi$, $|\psi| = \aleph_1$.*

(c) *It is consistent, relative to the existence of a weakly compact cardinal, that every φ with $|\varphi| = \aleph_2$ and $\varphi \to (\omega)^1_\omega$, contains a ψ with $|\psi| = \aleph_1$ and $\psi \to (\omega)^1_\omega$.*

Problem 13. (GCH)
(a) $\omega_1^2 \to (\omega_1^2, 3)^2$,
(b) $\omega_2\omega \to (\omega_2\omega, 3)^2$?

For (a), Hajnal proved $(\kappa^+)^2 \not\to ((\kappa^+)^2, 3)^2$ if $\kappa^{<\kappa} = \kappa$ in [89], i.e., for κ regular under GCH. Baumgartner extended this to singular cardinals, he proved that if λ is singular, $2^\lambda = \lambda^+$, then $(\lambda^+)^2 \not\to ((\lambda^+)^2, 3)^2$. For non-successor cardinals, he further proved that if λ is strong limit singular then $\lambda^2 \to (\lambda^2, 3)^2$ holds iff $\mathrm{cf}(\lambda)^2 \to (\mathrm{cf}(\lambda)^2, 3)^2$. Further, if κ is regular and there is a κ-Suslin tree, then $\kappa^2 \not\to (\kappa^2, 3)^2$ ([6]).

The consistency of $\omega_1^2 \to (\omega_1^2, 3)^2$ is, as far as I know, open.

In [50], Erdős and Hajnal proved $\omega_1^2 \to (\omega_1\alpha, 3)^2$ for $\alpha < \omega_1$. Baumgartner and Hajnal in [11] showed that CH implies $\omega_1^2 \not\to (\omega_1\omega, 4)^2$, but $\omega_1^2 \to (\omega_1\omega, 3, 3)^2$ holds in ZFC. This implies, via forcing and compactness, that there is a *finite* K_4-free graph all whose edge coloring with two colors has a homogeneous triangle, see Problem 54.

(b) was solved by Shelah and Stanley in [186]. They proved that $\omega_2\omega \not\to (\omega_2\omega, 3)^2$ is consistent with the negation of CH. A sophisticated argument gives that CH implies $\omega_2\omega \to (\omega_2\omega, k)^2$ for $k < \omega$. GCH is consistent with both $\omega_3\omega_1 \not\to (\omega_3\omega_1, 3)^2$ and with $\omega_3\omega_1 \to (\omega_3\omega_1, k)^2$ ($k < \omega$), the latter modulo the consistency of a weakly compact cardinal.

Miyamoto in [146] gave a morass proof of $\omega_3\omega_1 \not\to (\omega_3\omega_1, 3)^2$, thereby confirming that this holds under V=L. Stanley, Velleman, and Morgan improved this argument by showing that if $2^{\aleph_1} = \aleph_2$ and there is a simplified $(\omega_2, 1)$-morass with linear limits, then $\omega_3\omega_1 \not\to (\omega_3\omega_1, 3)^2$ holds ([190]). This implies that if GCH and $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$ hold then either $\aleph_2$ or $\aleph_3$ is inacessible in L.

3. Square bracket partition relations

Problem 14. *Assume that $\aleph_0 \le r \le b \le a$ are cardinals. Then $a \not\to [b]^r_{b^r}$.*

Erdős and Hajnal proved $a \not\to [r]^r_{2^r}$ for $a, r \ge \aleph_0$ cardinals. They specifically asked the following special case of Problem 14.

Problem 14/A. *Does $\kappa \not\to [\mathfrak{c}^+]^{\aleph_0}_{\mathfrak{c}^+}$ hold for every κ? Under GCH, does $\kappa \not\to [\aleph_2]^{\aleph_0}_{\aleph_2}$ hold for every κ?*

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It was pointed out by Laver ([134]), that the negation of this holds in a model of Kunen [126].

Problem 15. *If CH is not assumed, do we have*
(a) $2^{\aleph_0} \not\to [\aleph_1]^2_3$,
(b) $2^{\aleph_0} \not\to [2^{\aleph_0}]^2_3$, or
(c) $\aleph_1 \not\to [\aleph_1]^2_3$?

Erdős and Hajnal proved that CH implies $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$. Here they ask how much of this remains true under ZFC alone. We know, by Sierpiński, that $\mathfrak{c} \not\to [\aleph_1]^2_2$.

(a) Shelah in [176] proved that if the existence of a measurable cardinal is consistent then so is $\mathfrak{c} \to [\aleph_1]^2_3$. This was specific about $\aleph_1$ and $\mathfrak{c}$. In [182] he proved that if $\mu = \mu^{<\mu} < \theta < \kappa$, κ is strongly Mahlo, then a cardinal preserving forcing makes $2^\mu = \kappa$ and $\kappa \to [\theta]^2_{\sigma,2}$ for every $\sigma < \mu$.

For higher superscripts, he proved in [179] that if λ is ω-Mahlo, then in a ccc extension $\lambda = \mathfrak{c} \to [\aleph_1]^r_{h(r)}$ for some $h(r) < \omega$.

(b) Galvin and Shelah proved $2^{\aleph_0} \not\to [2^{\aleph_0}]^2_{\aleph_0}$ ([76]).

(c) First Blass obtained $\aleph_1 \not\to [\aleph_1]^2_3$ in [19]. This was quickly followed by Galvin and Shelah's $\aleph_1 \not\to [\aleph_1]^2_4$ ([76]). Finally, Todorcevic showed $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$ using a powerful new method, the Todorcevic walks ([204]).

Problem 16. *Let κ be a strongly inaccessible, not weakly compact cardinal. Does $\kappa \not\to [\kappa]^2_\kappa$ hold?*

Consider the following statements.
(a) $\Box(\kappa)$ and $\diamondsuit_k$ hold for every strongly inaccessible, not weakly compact cardinal.
(b) There is a κ-Suslin tree for every strongly inaccessible, not weakly compact cardinal κ.
(c) $\kappa \not\to [\kappa]^2_\kappa$ for every strongly inaccessible, not weakly compact cardinal κ.
(d) $\kappa \not\to [\kappa]^2_\omega$ for every strongly inaccessible, not weakly compact cardinal κ.

Here $(a) \to (b) \to (c) \to (d)$. Of course, (d) can be replaced with $\kappa \not\to [\kappa]^2_\gamma$ for any $3 \leq \gamma < \kappa$.

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Jensen proved (a) and therefore (b) in L (more exactly if $\diamondsuit(E)$ and $\square(E)$ hold where $E \subseteq \kappa$ is stationary and if C_α is in the square sequence then $acc(C_\alpha) \cap E = \emptyset$), from which R. A. Shore deduced (c) in L. ([188]).

Woodin first proved the independence of (a) by showing that if $2^\kappa > \kappa^+$ and Radin forcing is applied to a measure sequence of length κ^+, then in the resulting model κ is inacessible and $\diamondsuit_\kappa$ fails.

Omer Ben-Neria, Shimon Garti, and Yair Hayut in [17] proved that if κ is measurable with $o(\kappa) \geq \kappa^+$, $2^\kappa = 2^{\kappa^+}$, then in a generic extension κ is inacessible and Φ_κ fails.

Golshani ([78]) started with a κ which is $(\kappa+3)$-strong and gave a forcing extension in which κ is the least inaccessible and Φ_κ, consequently $\diamondsuit_\kappa$ fails.

Todorcevic ([204]) and independently, but slightly later Shelah ([176]), generalizing Todorcevic's argument for $\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$, proved that if $\kappa > \aleph_1$ is regular and there is a nonreflective stationary set in κ, then $\kappa \not\to [\kappa]^2_\kappa$.

Finally Rinot proved in [161] that if κ is regular and $\kappa \to [\kappa]^2_\kappa$, then κ is weakly compact in L.

As much as we know, (b), (c), and (d) can be true in ZFC.

Problem 17. (Erdős, Hajnal, Rado) *Assume that* $2 \leq r < \omega$, λ *is an infinite cardinal,* $\kappa_\alpha > r$ *are (possibly finite) cardinals* $(\alpha < \gamma)$*, and* $\lambda \not\to [\kappa_\alpha]^r_{\alpha<\gamma}$. *Does then* $2^\lambda \not\to [1+\kappa_\alpha]^{r+1}_{\alpha<\gamma}$ *hold?*

Todorcevic showed in [202] that if $\square_\lambda$ holds, $\lambda \not\to [\kappa_\alpha]^r_{\alpha<\gamma}$, then $\lambda^+ \not\to [1+\kappa_\alpha]^{r+1}_{\alpha<\gamma}$, assuming that each κ_α is infinite, regular $(\alpha > 0)$.

Problem 17/A. *Does* $2^{2^{\aleph_0}} \not\to [\aleph_1]^3_{\aleph_1}$ *hold? Under GCH, does* $\aleph_{n+1} \not\to [\aleph_1]^{n+2}_{\aleph_1}$ *hold* $(n < \omega)$*?*

In [58], Erdős, Hajnal, and Rado claimed that GCH implied $\aleph_2 \not\to [\aleph_1]^3_3$. Erdős and Hajnal reported in [52] that if κ is regular, $k \leq 4$, $\lambda \not\to [\kappa]^2_k$, then $2^\lambda \not\to [\kappa]^3_k$. In particular, $2^{\aleph_1} \not\to [\aleph_1]^3_4$.

Todorcevic in [207] proved $\aleph_2 \not\to [\aleph_1]^3_{\aleph_0}$ and $2^{\aleph_1} \not\to [\aleph_1]^3_{10}$.

Shelah noticed that if $2^\kappa \to [\kappa^+]^2_3$ then $(2^\kappa)^+ \to [\kappa^+]^3_3$. The consistency of the former statement is proved in [182].

Problem 18. $\omega^\omega \to [\omega^\omega]^2_{\aleph_0}$?

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In unpublished work Galvin proved that if φ is an order type, $\omega_1 \not\leq \varphi$, $\eta \not\leq \varphi$, then

$$\varphi \not\to [\omega, \omega^2, \omega^2, \omega^3, \omega^3, \omega^4, \omega^4, \dots]^2.$$

In particular, $\omega^\omega \not\to [\omega^\omega]^2_{\aleph_0}$.

Galvin also proved that $\eta \to [\eta]^2_3$. He also conjectured and Laver proved that

$$\begin{bmatrix} \eta \\ \eta \\ \eta \\ \vdots \\ \eta \end{bmatrix} \longrightarrow \begin{bmatrix} \eta \\ \eta \\ \eta \\ \vdots \\ \eta \end{bmatrix}^{1,1,\dots,1}_{t_k}.$$

for some number t_k (k is the number of rows) but obtained the minimal value of t_k wrong. That was finally calculated by Denis Devlin in [24] (see also [210], pp 143–148).

Problem 19. (GCH) $\aleph_2 \not\to [\aleph_1]^2_{\aleph_1,\aleph_0}$.

As it was quickly realized, $\aleph_2 \to [\aleph_1]^2_{\aleph_1,\aleph_0}$ is equivalent to $\aleph_2 \to [\aleph_1]^{<\omega}_{\aleph_1,\aleph_0}$, which is Chang's conjecture (CC). It is easy to see that the negation of CC is consistent: a Kurepa tree is a counterexample to it. The consistency of CC was proved by Jack Silver from the existence of an ω_1-Erdős cardinal. Todorcevic proved [207] that $\aleph_2 \to [\aleph_1]^3_{\aleph_1}$ is equivalent to CC. **Problem 19/A.** (GCH) *Does there exist a system $\{f_\alpha : \alpha < \omega_2\}$ with $f_\alpha : \omega_1 \to \omega$ such that $\{\xi < \omega_1 : f_\alpha(\xi) = f_\beta(\xi)\}$ is countable for $\alpha \neq \beta$?*

Problem 19/B. (GCH) *Does there exist a system $\{f_\alpha : \alpha < \omega_2\}$ with $f_\alpha : \omega_1 \to \omega$ such that if $\alpha \neq \beta$, then $\{\xi < \omega_1 : f_\alpha(\xi) = f_\beta(\xi)\}$ is a countable ordinal?*

Erdős and Hajnal notice that this is equivalent to the Kurepa hypothesis.

Problem 19/C. *Does either of the implications*
Problem 19 $\longrightarrow$ Problem 19/A, Problem 19/A $\longrightarrow$ Problem 19/B
hold?

Baumgartner proved in [5] that it is consistent from a strongly inacessible that Problem 19/A $\not\longrightarrow$ Problem 19/B. See also [147].

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Problem 19/D. *Does there exist a system $\mathcal{F}$, $|\mathcal{F}| = \aleph_2$ or $2^{\aleph_1}$ of almost disjoint stationary subsets of ω_1?*

That is, are there stationary subsets $\{S_\alpha : \alpha < \kappa\}$ of ω_1 such that $S_\alpha \cap S_\beta$ is countable ($\alpha \neq \beta$) for $\kappa = \aleph_2$ or $2^{\aleph_1}$?

An ideal I on ω_1 is κ-saturated if there are no sets $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(\omega_1)$ with $A_\alpha \notin I$ such that $A_\alpha \cap A_\beta \in I$ for $\alpha \neq \beta$. If $\kappa = \aleph_2$, then there are $\aleph_2$ stationary sets in ω_1 with pairwise nonstationary intersection iff there are $\aleph_2$ stationary sets with pairwise countable intersection.

First Magidor observed that if Jensen's $\diamondsuit$ principle holds, then there are $2^{\aleph_1}$ stationary sets in ω_1 with pairwise countable intersection.

In [126], Kunen proved that from a huge cardinal it is consistent that there is an ω_1-complete, $\aleph_2$-saturated ideal on ω_1.

In [191] J. R. Steel, R. A. van Wesep proved the consistency of ZFC plus the non-stationary ideal on ω_1 is $\aleph_2$-saturated from a model of $\mathrm{AD}_{\mathbb{R}}$ and θ is regular.

In [68], Foreman, Magidor, and Shelah proved the consistency of Martin's maximum (MM) from a supercompact cardinal, and showed that MM implies that the non-stationary ideal on ω_1 is $\aleph_2$-saturated.

Shelah proved from the consistency of a Woodin cardinal that the non-stationary ideal is $\aleph_2$-saturated.

In the last two models, CH fails. This is no accident: Woodin proved that if the nonstationary ideal on ω_1 is $\aleph_2$-saturated and a measurable cardinal exists, then CH fails ([213]).

Problem 19/E. (GCH) *Does there exist a family $\mathcal{F} \subseteq [\omega_\omega]^{\aleph_0}$, $|\mathcal{F}| = \aleph_{\omega+1}$, such that $|\mathcal{F}|X| \leq \aleph_0$ for every $X \in [\omega_\omega]^{\aleph_0}$?*

Here $\mathcal{F}|X = \{F|X : F \in \mathcal{F}\}$. If λ is a cardinal, let KH_λ denote the existence of a family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$, $|\mathcal{F}| > \lambda$ such that $|\mathcal{F}|X| \leq |X|$ for every infinite $X \subseteq \lambda$. An easy argument shows that under GCH, $\mathrm{KH}_{\aleph_\omega}$ is equivalent to the above statement. Erdős, Hajnal, and Milner proved in [57] that if GCH holds and $\omega < \mathrm{cf}(\lambda) < \lambda$ then KH_λ fails. In unpublished work, Prikry proved that in L, KH_λ holds for every $\omega = \mathrm{cf}(\lambda) < \lambda$ ([156]). Todorcevic improved upon this by showing that if $\omega = \mathrm{cf}(\lambda) < \lambda$, then $\square_\lambda$ implies KH_λ ([200], [202]). Yodorcevic proved that the Chang Conjecture $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_0)$ implies that $\mathrm{KH}_{\aleph_\omega}$ fails ([209], see also [206], [204], and [200]).

Recently Golshani proved that if κ is supercompact, then KH_λ fails for $\lambda \geq \kappa$ ([79]).

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The following problem concerns of the so called Milner-Rado paradox ([144]). This surprising statement states that $\alpha \not\to (\kappa^\omega)^1_\omega$ holds for every $\alpha < \kappa^+$.

Problem 20. *If κ, λ are infinite cardinals, $\kappa^+ < \lambda$, $\alpha < \lambda^+$ is an ordinal, then*

$$\alpha \not\to [\lambda^{\kappa^+}]^1_{\kappa^+,\kappa}.$$

The following is an important special case.

Problem 20/A. *Does $\alpha \not\to [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$ hold for every $\alpha < \omega_3$?*

Erdős and Hajnal proved the following partial results:
(1) $\alpha \not\to [\omega_2^{\omega_1}]^1_{\aleph_1,\aleph_0}$ $(\alpha < \omega_2^{\omega_2})$,
(2) if $\omega_2^{\omega_2} \not\to [\omega_2^{\omega_1}]^1$ then the answer to Problem 19/D is affirmative.

Problem 21. *Let $\alpha < \omega_3$. Does there exist a sequence $\langle f_\beta : \beta < \alpha \rangle$ of $\omega_1 \to \omega_1$ functions such that if $\beta_0 < \beta_1 < \alpha$ then the set $\{\xi < \omega_1 : f_{\beta_0}(\xi) \geq f_{\beta_1}(\xi)\}$ is nonstationary?*

Problem 21/A. *Let $\alpha < \omega_3$. Does there exist a sequence $\langle f_\beta : \beta < \alpha \rangle$ of $\omega_1 \to \omega_1$ functions such that if $\beta_0 < \beta_1 < \alpha$ then*

$$|\{\xi < \omega_1 : f_{\beta_0}(\xi) \geq f_{\beta_1}(\xi)\}| \leq \aleph_0?$$

In his thesis, Kunen proved that under GCH, the statement in Problem 21/A fails for large enough $a < \omega_3$. On the other hand for any $\alpha < \omega_3$, a forcing argument gives the consistent existence of such a sequence ([125]).

If $\kappa > \omega$ is regular, an easy recursion gives the existence of a sequence $\{h_\alpha : \alpha < \kappa^+\} \subseteq {}^\kappa\kappa$ such that $h_\beta <^* h_\alpha$, i.e., $\{\xi < \kappa : h_\beta(\xi) \geq h_\alpha(\xi)\}$ is bounded for $\beta < \alpha < \kappa^+$, see [65], pp 906–907 and chapter 22 in [123].

Here h_α is the *αth canonical function*, i.e., if $S \subseteq \kappa$ is stationary, $f : S \to$ ORD has $f(\xi) < h_\alpha(\xi)$ for $\xi \in S$, then there is stationary $S^* \subseteq S$, such that $f|S^* = h_\beta|S^*$ for some $\beta < \alpha$.

In 1976, Hajnal proved that in L there is no ω_2-th canonical function (unpublished, but see [101], pp 554–555). Jech and Shelah proved that if CH holds and θ is an ordinal, then in a cardinal and cofinality preserving extension for every $\alpha < \theta$ there is an α-th function [102].

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Deiser and Donder proved that the statement that c_{ω_1} (the constant ω_1 function on ω_1) is the ω_2nd canonical function on ω_1 is equiconsistent with the existence of an inaccessible limit of measurable cardinals [23]. This was extended by Larson and Shelah, who proved that the above condition suffices to give a model where c_{ω_1} is the ω_2nd canonical function and CH holds [131].

If κ, μ are cardinals, then $\kappa \Longrightarrow (\mu)^{<\omega}$ abbreviates the following statement: if $F : [\kappa]^{<\omega} \to 2$ with no $[X]^r \subseteq F^{-1}(0)$ for $X \in [\kappa]^{r+1}$ $(r < \omega)$, then there exist $r_0 < \omega$ and $Y \in [\kappa]^\mu$ such that $[Y]^r \subseteq F^{-1}(1)$ for $r_0 < r < \omega$.

Problem 22. *Does $\kappa \not\Longrightarrow (\omega)^{<\omega}$ hold for the least strongly inaccessible cardinal κ?*

Problem 23. *If $F_r : [\omega]^r \to \omega$ for every $1 \leq r < \omega$ then there are $S \in [\omega]^\omega$, $r_0 < \omega$, $g(r) < \omega$ $(r_0 < r)$ such that for $r_0 < r < \omega$, $x \in [S]^r$, we have $F(x) \neq g(r)$.*

This follows from the following theorem independently proved by J. E. Baumgartner and Erdős–Hajnal–Rado. If $a_r < b_r < \omega$ for $1 \leq r < \omega$, $a_r \to \infty$, and $f_r : [\omega]^r \to b_r$, then there are an infinite $A \subseteq \omega$ and $r_0 < \omega$ such that $|F_r[[A]^r]| \leq a_r$ for $r > r_0$. This is unpublished, but it was a problem in the 1968 M. Schweitzer Commemorative Contest, see [88], [192].

4. Polarized partition relations

Problem 24. (Erdős, Hajnal, Rado) (GCH) *Does*

$$\begin{pmatrix} \aleph_2 \\ \aleph_1 \end{pmatrix} \to \begin{pmatrix} \aleph_i & \aleph_j \\ \aleph_1 & \aleph_1 \end{pmatrix}^{1,1}$$

hold for $0 \leq i, j \leq 1$?

Prikry proved in [155] the consistency of GCH with

$$\begin{pmatrix} \aleph_2 \\ \aleph_1 \end{pmatrix} \not\to \begin{pmatrix} \aleph_0 \\ \aleph_1 \end{pmatrix}^{1,1}_{\aleph_0}.$$

His proof was the first example of forcing with side conditions. In [155], he also gives the consistency of the following stronger assertion with GCH.

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(∗) There are sets $A(\alpha,\xi) \subseteq \omega_1$ ($\alpha < \omega_2$, $\xi < \omega_1$) such that $\langle A(\alpha,\xi) : \xi < \omega_1\rangle$ is a partition of ω_1 ($\alpha < \omega_2$) and if $s \in [\omega_2]^{\aleph_0}$, $f : s \to \omega$, then

$$\left|\omega_1 - \bigcup\{A(\alpha, f(\alpha)) : \alpha \in s\}\right| \leq \aleph_0.$$

This is easily seen to imply the sqaure bracket polarized partition relation

$$\begin{bmatrix} \aleph_2 \\ \aleph_1 \end{bmatrix} \not\to \begin{bmatrix} \aleph_0 \\ \aleph_1 \end{bmatrix}^{1,1}_{\aleph_1}.$$

In [158] Prikry reports that in unpublshed work, Jensen deduced (∗) from $V = L$.

For the other direction, Laver ([134]) proved from the consistency of the existence of a huge cardinal that GCH is consistent with the existence of an $(\aleph_2, \aleph_2, \aleph_0)$ saturated ideal on ω_1 and the latter implies

$$\begin{pmatrix} \aleph_2 \\ \aleph_1 \end{pmatrix} \to \begin{pmatrix} \aleph_1 \\ \aleph_1 \end{pmatrix}^{1,1}_{\aleph_0}.$$

In unpublished work, Galvin generalized this to

$$\begin{pmatrix} \aleph_2 \\ \aleph_1 \end{pmatrix} \to \begin{pmatrix} \alpha \\ \aleph_1 \end{pmatrix}^{1,1}_{\aleph_0} \quad (\alpha < \omega_2)$$

(see the reference in [134] and see also [104]).

Problem 25. (Erdős, Hajnal, Rado) (GCH) *Does*

$$\begin{pmatrix} \aleph_{\omega+1} \\ \aleph_{\omega+1} \end{pmatrix} \to \begin{pmatrix} \aleph_{\omega+1} & \aleph_1 \\ \aleph_\omega & \aleph_\omega \end{pmatrix}^{1,1}$$

hold?

Garti proves the consistency of

$$\begin{pmatrix} \lambda^+ \\ \lambda \end{pmatrix} \not\to \begin{pmatrix} \lambda^+ & \aleph_1 \\ \lambda & \lambda \end{pmatrix}^{1,1}$$

for some strong limit singular λ with $\mathrm{cf}(\lambda) = \omega$ and notices that standard modification can give $\lambda = \aleph_\omega$ ([77]).

Problem 26. *Does*

$$\begin{pmatrix} \mathfrak{c} \\ \aleph_0 \end{pmatrix} \not\to \begin{pmatrix} \mathfrak{c} \\ \aleph_0 \end{pmatrix}_2^{1,1}$$

hold without the assumption of CH?

To give a model of $\mathfrak{c} = \aleph_2$ and the negative relation is fairly standard: one can add $\aleph_2$ Cohen reals or use a model of MA_{ω_1}.

The positive statement follows from the existence of an $\aleph_1$-generated selective ultrafilter on ω. Models for this (and not CH) were given by Kunen (unpublished) and later by Baumgartner and Laver [14] (cf. Problem 40).

Problem 27. (GCH) *Does*

$$\begin{pmatrix} \aleph_2 \\ \aleph_1 \end{pmatrix} \not\to \begin{pmatrix} \aleph_0 \\ \aleph_1 \end{pmatrix}_\omega^{1,1}$$

hold?

In unpublished work, Hajnal proved that GCH implies

$$\begin{pmatrix} \aleph_2 \\ \aleph_2 \end{pmatrix} \to \begin{pmatrix} \aleph_1 \\ \aleph_1 \end{pmatrix}_3^{1,1}$$

and

$$\begin{pmatrix} \aleph_3 \\ \aleph_2 \end{pmatrix} \to \left(\begin{pmatrix} \aleph_0 \\ \aleph_1 \end{pmatrix}_{\aleph_0} \begin{pmatrix} \aleph_2 \\ \aleph_2 \end{pmatrix} \right)^{1,1}$$

See also the remarks on Prikry's and Laver's results at Problem 24.

In [58] it is proved that GCH implies

$$\begin{pmatrix} \aleph_2 \\ \aleph_2 \end{pmatrix} \to \begin{pmatrix} \aleph_2 \\ \aleph_1 \end{pmatrix} \vee \begin{matrix} \aleph_1 \\ \aleph_2 \end{matrix}, \begin{matrix} \aleph_1 \\ \aleph_1 \end{matrix} \Bigg)^{1,1}.$$

With Prikry's method the consistency each of the following can be shown to be consistent with GCH:

$$\begin{pmatrix} \aleph_2 \\ \aleph_2 \end{pmatrix} \to \begin{pmatrix} \aleph_1 \\ \aleph_1 \end{pmatrix}_4^{1,1}$$

and

$$\begin{pmatrix} \aleph_2 \\ \aleph_2 \end{pmatrix} \not\to \left(\begin{matrix} \aleph_2 \\ \aleph_1 \end{matrix} \vee \begin{matrix} \aleph_1 \\ \aleph_2 \end{matrix}, \begin{pmatrix} \aleph_1 \\ \aleph_1 \end{pmatrix}_2 \right)^{1,1}.$$

Problem 28.

$$\begin{pmatrix} \aleph_1 \\ \aleph_1 \\ \aleph_1 \end{pmatrix} \to \begin{pmatrix} \aleph_0 \\ \aleph_0 \\ \aleph_0 \end{pmatrix}^{1,1,1}_{2}$$

Mills and Prikry proved ([140]) that

$$\begin{pmatrix} \aleph_1 \\ \aleph_1 \\ \aleph_1 \end{pmatrix} \not\to \begin{pmatrix} \aleph_0 \\ \aleph_0 \\ \aleph_0 \end{pmatrix}^{1,1,1}_{2}$$

and

$$\begin{bmatrix} \aleph_1 \\ \aleph_1 \\ \aleph_1 \end{bmatrix} \to \begin{bmatrix} \aleph_0 \\ \aleph_0 \\ \aleph_0 \end{bmatrix}^{1,1,1}_{3}$$

The consistency of

$$\begin{pmatrix} \mathfrak{c} \\ \mathfrak{c} \\ \mathfrak{c} \end{pmatrix} \to \begin{pmatrix} \aleph_0 \\ \aleph_0 \\ \aleph_0 \end{pmatrix}^{1,1,1}_{\aleph_1}$$

with $\mathfrak{c} = \aleph_3$ can be obtained with a forcing in [176]. Starting with a model of GCH, that forcing changes the value of $\mathfrak{c}$ to $\aleph_3$ with the property that each new set of ordinals of cardinality $\aleph_1$ contains an infinite old set. This easily implies the above partition relation.

Problem 29. (GCH) *Let λ be singular, $\lambda_\alpha \to \lambda$ $(\alpha < \mathrm{cf}(\lambda))$, $\lambda = \bigcup\{S_\alpha : \alpha < \lambda\}$ with $|S_\alpha| = \lambda_\alpha$. Let $F : [\lambda]^{<\omega} \to 2$. Does there exist $g : \mathrm{cf}(\lambda) \to \mathrm{cf}(\lambda)$ increasing, $S'_\alpha \in [S_{g(\alpha)}]^{\lambda_\alpha}$ such that if $s, t \in [\bigcup\{S'_\alpha : \alpha < \mathrm{cf}(\lambda)\}]^{<\omega}$, $|s| = |t|$, and $|s \cap S'_\alpha| = |t \cap S'_\alpha| \leq 1$ $(\alpha < \mathrm{cf}(\lambda))$, then $F(s) = F(t)$?*

Shelah proved the consistency of this for $\lambda = \aleph_\omega$ from infinitely many measurables in [172], then the consistency of a stronger statement from infinitely many strongly compacts in [173].

Problem 30. (GCH) *Does*

$$\aleph_{\omega+1} \not\to \left(\begin{pmatrix} \aleph_{\omega+1} \\ \aleph_\omega \end{pmatrix}, \aleph_1 \right)^2$$

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hold?

Problem 31. *Without CH can one prove*

$$\mathfrak{c} \not\to \left(\begin{pmatrix} \aleph_1 \\ \aleph_0 \end{pmatrix}, \begin{pmatrix} \aleph_1 \\ \aleph_0 \end{pmatrix} \right)$$

or at least

$$\mathfrak{c} \not\to \left(\begin{pmatrix} \aleph_1 \\ \aleph_1 \end{pmatrix}, \begin{pmatrix} \aleph_1 \\ \aleph_1 \end{pmatrix} \right)$$

The answer to the first question is no as MA_{ω_1} implies $\omega_1 \to (\omega_1, [\omega : \omega_1])^2$ (Laver). Szemerédi noticed that MA implies

$$\begin{pmatrix} \mathfrak{c} \\ \omega \end{pmatrix} \to \begin{pmatrix} \mathfrak{c} & \kappa \\ \omega & \omega \end{pmatrix}^{1,1} \quad (\kappa < \mathfrak{c}).$$

(See [105]).

Problem 32. (GCH) *Does there exist a graph X on ω_1 with $K_{\aleph_0,\aleph_1} \not\le X$ such that $\aleph_1 \not\to (X)^2_2$ holds?*

This would be a nice comment to

$$\aleph_1 \not\to \left(\begin{pmatrix} \aleph_1 \\ \aleph_0 \end{pmatrix}, \begin{pmatrix} \aleph_1 \\ \aleph_0 \end{pmatrix} \right)$$

which follows from CH.

In [167] Shelah proved the following. If $|X| = \omega_1$, $\mathrm{Col}(X) \le \omega$, then $\omega_1 \to (X)^2_n$ $(n < \omega)$. On the other hand, if $\diamondsuit$ holds and X is a graph with $|X| = \mathrm{Col}(X) = \omega_1$, then $\omega_1 \not\to (X)^2_2$. There is such a graph X with $K_{\aleph_0,\aleph_1} \not\le X$ in ZFC.

We notice that if X is a graph on ω_1 with $\mathrm{Col}(X) = \omega_1$ in the ground model then $\omega_1 \not\to (X)^2_2$ holds after forcing with $\mathrm{Add}(\omega, \omega_1)$. We pretend that the forcing $(P, \le)$ is the finite coloring poset, i.e., $p \in P$ if $\mathrm{Dom}(p) \in [\omega_1]^{<\omega}$, $\mathrm{Ran}(p) \subseteq 2$, $q \le p$ if $\mathrm{Dom}(q) \supseteq \mathrm{Dom}(p)$ and $p = q|\mathrm{Dom}(p)$. Assume that p forces that $f : \omega_1 \to \omega_1$ embeds X into color i. For each $\alpha < \omega_1$ let $p_\alpha \le p$ fix the value of $f(\alpha)$ as $g(\alpha)$. There is a decomposition $\omega_1 = \bigcup\{A_j : j < \omega\}$ such that if $\alpha, \beta \in A_j$ then p_α, p_β are compatible. There is $j < \omega$ such that $Y = X|A_j$ has $\mathrm{Col}(Y) = \omega_1$. Define the set mapping $h : A_j \to [A_j]^{<\omega}$ by $\beta \in h(\alpha)$ if $g(\beta)$ is contained in some pair in $\mathrm{Dom}(p_\alpha)$. By a result of Fodor's there are distinct $\alpha, \beta \in A_j$ such that $\{\alpha, \beta\} \in X$, $\alpha \notin h(\beta)$ and $\beta \notin h(\alpha)$,

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i.e., $\{g(\alpha), g(\beta)\}$ is not in $\mathrm{Dom}(p_\alpha)$ or $\mathrm{Dom}(p_\beta)$. Then there is a condition $q \leq p_\alpha, p_\beta$ such that $q(g(\alpha), g(\beta)) = 1 - j$, a contradiction.

With the above argument, one can show that after adding $\aleph_2$ Cohen reals, $\omega_1 \not\to (X)^2_2$ holds for every graph X with $|X| = \mathrm{Col}(X) = \omega_1$.

5. Set mappings

We denote by $(\kappa, a, b) \to \lambda$ if the following holds. Whenever $f : [\kappa]^a \to [\kappa]^{<b}$ is a set mapping, then there is a free set of cardinality λ.

Problem 33. *Is it true that if $\kappa \not\to (\aleph_0)^{<\omega}$, then there is a set mapping $f : [\kappa]^{<\omega} \to \kappa$ with no infinite free set? ($(\kappa, <\omega, 2) \not\to \aleph_0$)*

Baumgartner and independently, but later K. J. Devlin proved this in L ([25]). Devlin and Paris proved that if $(\kappa, <\omega, 2) \to \aleph_1$ holds for some κ, then $O^\sharp$ exists. Further, if $V = L[U]$ for some normal ultrafilter U, then $(\kappa, <\omega, 2) \to \lambda$ iff $\kappa \to (\lambda)^{<\omega}_2$ ([27]).

For $\kappa = \omega_\omega$ Koepke proved that the positive relation is equi-consistent with the existence of a measurable cardinal ([113]).

Problem 34.
(A) *Assume that κ is regular, not weakly compact. Does then exist a set mapping $[\kappa]^2 \to \kappa$ with no free set of cardinality κ? ($(\kappa, 2, 2) \not\to \kappa$)*
(B) (GCH) *Does there exist a set mapping $f : [\omega_2]^3 \to \omega_2$ with no uncountable free set? Does there exist a set mapping $f : [\omega_3]^3 \to [\omega_3]^{<\omega}$ with no free set of size $\aleph_2$?*

Various people pointed out that (A) is false, as $(\kappa, <\omega, \lambda) \to \kappa$ holds if κ is real valued measurable and $\lambda < \kappa$.

For (B), Erdős and Hajnal proved $(\exp_{r-1}(\mu)^+, r, \mu) \to \mu^+$ in [38]. Hajnal proved that if $k < \omega$, then GCH is consistent with $(\aleph_{k+2}, 3, 2) \not\to \aleph_{k+1}$ (cf. [98]).

The following are open: Is GCH is consistent with $(\aleph_2, 3, 2) \to \aleph_1$? Is GCH is consistent with $(\aleph_3, 4, 2) \not\to \aleph_1$?

Komjáth and Shelah proved in [121] that for $n < \omega$ it is consistent that $(\aleph_n, 2, 2) \not\to \aleph_1$ (but GCH fails).

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Problem 35. (Hajnal) (GCH) *Let $f : \omega_{\omega+1} \to [\omega_{\omega+1}]^{\leq\aleph_\omega}$ be a set mapping such that $|f(\alpha) \cap f(\beta)| < \aleph_\omega$ for $\alpha \neq \beta$. Does there exist a free set of cardinality $\aleph_{\omega+1}$?*

Problem 36. (Hajnal) *Let $f : \omega_1 \to [\omega_1]^{\leq\aleph_0}$ be a set mapping such that $|f(\xi) \cap f(\eta)| < \aleph_0$ for $\xi \neq \eta$. Does there exist a free set of type α for every $\alpha < \omega_1$?*

Shelah proved in [166] that if κ is regular, $\kappa^{<\kappa} = \kappa$, $f : \kappa^+ \to \mathcal{P}(\kappa^+)$ is a set mapping such that $|f(\xi) \cap f(\eta)| < \kappa$ for $\xi \neq \eta$, then for every $\alpha < \kappa^+$ there is a free set of order type α.

Problem 37. (Hajnal) *Does there exist an almost disjoint system $\mathcal{H} \subseteq [\lambda]^\kappa$, such that for every $S \in [\lambda]^{\kappa^+}$ there is $H \in \mathcal{H}$, $H \subseteq S$ where*
(A) $\kappa = \aleph_\omega$, $\lambda = \aleph_{\omega+1}$,
(B) $\kappa = \aleph_0$, $\lambda = \aleph_2$?

Problem 38. *Let $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a set mapping.*
(A) *Assume that $f(x)$ is nowhere dense for $x \in \mathbb{R}$. Does there exist an uncountable free set?*
(B) *Assume that $f(x)$ is closed and of measure ≤ 1 for $x \in \mathbb{R}$. Does there exist a 3-element free set?*
(C) *Let $f(x)$ be bounded and of outer measure ≤ 1 ($x \in \mathbb{R}$). Does there exist an infinite free set?*

(A) The following results have been proved.

Theorem. (Hechler, [99]) (CH) There is a set mapping $f : \mathbb{R} \to [\mathbb{R}]^\omega$ such that $f(x)$ is an omega-sequence converging to x and f has no uncountable free set.

Todorcevic extended this in [**?**], Chapter 1 this to the following. If $A \subseteq \mathbb{R}$, $|A| = \mathfrak{b}$, then there is $f : A \to \mathcal{P}(A)$, such that for any $x \in A$, $f(x)$ is a sequence converging to x, and f has no uncountable free set.

Theorem. (S. H.Hechler, [99]) After adding $\kappa \geq \aleph_2$ Cohen reals, the following is true. If $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set mapping with $f(x)$ of first category for $x \in \mathbb{R}$, then there is a second category free set of cardinality κ.

Theorem. (Bagemihl, [3]) *If $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$, with $f(x)$ nowhere dense ($x \in \mathbb{R}$), then there is an everywhere dense free set for f.*

Theorem. (F. Bagemihl, [4]) *If $X \subseteq \mathbb{R}$ is of second category, $f(x)$ is nowhere dense for $x \in X$, then there is a free set, dense in some interval.*

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Theorem. (Abraham, [1]) (MA_{ω_1}) *If* $\mathbb{R} \to [\mathbb{R}]^\omega$, $f(x)$ *is always a sequence converging to* x, *then there is an uncountable free set.*

Theorem. (Abraham [1], Fremlin [71], Newelski [150]) *If* $\kappa > \omega$, MA_κ *holds,* $A \subseteq \mathbb{R}$ *is of second category and* $f : A \to \mathcal{P}(\mathbb{R})$ *is a set mapping with* $f(x)$ *nowhere dense, then there is a free set of cardinality* κ.

(C) These are the corresponding results.

Theorem. (Erdős–Hajnal) *If* $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a set mapping such that* $f(x)$ *is bounded and* $\lambda^*(f(x)) \leq 1$ *for every* x, *then there is a* k*-element free set* $(k < \omega)$.

Theorem. (Newelski–Pawlikowski–Seredýnski, [151]) *If* $f : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a set mapping in which the Lebesgue measure of the closure of* $f(x)$ *is always at most 1, then there is an infinite free set.*

A family $\mathcal{F} \subseteq \mathcal{P}(S)$ has property B, if there is a set $T \subseteq S$, such that $\emptyset \neq A \cap T \neq A$ for every $A \in \mathcal{F}$. This is equivalent to $\mathcal{F}$ having chromatic number 2. If κ is an infinite cardinal, then $\mathcal{F}$ has property $B(\kappa)$ if there is a $T \subseteq S$ such that $1 \leq |A \cap T| < \kappa$ holds for every $A \in \mathcal{F}$. Property B was introduced by E. W. Miller in [139], he proved that if $\mathcal{F}$ is a system of infinite sets with $|A \cap B| \leq n$ $(A \neq B \in \mathcal{F})$ then $\mathcal{F}$ has property B.

In [90], Hajnal gives the entertainig tale of how he and Erdős discovered Miller's paper by accident, and realized that a new method is used in it.

Problem 39. (GCH) *Assume that* $\mathcal{F} \subseteq [\omega_{\omega+1}]^{\aleph_1}$, $|\mathcal{F}| = \aleph_{\omega+1}$, *and* $|A \cap B| < \aleph_0$ *holds for* $A \neq B \in \mathcal{F}$. *Does* $\mathcal{F}$ *have property* B *or even* $B(\aleph_1)$ *?*

In [95], Hajnal, Juhász, and Shelah gave a consistent counterexample from a supercompact cardinal. In [96], they proved the following extension of this. If GCH holds and μ is a regular cardinal with a 2-huge cardinal above it, then in a μ-closed forcing extension GCH still holds and there is an $\mathcal{F} \subseteq [\mu^{+\mu+1}]^{\mu^+}$, with $|A \cap B| < \mu$ for $A \neq B \in \mathcal{F}$ such that $\mathcal{F}$ does not have property B.

Problem 40. *Is it true that every* $\mathcal{F}$ *with* $\mathcal{F} \subseteq [S]^{\aleph_0}$, $|\mathcal{F}| < \mathfrak{c}$, *has property* B?

It is easy to see, that the answer is 'yes' if Martin's axiom MA holds. On the other hand, as Kunen showed in unpublished work, if ω_1 Silver reals are

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added with a finite support iteration to a model with an arbitrary value of $\mathfrak{c}$, then we obtain a model where there is a family $\mathcal{F} \subseteq [\omega]^\omega$, $|\mathcal{F}| = \aleph_1$, $\mathcal{F}$ does not have propert B, and $\mathfrak{c}$ is as large as wanted. Another argument was given by Baumgartner and Laver ([14]): if Sacks reals are added by a countable support iteration of length ω_2 to a model of CH, then in the resulting model $2^{\aleph_0} = \aleph_2$ and the statement of Problem 40 is also negated.

Problem 41. (GCH)
(A) *Does there exist a graph X with $|X| = \aleph_{\omega+1}$,* $\mathrm{Chr}(X) > \aleph_0$*, such that if $Y \leq X$ has $|Y| \leq \aleph_\omega$, then* $\mathrm{Chr}(Y) \leq \aleph_0$*?*
(B) *Does there exist a graph X with* $|X| = \mathrm{Chr}(X) = \aleph_2$ *such that if $Y \leq X$ has $|Y| \leq \aleph_1$ then* $\mathrm{Chr}(Y) \leq \aleph_0$*?*

(A) In [**?**] Todorcevic proves that if on some regular cardinal κ there is a nonreflecting stationary set of ω-limits, then there is an uncountably chromatic graph all whose smaller subgraphs are countably chromatic. In [178] Shelah remarks that in the model of Ben-David and Magidor ([16]) GCH holds and if $1 \leq n < \omega$ and X is a graph with $|X| = \aleph_{\omega+1}$ all whose smaller subgraphs have chromatic number at most $\aleph_n$ then $\mathrm{Chr}(X) \leq \aleph_n$. That is, there is an ultrafilter D on $\omega_{\omega+1}$ such that $|{}^{\omega_{\omega+1}}\omega_n/D| = \aleph_n$ for $1 \leq n < \omega$ and there are $A_\xi \in D$ ($\xi < \omega_{\omega+1}$) with $|\{\xi : \alpha \in A_\xi\}| < \aleph_\omega$ for $\alpha < \omega_{\omega+1}$. This readily implies the claimed result.

Spencer Unger proves in [212] that if κ is supercompact and GCH holds, then in some forcing extension $\kappa = \aleph_{\omega_1}$, GCH holds, and if $0 < \alpha < \omega_1$, X is a graph of size $\aleph_{\omega_1+1}$, all whose subgraphs of size less than $\aleph_{\omega_1}$ have chromatic number at most $\aleph_{\alpha+1}$, then $\mathrm{Chr}(X) \leq \aleph_{\alpha+1}$.

(B) Baumgartner showed in [9] the consistency of the existence of a graph X with $|X| = \mathrm{Chr}(X) = \aleph_2$, such that $\mathrm{Chr}(Y) \leq \aleph_0$ for each subgraph Y of smaller cardinality.

In [67], Foreman and Laver proved the consistency of the statement that each graph X with $|X| = \mathrm{Chr}(X) = \aleph_2$ contains a subgraph Y with $|Y| = \mathrm{Chr}(Y) = \aleph_1$ (and GCH), from the consistency of a huge cardinal. Let κ be huge, $j : V \to M$ an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) = \lambda$, ${}^\lambda M \subseteq M$. They construct forcing notions P and Q, such that $|P| = \kappa$, P is κ-c.c., in $V[G_P]$, $\kappa = \omega_1$, $Q \in V[G_P]$ is λ-c.c., of cardinality λ, Q is $< \kappa$-closed, and $\lambda = \omega_2$ in $V[G_P, G_Q]$. Further, $j(P)$ splits as $P * Q * R$ where R is κ-centred. It is fairly easy to see that the statement holds in $V[G_P, G_Q]$ (but the constructions of P and Q are involved).

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In [161], Rinot proved that if $\lambda > \omega$ is a cardinal, $\square_\lambda$ and $2^\lambda = \lambda^+$ hold, $\omega < \kappa \leq \lambda^+$ is a cardinal, then there is a graph X on λ^+ such that $\mathrm{Col}(X) = \mathrm{Chr}(X) = \kappa$ and for all subgraphs Y with $|Y| \leq \lambda$ one has $\mathrm{Chr}(Y) \leq \omega$.

Problem 42. (GCH)
(A) *Does there exist a family $\mathcal{F} \subseteq [\omega_2]^{\aleph_0}$, $|\mathcal{F}| = \aleph_2$, with no property B such that every $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \leq \aleph_1$ has property B?*
(B) *Does there exist a graph X with $|X| = \aleph_2$, $\mathrm{Col}(X) > \omega$ such that $\mathrm{Col}(Y) \leq \omega$ for every subgraph $Y \leq X$ with $|Y| \leq \aleph_1$?*
(C) *Does there exist a family $\mathcal{F} \subseteq [\omega_2]^{\aleph_0}$ with no transversal such that every $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| \leq \aleph_1$, has a transversal?*

(A) The answer is clearly 'yes' if there is a nonreflecting stationary set $S \subseteq S^{\omega_2}_\omega$ and CH holds, as then $\diamondsuit(S)$ holds by Gregory ([80]).

(B) Again, a nonreflecting stationary $S \subseteq S^{\omega_2}_\omega$ gives an example. Levy collapsing a weakly compact to ω_2 gives a model where no such graph exists and if a supercompact is collapsed to ω_2 then every graph X with $\mathrm{Col}(X) > \omega_1$ contains a subgraph Y with $|Y| = \mathrm{Col}(Y) = \omega_1$.

(C) Let $\mathrm{PT}(\lambda)$ denote the statement that if $\mathcal{F}$ is a λ-sized family of sets of cardinality $\aleph_0$, each $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| < \lambda$ has a transversal, then so has $\mathcal{F}$.

From a nonreflecting stationary subset of $S^{\omega_2}_\omega$ one can find an example witnessing $\neg\mathrm{PT}(\omega_2)$. This is in fact true in ZFC, the example given by J. Truss is $\mathcal{F} = \{F_{\alpha,\beta} : \omega \leq \alpha < \omega_1 \leq \beta < \omega_2\}$ where $F_{\alpha,\beta} = \alpha \times \{\alpha, \beta\}$. This was extended by Milner and Shelah, showing that $\neg\mathrm{PT}(\lambda)$ implies $\neg\mathrm{PT}(\lambda^+)$ ([145]). Shelah then proved that if $\aleph_\omega$ is strong limit, then $\neg\mathrm{PT}(\aleph_{\omega+1})$ holds ([171]).

In a remarkable paper Shelah proved that $\mathrm{PT}(\lambda)$ holds for each singular λ ([169]). This holds for the coloring number and a large class of other notions.

In [137], Magidor and Shelah proved $\neg\mathrm{PT}(\aleph_{\omega+1})$ without any extra assumption. From the consistency of ω supercompact cardinals, they proved $\mathrm{PT}(\aleph_{\omega^2+1})$ consistent, and they also proved (from the same assumption) that the following is consistent: if λ is the first cardinal fixed point, then λ is fully compact, i.e., if $\mathcal{F}$ is a system of sets of cardinality $\aleph_0$, $|\mathcal{F}| \geq \lambda$, with no transversal, then there is a subfamily $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| = \lambda$, with no transversals, either.

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Problem 43. *Does there exist a regressive function $f : \omega_1 - \{0\} \to \omega_1$ such that if $\alpha < \omega_1$ is a limit ordinal, then there is a sequence $\alpha_0 < \alpha_1 < \cdots$ converging to α such that $\alpha_n = f(\alpha_{n+1})$ $(n < \omega)$?*

Baumgartner noticed that an easy construction gives a function as wanted. Let $f : \omega_1 - \{0\} \to \omega_1$ be a regressive function such that if $\gamma = 0$ or limit, then for every $\beta < \gamma + \omega$ the set $\{\max(\beta, \gamma) < \alpha < \gamma + \omega : f(\alpha) = \beta\}$ is infinite. A function like this can easily be constructed and it is equally easy to see that it is as required.

Problem 44. (CH) *There is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ with $K_3, K_{\aleph_0,\aleph_0} \not\leq X$.*

In [89], Hajnal shows using CH that there is a graph X on ω_1^2, which is triangle-free, omits $K_{\aleph_0,\aleph_0}$, and has no independent set of type ω_1^2. The latter obviously implies $\mathrm{Chr}(X) = \aleph_1$.

Problem 45.
(A) *If X is a graph with $\kappa = \mathrm{Chr}(X)$ infinite, then there is a triangle-free subgraph $Y \leq X$ with $\mathrm{Chr}(Y) = \kappa$.*
(B) *There is a function $f : \omega \to \omega$, $\lim f(k) = \infty$, such that each graph with $\mathrm{Chr}(X) = k$ contains a triangle-free subgraph Y with $\mathrm{Chr}(Y) = f(k)$.*

(A) Komjáth and Shelah proved that consistently there is a graph X with $|X| = \mathrm{Chr}(X) = \aleph_1$ all whose triangle-free subgraphs are countably chromatic ([119]).

(B) This was proved by Rödl ([163]).

Erdős suggested a more general conjecture, namely, if $3 \leq n < \omega$, the chromatic number of X is $\aleph_0$, then there is a subgraph Y of X, with $\mathrm{Chr}(Y) = \aleph_0$ and Y omittings circuits of length $3, 4, \ldots, n$ (see, e.g., in [36]).

Problem 46. *If X is an uncountably chromatic graph, then X contains all sufficiently long odd circuits.*

This was proved independently by Erdős–Hajnal–Shelah ([59]) and Thomassen ([195]).

Problem 47. *Let X be a graph with* $\mathrm{Chr}(X) = \aleph_0$*, N the set of lengths of circuits contained in X, does*

$$\sum_{i \in N} \frac{1}{i} = \infty$$

hold?

Gyárfás, Komlós, and Szemerédi proved that for any finite graph this sum is at least $c \log k$, where k is the least degree and $k \geq k_0$ for some value k_0 ([81]). As by the Erdős–de Bruijn theorem a graph with chromatic number $\aleph_0$ contains finite subgraphs with arbitrarily large finite chromatic number, which in turn contain finite subgraphs with arbitrarily large minimal degree (in a minimal graph of chromatic number k each degree is at least $k-1$), the statement follows.

An extension given also by Erdős and Hajnal has recently been solved by Liu and Montgomery, who proved that if $\mathrm{Chr}(X) = k$, then

$$\sum_{n \in N, n \equiv 1 \pmod 2} \frac{1}{n} \geq \left(\frac{1}{2} - o_k(1) \right) \log k$$

(see [135]).

In fact, they prove that for some $d_0 > 0$, if X is a finite graph with average degree $d \geq d_0$, then there is some $k \geq d/(10 \log^{10} d)$, such that N contains all even integers in $[\log^8 k, k]$.

Problem 48. (GCH)
(A) *There is a graph of cardinality $\aleph_2$, with coloring number $> \aleph_0$, all whose smaller subgraphs have coloring number at most $\aleph_0$.*
(B) *There is a graph of cardinality $\aleph_{\omega+1}$, with coloring number $> \aleph_1$, all whose subgraphs of cardinality $\leq \aleph_1$ have coloring number at most $\aleph_0$.*

Shelah proved in [167] that, if V=L holds, $\kappa > \omega$ is regular, not weakly compact, then there is a graph X with $|X| = \kappa$, $\mathrm{Col}(X) > \omega$, such that for each subgraph Y with $|Y| < \kappa$, one has $\mathrm{Col}(Y) \leq \omega$.

The complementary independence result is proved by Komjáth: if $\mu < \kappa$, κ is weakly compact (supercompact) then in some model $\kappa = \mu^{++}$ and for every graph X of cardinality κ (of any cardinality) if every subgraph Y of X with $|Y| < \kappa$ has $\mathrm{Col}(Y) \leq \mu$, then $\mathrm{Col}(X) \leq \mu$ ([114]).

In [127] Lambie-Hanson and Rinot proved that if $\mu < \kappa$, κ is regular, X is a graph on κ with $\mathrm{Col}(Y) \leq \mu$ for each $Y \leq X$ with $|Y| < \kappa$, then
(a) $\mathrm{Col}(X) \leq \mu^+$, unless $\kappa = \lambda^+$, $\mathrm{cf}(\lambda) = \mathrm{cf}(\mu)$,
(b) $\mathrm{Col}(X) \leq \mu^{++}$ anyway.
Further, if X is a graph on $\omega_{\omega+1}$ whose all smaller subgraphs have countable coloring number and the Chang conjecture $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds, then $\mathrm{Col}(X) \leq \omega_1$.

Problem 49. (GCH) *Is there a number $l < 2k-2$ such that if X is a graph $|X| = \aleph_{\omega+1}$ such that for every $Y \leq X$, $|Y| \leq \aleph_\omega$, one has $\mathrm{Col}(Y) \leq k$, then $\mathrm{Col}(X) \leq l$?*

In [47], Erdős and Hajnal proved that if X is an infinite graph, $1 \leq k < \omega$, each finite subgraph Y has $\mathrm{Col}(Y) \leq k$, then $\mathrm{Col}(X) \leq 2k-1$. This is sharp in the sense that for each $n < \omega$ there is a graph X of size $\aleph_n$ with each smaller subgraph Y having $\mathrm{Col}(Y) \leq k$, but $\mathrm{Col}(X) \leq 2k-1$. The problem asks if there is a similar example of size $\aleph_{\omega+1}$. These results show that the Erdős–de Bruijn theorem holds for the coloring number in a weaker form.

Problem 50. (GCH) *If X is a graph with $|X| \leq \aleph_{\omega+1}$ and $K_{\aleph_0,\aleph_2} \not\leq X$, then $\mathrm{Col}(X) \leq \aleph_1$.*

Shelah in [166] proved the following.

Theorem. (GCH) *The following are equivalent.*
(a) *There is a graph X, $|X| = \aleph_{\omega+1}$, $\mathrm{Col}(X) > \aleph_1$, $K_{\aleph_1,\aleph_0} \not\leq X$.*
(b) *There is a graph X, $|X| = \aleph_{\omega+1}$, $\mathrm{Col}(X) > \aleph_1$, $K_{\aleph_2,\aleph_0} \not\leq X$.*
(c) *There is a graph X, $|X| = \aleph_{\omega+1}$, $\mathrm{Col}(X) > \aleph_1$, for every subgraph $Y \leq X$, $|Y| \leq \aleph_2$, $\mathrm{Col}(Y) \leq \aleph_1$ holds.*
(d) *There is a stationary set $S \subseteq S^{\omega+1}_1$ and there are sets $\{A_\alpha : \alpha \in S\}$ such that $A_\alpha \subseteq \alpha$, $\mathrm{tp}(A_\alpha) = \omega_1$, $\sup(A_\alpha) = \alpha$, and $|A_\alpha \cap A_\beta| < \omega$ $(\alpha \neq \beta)$.*

The consistency of (d) was then proved by Hajnal, Juhász, and Shelah in [95] from the consistency of the existence of a supercompact.

Problem 51. *There is a graph X with $|X| = \mathfrak{c}^+$ with $K_\omega \not\leq X$ such that if the edges of X are colored with $\aleph_0$ colors then in some color there is a K_k for each $k < \omega$.*

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Problem 52. *There is a graph X with $|X| = \mathfrak{c}^+$ with $K_{\aleph_1} \not\leq X$ such that if the edges of X are colored with $\aleph_0$ colors then in some color there is a K_ω.*

The relevance of $\mathfrak{c}^+$ is that no graph of cardinality $\leq \mathfrak{c}$ may have those properties as $K_\mathfrak{c}$ is the union of $\aleph_0$ bipartite graphs. For more comments see Problem 53.

Problem 53. *Is there a graph X with $K_4 \not\leq X$ with the property that in every countable coloring of the edges of X there is a monocolored triangle?*

Shelah proved the consistency of the existence of such a graph in [177]. The existence of a graph like this in ZFC is open. In [120], Komjáth and Shelah gave the consistency of the following. If X is a graph, μ a cardinal, then there is a graph Y such that $Y \to (X)^2_\mu$ and if $K_\alpha \not\leq X$, then $K_\alpha \not\leq Y$.

Erdős and Hajnal remarked that if κ is infinite, then there is a graph X omitting $K_{(2^\kappa)^+}$ with $X \to (\kappa^+)^2_\kappa$: $(2^\kappa)^+ \not\to ((2^\kappa)^+, (2^\kappa)^+)^2$ and $(2^\kappa)^+ \to ((2^\kappa)^+, (\kappa^+)^2_\kappa)^2$. If X consists of the edges of color 1 in the coloring witnessing the first example, then we are done.

In [115], I modified this as follows. Given κ, let λ be such that $\lambda^\kappa < \lambda^{\kappa^+}$ (e.g., $\lambda = \exp_{\kappa^+}(\kappa)$). Then partition theory gives $\lambda^+ \to (\lambda^+, (\kappa^+)_\kappa)^2$ (Erdős–Rado) and $\lambda^{\kappa^+} \not\to (\lambda^+, \kappa^{++})^2$. By the condition imposed on λ, we also have $\lambda^+ \not\to (\lambda^+, \kappa^{++})^2$. If X is the graph of the edges of color 1 in the latter example, then $K_{\kappa^{++}} \not\leq X$ and $X \to (\kappa^+)^2_\kappa$.

Problem 54. *If $2 \leq n < \omega$, $3 \leq k < \omega$, then there is a finite graph X with $K_{k+1} \not\leq X$ and with the property that if the edges of X are colored with n colors, then there is a monocolored K_k.*

An example for the simplest interesting case, $n = 2$, $k = 3$ was given by Jon Folkman in [64]. His proof was specific for the values of n, k and the given graph had a stellar number of vertices.

Erdős asked repeteadly if there is a significatly smaller such graph, with, say, less than 10^{10} vertices. Frankl and Rödl constructed an example with roughly 10^{11} vertices. In the years around 2000 this was first decreased to around 10,000 by L. Lu, then to 941 by Dudek and Rödl ([29]), then to 786 by Lange, Radziszowski, and Xu in [128].

The general case was then proved by Nesteril and Rödl, in fact, they proved that if Y is a finite graph containing no K_{k+1}, $r < \omega$, then there is a finite graph X omitting K_{k+1} such that if the edges of X are colored with r colors, then there is a monocolored, induced copy of Y ([148],[149]).

Amazingly, there are pure existence proofs of graphs asked in the Problem. One is given Shelah's consistent existence of a graph as in Problem 53 ([177]). Namely, that graph X has the (weaker) property that, if the edges are colored with 2 colors, then always exists a monocolored triangle. By compactness, there is a finite subgraph Y of X with this property. As by forcing no finite graph can be added, Y is already in the ground model.

For the Folkman case there is an even trickier argument. In [11] Baumgartner and Hajnal proved that $\omega_1^2 \to (\omega_1\omega, 3, 3)^2$, and under CH, the relation $\omega_1^2 \not\to (\omega_1\omega, 4)^2$ holds. The proof of the second statement is relatively straightforward, that of the first one is quite complicated. The second statement can be reformulated as follows. There is a K_4-free graph X on ω_1^2 with no indepedent set of type $\omega_1\omega$. By the first theorem each such graph has the property that under any 2-coloring of the edges there is a monochromatic triangle. Again, then X is a K_4-free graph having monochromatic triangles in any 2-colorings of the edges, by compactness X has a finite subgraph Y with similar properties. So, Y exists in any model of CH. As CH can be added by forcing, each model has a forcing extension having the above Y, and then each model must have it. Hajnal was very proud of this argument all his life.

It is unknown if $\omega_1^2 \to (\omega_1\omega, 3, 3, 3)^2$ holds or at least if it is consistent that $\alpha \not\to (\beta, 4)^2$ and $\alpha \to (\beta, 3, 3, 3)^2$, for some ordinals α, β.

In order to formulate the following problem, we write $a \to [b, c]_e^d$ if the following holds. If $\mathcal{F} \subseteq \mathcal{P}(a)$, $|\mathcal{F}| = e$, then either there is $S \in [a]^b$, such that for every $X \in [S]^d$, there is an $A \in \mathcal{F}$, $X \subseteq A$, or else there is $T \in [a]^c$, $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| = e$, $T \cap \bigcup \mathcal{F}' = \emptyset$.

Problem 55.
(A) $\aleph_2 \to [\aleph_2, a]_{\aleph_2}^{\aleph_0}$ for $a = \aleph_1$ or $a = \aleph_2$?
(B) $\aleph_{\omega+1} \to [\aleph_{\omega+1}, \aleph_0]_{\aleph_{\omega+1}}^{\aleph_o}$?
(C) $\aleph_{\omega_1} \to [\aleph_{\omega_1}, \aleph_0]_{\aleph_1}^{\aleph_0}$?

These problems were raised in [56]. In [52] Erdős and Hajnal mention that (B) and (C) can be proved relatively easily. I proved that (A) is false, i.e., GCH is consistent with the negation of the statement in (A) with $a = \aleph_2$ ([117]).

In [45] Erdős and Hajnal considered the following problem. Let κ be a cardinal, f a function on $[\kappa]^k$ with $f(s)$ a Lebesgue measurable subset of $[0,1]$, $\lambda(f(s)) \geq u$. Is there a subset $A \in [\kappa]^\mu$ with $\bigcap\{f(s) : s \in [A]^k\} \neq \emptyset$? We write $(\kappa, u)^k \to \mu$ if the answer is 'yes'.

Problem 56. *Does* $(\mathfrak{c}, u)^2 \to \aleph_1$ *hold for some* $u > \frac{1}{2}$ *in ZFC ?*

Problem 57.
(A) $(\aleph_1, u)^3 \to 4$ for $u > 0$?
(B) $(\aleph_0, u)^2 \to K_{\aleph_0,\aleph_0}$ for $u > \frac{1}{2}$?

Problem 58. *Let* S *be a set with* $|S| > \mathfrak{c}$. *Does there exist a partition* $[S]^{\aleph_0} = \bigcup\{I_\xi : \xi < \mathfrak{c}\}$ *such that if* $\{A_n : n < \omega\} \subseteq [S]^2$ *are disjoint, then for every* $\xi < \mathfrak{c}$ *there is an* $X \in I_\xi$ *which is a transversal of* $\{A_n : n < \omega\}$*?*

This is solved in [75].

Problem 59. Does

$$\begin{pmatrix} \aleph_0 \\ \aleph_1 \end{pmatrix} \to \begin{pmatrix} 1 & \aleph_0 \\ 4 & \aleph_0 \end{pmatrix}^{1,2}$$

hold?

Problem 60. Does

$$\begin{pmatrix} \aleph_1 \\ \aleph_0 \end{pmatrix} \to \begin{pmatrix} 1 & \aleph_0 \\ \aleph_0 & \aleph_0 \end{pmatrix}^{1,2}$$

hold?

Problem 61. *Do there exist circuitfree graphs* $\{X_n : n < \omega\}$ *on* ω_1 *such that if* $A \in [\omega_1]^{\aleph_1}$, *then* $\{n < \omega : X_n \cap [A]^2 = \emptyset\}$ *is finite?*

As Erdős and Hajnal remarked in [52], CH implies a 'yes' answer. In [116] we give this (unpublished) result and show that Martin's axiom implies a 'no' answer.

Problem 62. *Assume that* $2 < r < \omega$ *and* $\lambda = \exp_{r-1}(\aleph_0)$. *Is there a coloring* $F : [\lambda]^r \to 2$, *such that*
(1) *there is no homogeneous set of size* $\aleph_1$, *and*
(2) *if* $S \in [\lambda]^{\aleph_1}$, $n < \omega$, $i < 2$, *then there is an* n*-element subset of* S, *homogeneous in color* i*?. Or,*

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(2') *if $S \in [\lambda]^{\aleph_1}$, $i < 2$, then there is an infinite subset of S, homogeneous in color i?*

We say that a set system $\mathcal{F}$ posesses property $B(a, b)$ if for every $\mathcal{F}' \in [\mathcal{F}]^a$, $b' < b$, there is $\mathcal{F}'' \in [\mathcal{F}']^{b'}$ with $\bigcap \mathcal{F}'' \neq \emptyset$.

$(m, n) \to (a, b)$ denotes the statement that the Cartesian product of two systems with $B(a, b)$ has $B(m, n)$.

Problem 63. *Let $3 \leq r < \omega$. Does*

$$(\exp_{r-1}(\aleph_0), r+1) \not\to (\aleph_1, \aleph_0)$$

or

$$(\exp_{r-1}(\aleph_0), r+1) \not\to (\aleph_1, \aleph_1)$$

hold?

Problem 64. *Are there systems $\mathcal{F}_1$, $\mathcal{F}_2$ having property $B(\aleph_1, \aleph_0)$ such that $\mathcal{F}_1 \times \mathcal{F}_2$ does not have $B(\exp_k(\aleph_0), \aleph_0)$ for any $k < \omega$?*

Problem 65. *Assume that $F_n : \omega_1 \times \omega \to 2$ $(n < \omega)$. Do there exist $A \in [\omega_1]^\omega$, $i < 2$, $B_k \in [\omega]^\omega$, $n_0 < n_1 < \cdots$ such that $A \times B_k$ is homogeneous in color i for F_{n_k}?*

As it was remarked in [52], this follows immediately from

$$\begin{pmatrix} \omega_1 \\ \omega^2 \end{pmatrix} \to \begin{pmatrix} \omega \\ \omega^2 \end{pmatrix}_2^{1,1}.$$

of [10].

Problem 66. (Erdős, Hajnal, Milner) *Let X be a graph on an ordered set of ordinal ω_1^ρ for some $\rho < \omega_2$. Is it true that if X does not contain an infinite path, then there is an independent set of type ω_1^ρ?*

Problem 67. (Erdős, Hajnal, Milner) *Let X be a graph on the ordered set V of type θ. Assume that X does not contain a C_4 and $\mathrm{tp}(V - \{v\}) = \theta$ for every $v \in V$. Does X have an independent set of type θ?*

Laver gave an affirmative answer to this. He also proved that if φ is a σ-scattered order type, i.e., $\varphi \not\to (\eta)^1_{\aleph_0}$, then $\varphi \to [\varphi]^1_n$ holds for some $n < \omega$ ([132]).

Problem 68. (GCH) *Assume that $F : [\omega_1]^2 \to 3$ is such that F assumes all values on any uncountable subset of ω_1. Do there exist $\alpha < \beta < \gamma < \omega_1$ with $\{F(\alpha, \beta), F(\alpha, \gamma), F(\beta, \gamma)\} = \{0, 1, 2\}$?*

Raised in [49]. In [91], Hajnal attributes this beautiful question to Erdős.

Shelah in [168] proved that CH gives a counterexample. In fact, this is an easy application of the existence of a Lusin set.

Shelah also proved, that if V=L, then for every regular cardinal κ, there is a function $F : [\kappa^+]^2 \to \kappa^+$ witnessing $\kappa^+ \not\to [\kappa^+]^2_{\kappa^+}$ with no triangles of three colors. Todorcevic in [196] extended this to all regular, not weakly compact cardinals.

In [116] I proved that a κ-Suslin tree implies the existence of a function $F : [\kappa]^2 \to \kappa$ witnessing $\kappa \not\to [\kappa]^2_\kappa$ with no 3-colored triangles. On the other hand, if such a coloring exists, then there is a κ-Aronszajn tree.

Erdős asked the following finite version of the problem. Is there an $\varepsilon > 0$ such that the following holds? If $F : [n]^2 \to 3$ is such that in every $A \subseteq n$, $|A| \geq n^\varepsilon$, all three colors occur, does then F necessarily contain a 3-colored triangle? This was answered by Shelah in [168] with the value $\varepsilon = 1/12$. The bound was improved by Fox, Grinshpun, and Pach in [70] to $|A| \geq n^{1/3} \log^2 n$ and this, apart from a constant, is sharp.

Problem 69. *Let X be a graph on ω_1. Assume that for every $A \in [\omega_1]^{\aleph_1}$ there is a finite $s \subseteq A$ such that each element of $A - s$ is joined to some element of s. Does X nevessarily contain $K_{\aleph_1}$?*

In [116] I proved that both this statement and its negation are consistent.

If $\langle V, X\rangle$ is a graph, $\mathcal{F}$ a set system, we say that $\mathcal{F}$ κ-represents X iff there is a bijection $f : V \to \mathcal{F}$ such that

$$\{x, y\} \in X \text{ iff } |f(x) \cap f(y)| < \kappa.$$

Erdős and Hajnal proved that if κ is regular, then every graph with at most κ^+ vertices can be κ-represented by some $\mathcal{F} \subseteq \mathcal{P}(\kappa)$.

Problem 70. (GCH) *Let X be a graph on $\aleph_{\omega+1}$ vertices. Can it be $\aleph_\omega$-represented by a family $\mathcal{F} \subseteq \mathcal{P}(\omega_\omega)$?*

The above mentioned regular case was finally published in [61] (Lemma 6).

Let A be a set and $\mathcal{F} \subseteq \mathcal{P}(A)$. Let $a = \{a_\xi : \xi < \varphi\}$ be a set of elements of A. We say that $\mathcal{F}$ *strongly cuts* a if for each $\xi < \varphi$ there is an $A_\xi \in \mathcal{F}$ such that $A_\xi \cap a = \{a_\eta : \eta < \xi\}$.

In [60] Erdős and Makkai proved that if A is infinite, $|\mathcal{F}| > |A|$, then there is a sequence of type ω which is either strongly cut by $\mathcal{F}$ or is strongly cut by the set of complements of $\mathcal{F}$.

Problem 71. (Erdős, Makkai)
(A) *Assume $|A| = \aleph_1$, $|\mathcal{F}| > \aleph_1$. Does there exist a sequence of length ω strongly cut by $\mathcal{F}$?*
(B) *Assume $|A| = \aleph_1$, $|\mathcal{F}| > \aleph_1$. Does there exist a sequence of length ξ, $\omega + 2 \leq \xi \leq \omega_1$, which is strongly cut by either $\mathcal{F}$ or by the set of complements?*

Concerning (A), Shelah proved that for every infinite cardinal λ there is a set S, $|S| = \lambda$, and a set system $\mathcal{H} \subseteq \mathcal{P}(S)$ with no sequences $a = \{a_i : i < \omega\} \subseteq S$ and $\{H_i : i < \omega\} \subseteq \mathcal{H}$ such that $H_i \cap a = \{a_j : j < i\}$ $(i < \omega)$. Shelah further proved that, if GCH holds, $|A| = \kappa^{+3}$, $\mathcal{F} \subseteq \mathcal{P}(A)$, $|\mathcal{F}| = \kappa^{+4}$, then there is a sequence of elements of A, of length κ^+, which is strongly cut either by $\mathcal{F}$ or by the set of complements in $\mathcal{F}$ ([165]).

As for (B), one can easily see that if $\binom{\aleph_2}{\aleph_1} \not\to \binom{\aleph_0}{\aleph_1}^{1,1}_2$ holds, then there is a set system $\mathcal{F} \subseteq \mathcal{P}(\omega_1)$, $|\mathcal{F}| = \aleph_2$ so that no sequence of length ω_1 is strongly cut by $\mathcal{F}$ or by the set of complements of $\mathcal{F}$ (cf Prikry's result in Problem 24).

The same holds if CH fails by an easy construction. Namely, if $\mathcal{F} \subseteq [\omega]^\omega$ is an almost disjoint set system, $|\mathcal{F}| = \aleph_2$, then $\mathcal{F}$ establishes that (B) fails with $\xi = \omega + 2$. The consistency of the positive statement is not known.

Problem 72. *Let X be a graph of size $\aleph_1$ which does not contain a $K_{\aleph_1}$. Does then the complement of X contain a topological $K_{\aleph_1}$?*

This extends a result in [44], that for every $F : [\omega_1]^2 \to k$ (k finite) there is a monocolored topological $K_{\aleph_1}$. In [52] Erdős and Hajnal notice that the comparison graph of a Suslin tree is a counterexample. Komjáth and Shelah in [122] prove that the statement holds exactly if there is no Suslin tree. They further prove that the statement that if $F : [\omega_2]^2 \to \omega$, then there is a monocolored topological $K_{\aleph_2}$ is both consistent and independent.

Recently, a paper by Bergfalk, Hrušak, and Shelah considers a similar problem: if κ is an infinite cardinal, $F : [\kappa]^2 \to \mu$, is there a monocolored κ-connected graph [18]? It is easy to see that a yes answer gives a monochromatic topological K_κ.

The answer is 'yes', if μ is finite, by the proof of Erdős–Hajnal [44]. If $\kappa = \aleph_2$, $\mu = \aleph_0$, both answers are consistent. Bergfalk, Hrušak, and Shelah also deduce a negative answer from a form of $\square$, considered by Todorcevic. Finally, they prove that if $\lambda^\kappa = \lambda$, $F : [\lambda^+]^2 \to \kappa$, then there is a λ-connected, monochromatic subgraph of size λ.

Problem 73. *If κ is a strongly inaccessible, not weakly compact cardinal, then there is a set system $\mathcal{F} \subseteq [\kappa]^{<\kappa}$, $|\mathcal{F}| = \kappa$, such that $A \not\subseteq B$ for $A \neq B \in \mathcal{F}$, and if $\mathcal{F}' \in [\mathcal{F}]^\kappa$, then $|\kappa - \bigcup \mathcal{F}'| < \kappa$.*

The affirmative answer (based on the existence of a κ-Aronszajn tree) was given by Erdős and Hajnal in [53].

Problem 74. (Erdős, Rado) (GCH) *Is there a universal graph of cardinality $\aleph_\omega$ in which every vertex has degree $< \aleph_\omega$?*

This was answered in the affirmative by Shelah in [166] for any singular cardinal in place of $\aleph_\omega$. Next, Komjáth and Pach discussed under GCH when a universal graph of cardinality κ omitting $K_{n,\lambda}$ (n finite, $\omega \leq \lambda \leq \kappa$) exists ([118]). Finally, in [185], Shelah completely settled (again under GCH) when a universal $K_{\alpha,\beta}$-omitting graph of size κ exists.

Problem 75. (Erdős, Milner) (GCH) *Let $\mathcal{F} \subseteq [\omega_\omega]^{\aleph_0}$ be a family, $|\mathcal{F}| = \aleph_{\omega+1}$. Does there exist a partition $\omega_\omega = A \cup B \cup C$ such that $|C| \leq \aleph_0$, and both $A \cup C$ and $B \cup C$ contain $\aleph_{\omega+1}$ elements of $\mathcal{F}$?*

Shelah remarked that an ω_1-descendingly complete uniform ultrafilter on ω_ω gives a counterexample.

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Problem 76. (Erdős) *If $\kappa \leq \mathfrak{c}$ is an uncountable cardinal, can there be a set $\mathcal{F}$ of κ entire functions, such that*

$$\left|\{f(a) : f \in \mathcal{F}\}\right| < \kappa$$

holds for every $a \in \mathbb{C}$?

This was asked by J. Wetzel for $\kappa = \aleph_1$. It is easy to see that an affirmative answer can only hold for $\kappa = \mathfrak{c}$. Erdős proved that the $\kappa = \aleph_1$ case is equivalent to the continuum hypothesis ([35]).

Erdős's problem has recently been settled by Kumar and Shelah in [124]. First they show that a 'no' answer is consistent with any possible value of $\mathfrak{c}$ with $\mathrm{cf}(\mathfrak{c}) > \omega_1$. In fact, this holds if $\aleph_1$ Cohen reals are added to a model of $\mathrm{cf}(\mathfrak{c}) > \omega_1$. The proof utilizes the fact that a nonzero entire function cannot have more roots in a larger universe. Then they prove that a 'yes' answer is consistent with $\mathfrak{c} = \aleph_{\omega_1}$.

Problem 77. (de Groot, Efimov, Isbell) *Does there exist a Hausdorff space of $\mathfrak{c}^+$ points with no uncountable discrete subspace?*

In [92] Hajnal and Juhász proved that a Hausdorff space with no discrete subspace of cardinality $> \kappa$ has at most 2^{2^κ} points. In [198], Todorcevic showed the consistency of the statement that if a Hausdorff space does not contain uncountable discrete subspaces then it has at most $\mathfrak{c}$ points.

Problem 77/A. (Hajnal, Juhász) (GCH) *Let $\langle R, < \rangle$ be an ordered set with $|R| = \aleph_2$ such that each point has character $\aleph_0$. Do necessarily exist $\aleph_2$ disjoint open intervals of $\langle R, < \rangle$?*

Jensen proved that a tree with certain properties, whose existence follows from $V = L$, implies a counterexample. In [63], Fleissner gives Jensen's argument and further proves that such a tree can be constructed from an ω_2-Suslin-tree.

Problem 78. (Hajnal, Juhász) *Does there exist a hereditarily separable Hausdorff space of cardinality greater than $\mathfrak{c}$?*

Juhász and Shelah in [107] prove the following result. Assume GCH and that $\aleph_1 < \lambda < \mu$ are regular cardinals. Then in some cardinal preserving extension $\lambda^{<\lambda} = \lambda = \mathfrak{c}$, $2^\lambda = \mu$, and there is a hereditarily separable Hausdorff

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space $X \subseteq {}^{\lambda}2$, $|X| = \mu$. Todorcevic proved in [198] that PFA implies a NO answer.

Problem 79. (Hajnal, Juhász) (GCH) *Does there exist a regular space of size $\aleph_1$, all whose subspaces of cardinality $\aleph_1$ have weight $\aleph_2$?*

Hajnal and Juhász announced in [93] and published in [94] the consistency of a 'yes' answer. They formulated a combinatorial statement implying the existence of a space as required and then gave a forcing proof for the consistency of the statement. In [108], Kanamori shows that the statement follows from Silver's principle W, holding in L.

Problem 80. (Hajnal) *Let κ be the first weakly inaccessible cardinal, $S \subseteq \kappa$ a stationary set. Does there exist a matrix $\{A_{\alpha,\beta} : \alpha < \beta \in S\}$ of subsets of κ, such that:*
(A) $A_{\alpha,\beta} \cap A_{\alpha,\gamma} = \emptyset$ $(\alpha < \beta < \gamma,\ \beta, \gamma \in S)$,
(B) $|\kappa - \bigcup\{A_{\alpha,\beta} : \alpha < \beta\}| < \kappa$ $(\beta \in S)$?

Hajnal proved this if κ is weakly inaccessible and not ω-weakly Mahlo or if κ is strongly inaccessible and not ω-strongly Mahlo ([86]). He also proved that no such matrix exists if κ is weakly compact. The results of [86] combined with results of Jensen's give that if V=L, then the statement holds for every regular, not weakly compact cardinal. Hajnal also proved that there is a matrix as above if and only if there is a stationary set $S \subseteq \kappa$ such that $S \cap \mu$ is nonstationary for every regular $\mu < \kappa$ ([86]).

In [138], Mekler and Shelah defined the large cardinal notion of a *reflection cardinal* and prove that if V=L, then no cardinal which is at most the first greatly Mahlo cardinal is reflection cardinal and the consistency of a reflection cardinal implies the consistency of a cardinal in which every stationary set reflects in a regular cardinal. They also proved that, assuming the consistency of a reflection cardinal, it is consistent that each stationary subset of the first ω-Mahlo cardinal reflects in a regular cardinal.

Recently, Inamdar and Rinot have used the main lemma of Hajnal's paper in [100].

Problem 81. (S. Ulam, [211]) *Are there $\aleph_1$ σ-additive 0-1 measures on ω_1 such that each subset is measurable with respect to one of them?*

Erdős and Alaoglu proved that countably many measures do not suffice, see [34].

In [155], Prikry deduced a "no" answer to Ulam's problem from his principle $(*)$ quoted at Problem 24. Indeed, let $\{A(\alpha,\xi) : \alpha < \omega_2, \xi < \omega_1\}$ be a set system as there. Assume that $\{\mu_\tau : \tau < \omega_1\}$ are measures such that each subset of ω_1 is measurable with respect to one of them. An easy argument gives that there are $Z \in [\omega_2]^{\aleph_0}$, $\tau < \omega_1$, $\xi_\alpha < \omega_1$ $(\alpha \in Z)$, $i < 2$, such that $\mu_\tau(A_{\alpha,\xi_\alpha}) = i$ $(\alpha \in Z)$.

If $i = 0$, set $X = \bigcup\{A(\alpha,\xi_\alpha) : \alpha \in Z\}$. Now $\mu_\tau(X) = 0$ and $|\omega_1 - X| \le \aleph_0$, therefore $\mu_\tau(\omega_1 - X) = 0$ and so $\mu_\tau(\omega_1) = 0$, a contradiction.

If $i = 1$, consider $Y = \bigcap\{A(\alpha,\xi_\alpha) : \alpha \in Z\}$. Again, $\mu_\tau(Y) = 1$ and Y is countable, a contradiction.

In [158], Prikry deduced a "no" answer to the problem from the existence of a Kurepa tree.

In [193], [194] Taylor proved that Ulam's problem is equivalent to the existence of a countably complete, $\aleph_1$-dense ideal on ω_1 and MA_{ω_1} implies the nonexistence of such an ideal.

In the 1970's Woodin deduced the consistency of the existence of a countably complete, $\aleph_1$-dense ideal on ω_1 from the consistency of $ZF + AD_{\mathbb{R}} + \theta$ is regular. Later he proved this from the consistency of an almost huge cardinal. Shelah had a different proof for the statement of the Problem around 1985 (see [**?**], [181]). The consistency of the statement that the stationary ideal on ω_1 is $\aleph_1$-dense was then shown to follow from the consistency of the existence of infinitely many Woodin cardinals. In the other direction, Deiser and Donder proved that the consistency of Ulam's problem implies the consistency of the existence of an inaccessible stationary limit of measurable cardinals, see [23].

Finally, Woodin proved the following.

Theorem. (Woodin, [213]) *The following are equiconsistent:*
(1) ZF+AD,
(2) ZFC *and the stationary ideal on* ω_1 *is* $\aleph_1$*-dense,*
(3) ZFC *and there is a normal, uniform, countably dense ideal on* ω_1.

Problem 82. (L. Gillman) *Let I be a nonprincipal prime ideal on ω_1. Does there exist an $I' \subseteq I$, $|I'| = \aleph_1$ such that $\bigcup I'' = \omega_1$ for every infinite $I'' \subseteq I'$?*

In other words, is every uniform ultrafilter on ω_1 regular, i.e., (ω,ω_1)-

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regular? It was also raised by Keisler in [111].

Prikry gave an affirmative answer, assuming V=L, see [154]. His proof essentially shows that if V=L and κ is regular, then all uniform ultrafilters on κ^+ are (κ, κ^+)-regular. Benda showed that a weak form of Kurepa Hypothesis suffices ([15]). Ketonen relaxed the assumption to $\neg O^\sharp$ ([112]). This was further relaxed to $\neg L^\mu$ by Jensen (see [28]).

On the positive direction, Magidor proved that if the existence of a huge cardinal is consistent then so is that there is a nonregular ultrafilter on $\aleph_2$ ([136]). Finally, Foreman, Magidor, and Shelah proved the consistency of a nonregular ultrafilter on $\aleph_1$ from the same hypothesis ([69]).

New problems from [52]

Problem I. (GCH)
(a) *If $F : [\omega_2]^2 \to 2$ establishes $\aleph_2 \not\to (\aleph_2)^2_2$, then there is $Z \in [\omega_2]^{\aleph_1}$ such that $F|Z$ establishes $\aleph_1 \not\to (\aleph_1)^2_2$.*
(b) *If φ is an order type, $|\varphi| = \aleph_2$, $\omega_2, \omega_2^* \not\leq \varphi$ then there is an order type $\psi \leq \varphi$, $|\psi| = \aleph_1$, $\omega_1, \omega_1^* \not\leq \psi$.*

Sierpiński's argument shows that (a) implies (b): if $(\omega_2, \prec)$ is a counterexample to (b), set $F(\alpha, \beta) = 1$ for $\alpha < \beta < \omega_2$ iff $\alpha \prec \beta$. It is easy to check that (a) holds in the model of Foreman and Laver in [67] (from a huge cardinal) where GCH also holds.

Devlin constructed a counterexample to (b) from a Kurepa tree without Aronszajn subtrees, whose existence in L was earlier proved by Jensen. Combining methods of Mitchell and Silver and using the existence of a Ramsey cardinal, Devlin also proved the existence of a model of $\mathfrak{c} = \aleph_2$, CC, and (b). See [26].

In [196], Todorcevic proved that the negation of (b) holds exactly if either there is a Kurepa tree with no ω_1-Aronszajn subtree or there is an ω_2-Aronszajn tree with no ω_1-Aronszajn or ω-Cantor subtree. He also proves that $\Box_\kappa$ implies the existence of a κ^+-Aronszajn tree with neither λ-Aronszajn subtree ($\lambda \leq \kappa$), nor ν-Cantor subtree (any ν). Further, from the existence of a weakly compact cardinal, GCH is consistent with the folllowing statement: if $|\varphi| = \aleph_2$ and $\omega_2, \omega_2^* \not\leq \varphi$, then φ contains each order type of cardinality $\aleph_1$.

Problem II. *Does there exist a partition $F : [\omega_1]^2 \to 2$ establishing $\aleph_1 \not\to (\aleph_1)^2_2$ such that for every $Z \in [\omega_1]^{\aleph_1}$, $i < 2$, there are $A, B \in [Z]^{\aleph_1}$, with $F(x, y) = i$ $(x \in A, y \in B)$.*

Galvin and Shelah answered this in the positive, see [76].

Problem III. (GCH) *If $F : [\omega_2]^2 \to \omega$, then there are $A \in [\omega_2]^{\aleph_2}$, $n < \omega$, such that for every $B \in [A]^{\aleph_2}$, F assumes the value n on $[B]^2$.*

Problem IV. *Let $X \subseteq [\omega_1]^2$ be such that for each $S \in [\omega_1]^{\aleph_1}$, $\alpha < \omega_1$, there is $T \subseteq S$, $\mathrm{tp}(T) = \alpha$, $[T]^2 \subseteq X$. Decompose X as $X = X_0 \cup \cdots \cup X_{n-1}$ $(n < \omega)$. Does there exist an $i < n$ such that for every $\alpha < \omega_1$, there is T, $\mathrm{tp}(T) = \alpha$, $[T]^2 \subseteq X_i$?*

Erdős and Prikry raised the following question. If $F : [\omega]^\omega \to \omega$, then there are $A, B \in [\omega]^\omega$, $A \neq B$, $F(A) = F(B) = F(A \cup B)$.

This was proved by György Elekes in [30], [31].

Problem V. (Erdős–Prikry) *Assume $F : [\omega_1]^{\aleph_1} \to \omega_1$. Do there exist A, B with $F(A) = F(B) = F(A \cup B)$?*

This is claimed in the paper of Elekes, Erdős, and Hajnal in [32]. Their paper contains a large number of observations, most of them without proof. See also the paper of Elekes, Hajnal, and Komjáth [33].

Problem VI. (GCH) *Does there exist a coloring $F : [\omega_2]^2 \to \omega_1$ such that for each $A \in [\omega_2]^{\aleph_2}$ there is $B \in [A]^{\aleph_1}$ with $F|[B]^2$ injective?*

There is such an F if $2^{\aleph_1} = \aleph_2$.

Namely, there is a Lusin set $A \subseteq {}^{\omega_1}\omega_1$, that is, if $B \subseteq A$ has $|B| = \aleph_2$, then there are $s \in {}^{<\omega_1}\omega_1$ and $g_\xi \in B$ with $s\xi < g_\xi$ $(\xi < \omega_1)$. For the construction of A, enumerate

$$^{({}^{<\omega_1}\omega_1)}\omega_1$$

as $\{h_\alpha : \alpha < \omega_2\}$. Then define $A = \{f_\alpha : \alpha < \omega_2\} \subseteq {}^{\omega_1}\omega_1$ by recursion on α such that

(1) $f_\alpha \neq f_\beta$ $(\beta < \alpha)$ and

(2) there is $\xi < \omega_1$, such that $f_\alpha(\xi) = h_\beta(f_\alpha|\xi)$ $(\beta < \alpha)$.

It is easy to see that A is as required.

Finally define $F : [A]^2 \to \omega_1$ as follows. If $g, h \in A$, ξ is the first difference, then $F(g, h) = \{g(\xi), h(\xi)\} \in [\omega_1]^2$. If $B \in [A]^{\aleph_2}$, then by the above property

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there are $s \in$ and $g_\xi \in B$ $(\xi < \omega_1)$ such that $s\xi < g_\xi$ and so on $[\{g_\xi : \xi < \omega_1\}]^2$ F is a bijection to $[\omega_1]^2$.

Problem VII. *If $X \subseteq [\omega_1]^2$, then either*
(1) *there is $A \subseteq \omega_1$, $\mathrm{tp}(A) = \omega_1$, $[A]^2 \subseteq X$, or else*
(2) *there is $B \subseteq \omega_1$, $\mathrm{tp}(B) = \omega^2$, no infinite $C \subseteq B$ has $[C]^2 \subseteq X$.*

Problem VIII. (GCH) *If $X \subseteq [\omega_2]^2$, then either*
(1) *there is $A \subseteq \omega_2$, $\mathrm{tp}(A) = \omega_1 + \omega$, $[A]^2 \subseteq X$, or else*
(2) *there is $B \subseteq \omega_2$, $\mathrm{tp}(B) = \omega_1\omega$, no infinite $C \subseteq B$ has $[C]^2 \subseteq X$.*

Problems of this type for cardinals were investigated in [58]. The ordinal version was also independently discovered by Galvin and Hajnal.

The last two problems were raised when Erdős and Hajnal started working on the paper [54].

We say that $F : [\kappa]^2 \to 2$ *establishes* $\kappa \not\to (\lambda, \mu)^2$ if there is no $A \in [\kappa]^\lambda$ homogeneous in color 0 and there is no $B \in [\kappa]^\mu$ homogeneous in color 1. We use this piece of notation for other partition relations.

If X, Y are graphs, then X embeds into Y $(X \le Y)$ if X is isomorphis to a subgraph of Y. X weakly embeds into Y $(X \le^w Y)$ if X embeds into either Y or Y's complement.

Erdős and Hajnal proved the following results.

Theorem 1. *If $F : [\omega_1]^2 \to 2$ establishes $\aleph_1 \not\to ([\aleph_0, \aleph_1])^2_2$, then $F^{-1}(0)$ embeds every countable graph.*

Theorem 2. *The following are equivalent.*
(1) *Suslin's hypothesis.*
(2) *C_4 embeds into $F^{-1}(0)$ for every F establishing $\aleph_1 \not\to (\aleph_1)^2_2$.*

Theorem 3. *X weakly embeds into $F^{-1}(0)$ where X is the disjoint union of countably many $K_{\aleph_0}$ and F establishes $\aleph_1 \not\to (\aleph_1)^2_2$.*

Theorem 4. *Assume SH and let X be a finite graph. Then*
(a) *if f establishes $\aleph_1 \not\to (\aleph_1)^2_2$, then $X \le^w f^{-1}(0)$ iff X is the comparison graph of a tree,*
(b) *if f establishes $\aleph_1 \not\to (\aleph_1)^2_2$ then $(V, X) \le f^{-1}(0)$ where $V = \{x_i : i < n\} \cup \{y_i : i \in M \subseteq [0, n)\}$, X is the comparison graph of the following partially ordered set on V: $x_0 < \cdots < x_{n-1}$, $x_i < y_i$ $(i \in M)$, the y_i's are incomparable.*

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Theorem 5. *$X \not\leq^w f^{-1}(0)$ where f establishes the Sierpiński partition on $\mathbb{R}$ and X is the comparison graph of a normal tree of height ω.*

Problem IX. (GCH) *If F establishes $\lambda^+ \not\to (K_{\lambda,\lambda^+})^2_2$, then for what $\kappa \leq \lambda$ does F embed every $f : [\kappa]^2 \to 2$?*

Problem X. (1) *Let $f : [\mathbb{R}]^2 \to \{0,1\}$ be the Sierpiński partition. If $X \leq f^{-1}(0)$ is finite (or even countable) then $X \leq g^{-1}(0)$ for each g establishing $\aleph_1 \not\to ([\aleph_1, \aleph_1])^2_2$*
(2) *Characterize Theorem 4 above under SH.*
(3) *Assume* ¬SH. *Characterize all countable graphs X with $X \leq^w f^{-1}(0)$ for all f establishing $\aleph_1 \not\to (\aleph_1)^2_2$.*

In [168], Shelah proved that if $\kappa, \lambda\, \tau$ all have cofinality greater than ω, F establishes $\kappa \not\to (K_{\lambda,\tau})^2_2$, and a Cohen real is added, then F retains this property and there is an $f : [\omega_1]^2 \to 2$ which does not embed into F.

References

[1] U. Abraham: Free sets for nowhere-dense set mappings, *Israel Journal of Mathematics*, **39**(1981), 167–176.

[2] U. Abraham, M. Magidor: Cardinal arithmetic, in: *Handbook of Set Theory*, (M. Foreman, A. Kanamori, eds), II, Springer, 2010, 1149–1227.

[3] F. Bagemihl: The existence of an everywhere dense independent set, *Michigan Math. Journal*, **20**(1973), 1–2.

[4] F. Bagemihl: Independent sets and second category, *Order*, **7**(1990), 65–66.

[5] J. E. Baumgartner: $ADT \not\Longrightarrow KH$, unpublished notes, cca 1971.

[6] J. E. Baumgartner: Partition relations for uncountable cardinals, *Israel Journal of Mathematics*, **21**(1975), 296–307.

[7] J. E. Baumgartner: Almost-disjoint sets, the dense set problem and the partition calculus, *Annals of Math. Logic*, **10**(1976), 401–439.

[8] J. E. Baumgartner: A new class of order types, *Annals of Math. Logic*, **9**(1976), 187–222.

https://doi.org/10.1017/bsl.2025.1 Published online by Cambridge University Press

[9] J. E. Baumgartner: Generic graph construction, *Journal of Symb. Logic*, **49**(1984), 234–240.

[10] J. E. Baumgartner, A. Hajnal: A proof (involving Martin's axiom) of a partition relation, *Fundamenta Mathematicæ*, **LXXVIII**, (1973), 193–203.

[11] J. E. Baumgartner, A. Hajnal: A remark on partition relations for infinite ordinals with an application to finite combinatorics, in: *Logic and combinatorics (Arcata, Calif., 1985)*, Contemp. Math., **65**, Amer. Math. Soc., Providence, RI, 1987, 157–167.

[12] J. E. Baumgartner, A. Hajnal: Polarized partition relations, *Journal of Symb. Logic*, **66**(2001), 811–821.

[13] J. E. Baumgartner, A. Hajnal, S. Todorčević: Extensions of the Erdős-Rado theorem, in: *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **411**, Kluwer Acad. Publ., Dordrecht, 1993. 1–17.

[14] Baumgartner, R. Laver: Iterated perfect set forcing, *Annals of Math. Logic*, **17**(1979), 271–288.

[15] M. Benda, J. Ketonen: Regularity of ultrafilters, *Israel Journal of Mathematics*, **17**(1974), 231–240.

[16] Sh. Ben-David, M. Magidor: The weak $\square$ is really weaker than the full $\square$, *Journal of Symb. Logic*, **51**(1986), 1029–1033.

[17] Omer Ben-Neria, Shimon Garti, Yair Hayut: Weak prediction principles, *Fundamenta Mathematicæ*, **245**(2019), 109–125.

[18] Jeffrey Bergfalk, Michael Hrušák, Saharon Shelah: Ramsey theory for highly connected monochromatic subgraphs, *Acta Math. Hung.*, **163**(2021), 309–322.

[19] A. Blass: Weak partition relations, *Proc. Amer. Math. Soc.*, **35**(1972), 249–253.

[20] C. C. Chang: A partition theorem for the complete graph on ω^ω, *Journal of Combinatorial Theory*, **12**(1972), 396–452.

[21] J. Czipszer, P. Erdős, A. Hajnal: Some extremal problems on infinite graphs, *Publ. Math. Inst. Hung. Acad. Sci.*, **7**(1962), 441–457.

[22] Carl Darby, Jean A. Larson: Multicolored graphs on countable ordinals of finite exponent, Set theory (Piscataway, NJ, 1999), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 58, Amer. Math. Soc., Providence, RI, 2002, 31–43.

[23] O. Deiser, D. Donder: Canonical functions, non-regular ultrafilters and Ulam's problem on ω_1, *Journal of Symb. Logic*, **68**(2003), 713–739.

[24] Denis C. Devlin: *Some partition theorems and ultrafilters on* ω, Ph. D. thesis, Dartmouth College, July 1979.

[25] K. J. Devlin: Some weak versions of large cardinal axioms, *Annals of Math. Logic*, **5**(1972–1973), 291–325.

[26] K. J. Devlin: A note on a problem of Erdős and Hajnal, *Disc. Math.*, **11**(1975), 9–22.

[27] K. J. Devlin, J. Paris: More on the free subset problem, *Annals of Math. Logic*, **5**(1972–1973), 327–336.

[28] H.-D. Donder, R. B. Jensen, B. J. Koppelberg: Some applications of the core model, in: *Set Theory and Model Theory*, Proc. Bonn, 1979, *Lecture Notes in Mathematics*, **872**, Springer, Berlin, 1981, 55–97.

[29] A. Dudek, V. Rödl: On the Folkman number $f(2,3,4)$, *Exp. Math.*, **17**(2008), 63–67.

[30] G. Elekes: Colouring infinite subsets of ω, Colloquia Mathematica Societatis János Bolyai, **10.**, *Infinite and Finite Sets*, Keszthely, Hungary, 1973, 393–396.

[31] G. Elekes: On a partition property of infinite subset of a set, *Periodica Math. Hung.*, **5**(1974), 215–218.

[32] G. Elekes, P. Erdős, A. Hajnal: On some partition properties of families of sets, *Studia Sci. Math. Hungar.*, **13**(1978), 151–155.

https://doi.org/10.1017/bsl.2025.1 Published online by Cambridge University Press

[33] G. Elekes, A. Hajnal, P. Komjáth: Partition theorems for the power set, in: *Sets, graphs and numbers (Budapest, 1991)*, Colloq. Math. Soc. János Bolyai, **60**, North-Holland, Amsterdam, 1992, 211–217.

[34] P. Erdős: Some remarks on set theory, *Proc. Amer. Math. Soc.*, **1**(1950), 127–141.

[35] P. Erdős: An interpolation problem associated with the continuum hypothesis, *Michigan Math. J.*, **11**(1964), 9–10.

[36] P. Erdős: Problems and results in chromatic graph theory, in: *Proof Techniques in Graph Theory*, Proc. Second Ann Arbor Graph Theory Conference, Ann Arbor, MI, 1968, Academic Press, 1969, 27–35.

[37] P. Erdős, G. Fodor, A. Hajnal: On the structure of inner set mappings, *Acta Sci. Math.*, **20**(1959), 81–90.

[38] P. Erdős, A. Hajnal: On the structure of set mappings, *Acta Math. Acad. Sci. Hung.*, **9**(1958), 111–131.

[39] P. Erdős, A. Hajnal: Some remarks on set theory, VIII: *Michigan Math. J.*, **7**(1960), 187–191.

[40] P. Erdős, A. Hajnal: Some remarks on set theory, VII: *Acta Sci. Math.*, **21**(1960), 154–163.

[41] P. Erdős, A. Hajnal: On a property of families of sets, *Acta Math. Acad. Sci. Hung.*, **12**(1961), 87–123.

[42] P. Erdős, A. Hajnal: Some remarks concerning our paper "On the structure of set mappings"—Nonexistence of a two-valued σ-measure for the first uncountable inaccessible cardinal, *Acta Math. Acad. Sci. Hung.*, **13**(1962), 223–226.

[43] P. Erdős, A. Hajnal: On a classification of denumerable order types and an application to the partition calculues, *Fund. Math.*, **5**(1962), 117–129.

[44] P. Erdős, A. Hajnal: On complete topological subgraphs of certain graphs, *Annales Univ. Sci. Budapest*, **7**(1964), 143–149.

[45] P. Erdős, A. Hajnal: Some remarks on set theory, IX: *Michigan Math. J.*, **11**(1964), 107–127.

[46] P. Erdős, A. Hajnal: On a problem of B. Jónnson, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, **14**(1966), 19–23.

[47] P. Erdős, A. Hajnal: On chromatic number of graphs and set systems, *Acta Math. Acad. Sci. Hung.*, **17**(1966), 61–99.

[48] P. Erdős, A. Hajnal: On decompositions of graphs, *Acta Math. Acad. Sci. Hung.*, **18**(1967), 359–377.

[49] P. Erdős, A. Hajnal: On the chromatic number of infinite graphs, In: *Graph Theory Symposium, Tihany, Hungary*, 1966, 83–98.

[50] P. Erdős, A. Hajnal: Some results and problems for certain polarized partitions, *Acta Math. Acad. Sci. Hung.*, **21**(1970), 369–392.

[51] P. Erdős, A. Hajnal: Unsolved problems in set theory, in: *Proc. Symp. Pure Math*, **XIII**, Providence, R. I., 1971, 17–48.

[52] P. Erdős, A. Hajnal: Unsolved and solved problems in set theory, *Proceedings of the Tarski Symposium*, (Proc. Sympos. Pure Math., **XXV**, Univ. California, Berkeley, Calif., 1971, Amer. Math. Soc., Providence, R.I., 1974, 269–287.

[53] P. Erdős, A. Hajnal: Some remarks on set theory XI, Collection of articles dedicated to Andrzej Mostowski on the occasion of his sixtieth birthday, III., *Fund. Math.*, **81**(1974), 261–265.

[54] P. Erdős, A. Hajnal: Embedding theorems for graphs establishing negative partition relations, *Period. Math. Hungar.*, **9**(1978), 205–230.

[55] P. Erdős, A. Hajnal, A. Máté, R. Rado: *Combinatorial Set Theory: Partition Relation for Cardinals*, North-Holland, 1984.

[56] P. Erdős, A. Hajnal, E. C. Milner: On the complete subgraphs of graphs defined by systems of sets, *Acta Math. Acad. Sci. Hung.*, **17**(1966), 159–229.

[57] P. Erdős, A. Hajnal, E. C. Milner: On sets of almost disjoint subsets of a set, *Math. Acad. Sci. Hung.*, **19**(1968), 209–218.

https://doi.org/10.1017/bsl.2025.1 Published online by Cambridge University Press

[58] P. Erdős, A. Hajnal, R. Rado: Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hung.*, **16**(1965), 93–196.

[59] P. Erdős, A. Hajnal, S. Shelah: On some general properties of chromatic numbers, *Topics in topology*, (Proc. Colloq., Keszthely, 1972), Colloq. Math. Soc. János Bolyai, **8**, North-Holland, Amsterdam, 1974, 243–255.

[60] P. Erdős, M. Makkai: Some remarks on set theory, X, *Studia Sci. Math. Hung.*, **1**(1966), 157–159.

[61] P. Erdős, E. C. Milner, R. Rado: Intersection theorems for systems of sets, III, Collection of articles dedicated to the memory of Hanna Neumann, IX, *Journ. Austral. Math. Soc.*, **18**(1974), 22–40.

[62] P. Erdős, R. Rado: A partition calculus in set theory, *Bull. Amer. Math. Soc.*, **62**(1956), 427–489.

[63] W. G. Fleissner: Some spaces related to topological inequalities proven by the Erdős–Rado theorem, *Proc. Amer. Math. Soc.*, **71**(1978), 313–320.

[64] J. Folkman: Graphs with monochromatic complete subgraphs in every edge coloring, *SIAM Journal of App. Math.*, **18**(1970), 19–24.

[65] M. Foreman: Ideals and generic elementary embeddings, in: Handbook of set theory, (ed. M. Foreman, A. Kanamori), Springer, 2010, 885–1147.

[66] M. Foreman, A. Hajnal: A partition relation for successors of large cardinals, *Math. Ann.*, **325**(2003), 583–623.

[67] M. Foreman, R. Laver: Some downward transfer properties of $\aleph_2$, *Adv. in Math.*, **67**(1988), 230–238.

[68] M. Foreman, M. Magidor, S. Shelah: Martin's maximum, saturated ideals, and nonregular ultrafilters, I, *Annals of Math.*, **127**(1988), 1–47.

[69] M. Foreman, M. Magidor, S. Shelah: Martin's maximum, saturated ideals, and nonregular ultrafilters, II, *Annals of Math.*, **127**(1988), 521–545.

[70] J. Fox, A. Grinshpun, J. Pach: The Erdős–Hajnal conjecture for rainbow triangles, *Journal of Combinatorial Theory*, (B), **111**(2015), 75–125.

https://doi.org/10.1017/bsl.2025.1 Published online by Cambridge University Press

[71] D. H. Fremlin: *Consequences of Martin's axiom*, Cambridge University Press, 1984.

[72] D. H. Fremlin: Real-valued measurable cardinals, in: *Set Theory of the Reals*, (Haim Judah, ed.), Israel Math. Conf. Proceedings, **6**, 1993, 151–304.

[73] F. Galvin: On a partition theorem of Baumgartner and Hajnal, Colloquia Mathematica Societatis János Bolyai, **10.**, *Infinite and Finite Sets*, Keszthely, Hungary, 1973, 711–729.

[74] Fred Galvin, Jean Larson: Pinning countable ordinals. Collection of articles dedicated to Andrzej Mostowski on his sixtieth birthday, VIII. *Fundamenta Mathematicæ*, **82**(1974/75), 357–361.

[75] F. Galvin, K. Prikry: Borel sets and Ramsey's theorem, *Journal of Symb. Logic*, **38**(1973), 193–198.

[76] F. Galvin, S. Shelah: Some counterexamples in partition calculus, *Journal of Combinatorial Theory*, (A) **15**(1973), 167–174.

[77] S. Garti: Unbalanced polarized relations, *Proc. Amer. Math. Soc.*, **149**(2021), 1281–1287.

[78] M. Golshani: (Weak) diamond can fail at the least inaccessible cardinal, *Fundamenta Mathematicæ*, **256**(2022), 113–129.

[79] M. Golshani: Generalized Kurepa hypothesis at singular cardinals, *Periodica Math. Hung.*, **78**(2019), 200–202.

[80] J. Gregory: Souslin trees and the generalized continuum hypothesis, *Journal of Symb. Logic*, **41**(1976), 663–671.

[81] A. Gyárfás, J. Komlós, E. Szemerédi: On the distribution of cycle lengths in graphs, *Journal of Graph Theory*, **8**(1984), 441–462.

[82] Labib Haddad, Gabriel Sabbagh: Sur une extension des nombres de Ramsey aux ordinaux, *C. R. Acad. Sci. Paris Sér. A-B*, **268**(1969), A1165–A1167.

[83] Labib Haddad, Gabriel Sabbagh: Calcul de certains nombres de Ramsey généralisés, *C. R. Acad. Sci. Paris Sér. A-B*, **268**(1969), A1233–A1234.

[84] Labib Haddad, Gabriel Sabbagh: Nouveaux résultats sur les nombres de Ramsey généralisés, *C. R. Acad. Sci. Paris Sér. A-B*, **268**(1969), A1516–A1518.

[85] A. Hajnal: Some results and problems in set theory, *Acta Math. Sci. Hung.*, **11**(1960), 277–298.

[86] A. Hajnal: Ulam matrices for inaccessible cardinals, *Bull. Acad. Pol. des Sci.*, **17**(1969), 683–688.

[87] A. Hajnal: On some combinatorial problems involving large cardinals, *Fundamenta Mathematicæ*, **69**(1970), 39–53.

[88] A. Hajnal: Report on the Miklós Schweitzer Memorial Contest in 1968, *Mat. Lapok*, **20**(1969), 145–171 (in Hungarian).

[89] A. Hajnal: A negative partition relation, *Proc. Nat. Acad. Sci.*, **68**(1971), 142–144.

[90] A. Hajnal: Paul Erdős' set theory, in: *The mathematics of Paul Erdős*, *Algorithms Combin.*, **14**, Springer, Berlin, 1997, II, 352–393.

[91] A. Hajnal: Rainbow Ramsey theorems for colorings establishing negative partition relations, *Fundamenta Mathematicæ*, **198**(2008), 255–262.

[92] A. Hajnal, I. Juhász: On discrete subspaces of topological spaces, *Indag. Math.*, **29**(1967), 343–356.

[93] A. Hajnal, I. Juhász: Two consistency results in topology, *Bull. Amer. Math. Soc.*, **73**(1972), 711.

[94] A. Hajnal, I. Juhász: A consistency result concerning hereditarily α-Lindelöf spaces, *Acta Math. Acad. Sci. Hung.*, **24**(1973), 307–312.

[95] A. Hajnal, I. Juhász, S. Shelah: Splitting strongly almost disjoint families, *Transactions of the American Mathematical Society*, **295**(1986), 369–387.

[96] A. Hajnal, I. Juhász, S. Shelah: Strongly almost disjoint families, revisited, *Fundamenta Mathematicæ*, **163**(2000), 13–23.

[97] A. Hajnal, P. Komjáth: Embedding graphs into colored graphs, *Transactions of the American Mathematical Society*, **307**(1988), 395–409.

[98] A. Hajnal, A. Máté: Set mappings, partitions, and chromatic numbers, Logic Colloquium '73, Bristol, 1973, Studies in Logic and the Foundations of Mathematics, **80**, North-Holland, Amsterdam, 1975. 347–379.

[99] S. H. Hechler: On two problems in combinatorial set theory, *Bull Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.*, **20**(1972).

[100] T. Inamdar, A. Rinot: Was Ulam right? I. Basic theory and subnormal ideals, *Topology App.*, Paper No. 108287, 53 pp.

[101] T. Jech: *Set Theory*, The Third Millenium Edition, Revised and Expanded, Springer Monographs in Mathematics, Springer, 2003.

[102] T. Jech, S. Shelah: A note on canonical functions, *Israel Journal of Mathematics*, **68**(1989), 376–380.

[103] Albin L. Jones: A short proof of a partition relation for triples, *Electron. J. Combin.*, **7**(2000), res. paper 24, 9 pp.

[104] Albin L. Jones: A polarized partition relation for weakly compact cardinals using elementary substructures, *Journal of Symb. Logic*, **71**(2006), 1342–1352.

[105] Albin L. Jones: On a result of Szemerédi, *Journal of Symb. Logic*, **73**(2008), 953–956.

[106] Albin L. Jones: Even more on partitioning triples of countable ordinals, *Proc. Amer. Math. Soc.*, **146**(2018), 3529–3539.

[107] I. Juhász, S. Shelah: How large a hereditarily separable or hereditarily Lindelöf space can be ? *Israel Journal of Mathematics*, **53**(1986), 355–364.

[108] A Kanamori: On Silver's and related principles, in: *Logic Colloquium '80*, Studies in Logic and the Foundations of Mathematics, **108**, North-Holland, Amsterdam, 1982, 153–172.

[109] A. Kanamori: P,artition relations for successors of cardinals, *Adv. in Math.*, **59**(1986), 152–169.

https://doi.org/10.1017/bsl.2025.1 Published online by Cambridge University Press

[110] A. Kanamori: Erdős and set theory, *Bulletin of the Symbolic Logic*, **20**(2014), 449–490.

[111] H.J.Keisler: A survey of ultraproducts, in: *Logic, Methodology and Philosophy of Science*, Proc. of the 1964 Congress, (Y. Bar-Hillel, ed.), North-Holland, Amsterdam, 1965, 112–126.

[112] J. Ketonen: Nonregular ultrafilters and large cardinals, *Transactions of the American Mathematical Society*, **224**(1976), 61–73.

[113] Peter Koepke: The consistency strength of the free–subset property for ω_ω, *Journal of Symb. Logic*, **49**(1984), 1198–1204.

[114] P. Komjáth: The colouring number, *Journal of London Math. Soc.*, **54**(1987), 1–14.

[115] P. Komjáth: Erdős's work on infinite graphs, in: *Erdős Centennial*, (L. Lovász, I. Ruzsa, V. T. Sós, eds), Springer, 2013, 325–345.

[116] P. Komjáth: Notes on some Erdős–Hajnal problems, *Journal of Symb. Logic*, **86**(2021), 1116–1123.

[117] P. Komjáth: A comment on a problem of Erdős, Hajnal, and Milner, manuscript.

[118] P. Komjáth, J. Pach: Universal graphs without large bipartite subgraphs, *Mathematika*, **31**(1984), 282–290.

[119] P. Komjáth, S. Shelah: Forcing constructions for uncountably chromatic graphs, *Journal of Symb. Logic*, **53**(1988), 696–707.

[120] P. Komjáth, S. Shelah: A consistent edge partition theorem for infinite graphs, *Acta Math. Hung.*, **61**(1993), 115–120.

[121] P. Komjáth, S. Shelah: Two consistency results on set mappings, *Journal of Symb. Logic*, **65**(2000), 333–338.

[122] P. Komjáth, S. Shelah: Monocolored topological complete graphs in colorings of uncountable complete graphs, *Acta Math. Hung.*, **163**(2021), 71–84.

https://doi.org/10.1017/bsl.2025.1 Published online by Cambridge University Press

[123] P. Komjáth, V. Totik: *Problems and Theorems in Classical set Theory*, Springer, 2006.

[124] A. Kumar, S. Shelah: On a question about families of entire functions, *Fundamenta Mathematicæ*, **239**(2017), 279–288.

[125] K. Kunen: *Inacessibility porperties of cardinals*, Ph. D. thesis, Stanford University, 1968.

[126] K. Kunen: Saturated ideals, *Journal of Symb. Logic*, **43**(1978), 65–76.

[127] C. Lambie-Hanson, A. Rinot: Reflections on the coloring and chromatic numbers, *Combinatorica*, **39**(2019), 165–214.

[128] A. R. Lange, S. P. Radziszowski, X. Xu: Use of MAX-CUT for Ramsey arrowing triangles, arXiv:1207.3750

[129] J. Larson: *On some arrow relations*, Ph. D. Thesis, Dartmouth College, Hanover, New Hampshire, 1972.

[130] Jean A. Larson: A short proof of a partition theorem for the ordinal ω^ω, *Annals of Math. Logic*, **6**(1973/74), 129–145.

[131] Paul Larson, S. Shelah: Bounding by canonical functions, with CH, *Jornal of Math. Logic*, **3**(2003), 193–215.

[132] R. Laver: An order type decomposition theorem, *Annals of Math.*, (2), **98**(1973), 96–119.

[133] R. Laver: Partition relations for uncountable cardinals $\leq 2^{\aleph_0}$, Colloquia Mathematica Societatis János Bolyai, **10.**, *Infinite and Finite Sets*, Keszthely, Hungary, 1973, 1029–1042.

[134] R. Laver: An $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal on ω_1, in: *Logic Colloquium '80*, Studies in Logic and the Foundations of Mathematics, **108**, North-Holland, Amsterdam, 1982, 173–180.

[135] Hong Liu, Richard Montgomery: A solution to Erdős and Hajnal's odd cycle problem, arXiv:2010.15802

[136] M. Magidor: On the existence of nonregular ultrafilters and the cardinality of ultrafilters, *Transactions of the American Mathematical Society*, **249**(1979), 97–111.

[137] M. Magidor, S. Shelah: When does almost free imply free? (For groups, transversals, etc.), *Journal of the American Mathematical Society*, **7**(1994), 769–830.

[138] A. H. Mekler, S. Shelah: The consistency strength of "every stationary set reflects", *Israel Journal of Mathematics*, **67**(1989), 353–366.

[139] E. W. Miller: On a property of families of sets, *Comptes Rendus Varsovie*, **30**(1937), 31–38.

[140] C. Mills, K. Prikry: Some recent results about partitions of ω_1, unpublished.

[141] E. C. Milner: Partition relations for ordinal numbers, *Canadian Journal of Mathematics*, **21**(1969), 317–334.

[142] E. C. Milner, K. Prikry: A partition theorem for triples, *Proc. Amer. Math. Soc.*, **97**(1986), 488–494.

[143] E. C. Milner, K. Prikry: A partition relation for triples using a model of Todorčević, Directions in infinite graph theory and combinatorics (Cambridge, 1989), *Disc. Math.***95**(1991), 183–191.

[144] E. C. Milner, R. Rado: The pigeonhole principle for ordinal numbers, *Journal of London Math. Soc.*, **15**(1965), 750–768.

[145] E. C. Milner, S. Shelah: Some theorems on transversals, Colloquia Mathematica Societatis János Bolyai, **10.**, *Infinite and Finite Sets*, Keszthely, Hungary, 1973, 1115–1126.

[146] T. Miyamoto: *Some results in Forcing*, Ph. D. Thesis, Dartmouth College, 1987.

[147] T. Miyamoto: Separating a family of weak Kurepa Hypotheses and the Transversal Hypothesis, http://www.seto.nanzan-u.ac.jp/st/nas/academia/vol005.pdf/05-025-038.pdf

[148] J. Nesteril, V. Rödl: A Ramsey graph without triangles exists for any graph without triangles, Colloquia Mathematica Societatis János Bolyai, **10.**, *Infinite and Finite Sets*, Keszthely, Hungary, 1973, 1127–1132.

https://doi.org/10.1017/bsl.2025.1 Published online by Cambridge University Press

[149] J. Nesetril, V. Rödl: The Ramsey property for graphs without forbidden complete subgraphs, *Journal of Combinatorial Theory*, (B), **20**(1976), 243–249.

[150] L. Newelski: Free sets for some special set mappings, *Israel Journal of Mathematics*, **58**(1987), 213–224.

[151] L. Newelski, J. Pawlikowski, W. Seredyński: Infinite free set for small measure set mappings, *Proc. Amer. Math. Soc.*, **100**(1987), 335–339.

[152] Eva Nosal: On partition relation for ordinal numbers, *Journal of London Math. Soc.*, (2), **8**(1974), 306–310.

[153] Eva Nosal: Partition relations for denumerable ordinals, *Journal of Combinatorial Theory*, (B), **27**(1979), 190–197.

[154] K. Prikry: On a problem of Gillman and Keisler, *Annals of Math. Logic*, **2**(1970), 179–187.

[155] K. Prikry: On a problem of Erdős, Hajnal and Rado, *Disc. Math.*, **2**(1972), 51–59.

[156] K. Prikry: Kurepa's hypothesis for singular cardinals, unpublished notes, 1972.

[157] K. Prikry: On a set-theoretic partition problem, *Duke Math. J.*, **39**(1972), 77–83.

[158] K. Prikry: Kurepa's hypothesis and a problem of Ulam on families of measures, *Monatsh. Math.*, **81**(1976), 41–57.

[159] D. Raghavan, S. Todorcevic: Suslin trees, the bounding number, and partition relations, *Israel Journal of Mathematics*, **225**(2018), 771–796.

[160] J. R. Rebholz: Some consequences of the morass and diamond, *Annals of Math. Logic*, **7**(1974), 361–385.

[161] A. Rinot: Chain conditions of products, *Bulletin of the Symbolic Logic*, **20**(2014), 293–314.

[162] A. Rinot: Chromatic numbers of graphs—large gaps, *Combinatorica*, **35**(2015), 215–233.

https://doi.org/10.1017/bsl.2025.1 Published online by Cambridge University Press

[163] V. Rödl: On the chromatic number of subgraphs of a graph, *Proc. Amer. Math. Soc.*, **64**(1977), 370–371.

[164] R. Schipperus: Countable partition ordinals, *Annals of Pure and Appl. Logic*, **161**(2010), 1195–1215.

[165] S. Shelah: A combinatorial problem: stability and order for models and theories in infinitary languages, *Pacific J. Math.*, **41**(1972), 247–261.

[166] S. Shelah: Notes on combinatorial set theory, *Israel Journal of Mathematics*, **14**(1973), 262–277.

[167] S. Shelah: Notes on partition calculus, Colloquia Mathematica Societatis János Bolyai, **10.**, *Infinite and Finite Sets*, Keszthely, Hungary, 1973 1257–1276.

[168] S. Shelah: Colouring without triangles and partition relations, *Israel Journal of Mathematics*, **20**(1975), 1–12.

[169] S. Shelah: A compactness theorem in singular cardinals, free algebras, Whitehead problem, and transversals, *Israel Journal of Mathematics*, **21**(1975), 319–349.

[170] S. Shelah: A compactness theorem in singular cardinals, free algebras, Whitehead problem, and transversals, *Israel Journal of Mathematics*, **21**(1975), 319–349.

[171] S. Shelah: On successors of singular cardinals, Logic Coll. '78, (M. Boffa, D. van Dalen,, K. McAloon, eds), North-Holland, Amsterdam, 1979, 357–380.

[172] S. Shelah: Independence of strong partition relation for small cardinals and the free-subset problem, *Journal of Symb. Logic*, **45**(1980), 505–509.

[173] S. Shelah: $\aleph_\omega$ may have a strong partition relation, *Israel Journal of Mathematics*, **38**(1981), 283–288.

[174] S. Shelah: *Proper Forcing*, Lecture Notes in Mathematics, **940**, Springer, 1982.

[175] S. Shelah: Iterated forcings and normal ideals on ω_1, *Israel Journal of Mathematics*, **60**(1987), 345–380.

[176] S. Shelah: Was Sierpiński right? *Israel Journal of Mathematics*, **62**(1988), 355–380.

[177] S. Shelah: Consistency of positive partition theorems for graphs and models, in *Set theory and its applications*, (Toronto, ON, 1987) (1989), 167–193. (sh289)

[178] S. Shelah: Incompactness for chromatic numbers of graphs, in: *A Tribute to Paul Erdős*, (A. Baker, B. Bollobás, A. Hajnal, eds), Cambridge University Press, Cambridge, 1990, 361–371.

[179] S. Shelah: Strong Partition Relations Below the Power Set: Consistency, Was Sierpiński Right, II?, *Proc Conference on Set Theory and its Applications in honor of A.Hajnal and V.T.Sós*, Budapest, 1/91 (1991) 637–638.

[180] S. Shelah: On CH+$2^{\aleph_1} \to (\alpha)^2_2$ for $\alpha < \omega_2$, 281–289.

[181] S. Shelah: *Proper and Improper Forcing*, Springer, 1998.

[182] S. Shelah: Was Sierpiński right? IV, *Journal of Symb. Logic*, **65**(2000), 1031-1054.

[183] S. Shelah: A partition relation using strongly compact cardinals, *Proc. Amer. Math. Soc.*, **131**(2003), 2585–2592.

[184] S. Shelah: The Erdős-Rado arrow for singular, *Canadian Math. Bull.*, **52**(2009), 127–131.

[185] S. Shelah: Universality among graphs omitting a complete bipartite graph, *Combinatorica*, **32**(2012), 325–362.

[186] S. Shelah, L. Stanley: A theorem and some consistency results in partition calculus, *Annals of Pure and Appl. Logic*, **36**(1987), 119–152.

[187] S. Shelah, L. Stanley: More consistency results in partition calculus, *Israel Journal of Mathematics*, **81**(1993), 97–110.

[188] R. Shore: Square bracket relations in L, *Fundamenta Mathematicæ*, **84**(1974), 101–106.

[189] E. Specker: Teilmengen von Mengen mit Relationen, *Comment. Math. Helv.*, **31**(1957), 302–314.

[190] Lee Stanley, Dan Velleman, and Charles Morgan: $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$ requires an inaccessible, *Proc. Amer. Math. Soc.*, **111**(1991), 1105–1118.

[191] J. R. Steel, R. A. van Wesep: Two consequences of determinacy consistent with choice, *Transactions of the American Mathematical Society*, **272**(1982), 67–85.

[192] *Contests in Higher Mathematics*, (Gábor J. Székely, ed.), Problem books in mathematics, Springer, 1996.

[193] Alan D. Taylor: Regularity properties of ideals and ultrafilters, *Annals of Math. Logic*, **16**(1979), 33–55.

[194] Alan D. Taylor: On saturated sets of ordinals and Ulam's problem, *Fundamenta Mathematicæ*, **109**(1980), 37–53.

[195] C. Thomassen: Cycles in graphs of uncountable chromatic number, *Combinatorica*, **3**(1983), 133–134.

[196] S. Todorcevic: Trees, subtrees, and order types, *Annals of Math. Logic*, **20**(1981), 233–268.

[197] S. Todorcevic: Reals and positive partition relations, Logic, Methodology, and Philosophy of Science, VII, Salzburg, 1983, Studies of Logic, **114**, North-Holland, Amsterdam, New York, 1986, 159–169.

[198] S. Todorcevic: Forcing positive partition relations, *Transactions of the American Mathematical Society*, **280** (1983), 703–720.

[199] S. Todorcevic: On a conjecture of R. Rado, *Journal of London Math. Soc.*, **27**(1983), 1–8.

[200] S. Todorcevic: Trees and linearly ordered sets, in: *Handbook of Set-theoretic Toplogy*, North-Holland, Amsterdam, 1984, 235–293.

[201] S. Todorcevic: Some partitions of 3-dimensional combinatorial cubes, *Journal of Combinatorial Theory*(A), **68**(1984), 410–437.

[202] S. Todorcevic: Aronszajn trees and partitions, *Israel Journal of Mathematics*, **52**(1985), 53–58.

[203] S. Todorcevic: Partition relations for partially ordered set, *Acta Math.*, **155**(1985), 1–25.

[204] S. Todorcevic: Partitioning pairs of countable ordinals, *Acta Math.*, **159**(1987), 261–294.

[205] S. Todorcevic: *Partittion Problems in Topology*, Contemp. Math., **84**, AMS, 1989.

[206] S. Yodorcevic: Partitioning pairs of countable sets, *Proc. Amer. Math. Soc.*, **111**(1991), 841–844.

[207] S. Todorcevic: Some partitions of three-dimensional combinatorial cubes, *Journal of Combinatorial Theory*, (A), **68**(1994), 410–437.

[208] S. Todorcevic: Countable chain condition in partition calculus, *Disc. Math.*, **88**(1998), 105–223.

[209] S. Todorcevic: *Walks om Ordinals amd Their Characteristics*, Progr. Math., **263**, Birkhäuser, 2007.

[210] S. Todorcevic: *Introduction to Ramsey Spaces*, Annals of Mathematics Studies, Princeton University Press, 2010.

[211] St. M. Ulam: *A Collection of Mathematical Problems*, New York: Interscience Publications, 1960, reprinted: *Problems in Modern Mathematics*, Dover, Mineola, New York, 2004

[212] S. Unger: Compactness for the chromatic number at $\aleph_{\omega_1+1}$, manuscript.

[213] W. Hugh Woodin: *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal*, Walter de Gruyter, 1999.

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