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THE SHARP BOUND OF THE SECOND HANKEL DETERMINANT OF LOGARITHMIC COEFFICIENTS FOR STARLIKE AND CONVEX FUNCTIONS

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Abstract

Let $\mathcal{S}$ denote the class of univalent functions in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The logarithmic coefficients γ_n of $f \in \mathcal{S}$ are defined by $F_f(z) := \log(f(z)/z) = 2\sum_{n=1}^{\infty} \gamma_n z^n$. The second Hankel determinant for logarithmic coefficients is defined by

$$H_{2,2}(F_f/2) = \begin{vmatrix} \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_4 \end{vmatrix} = \gamma_2\gamma_4 - \gamma_3^2.$$

We obtain sharp upper bounds of the second Hankel determinant of logarithmic coefficients for starlike and convex functions.

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1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Here, $\mathcal{H}$ is a locally convex topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ such that $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions which are univalent (that is, one-to-one) in $\mathbb{D}$. If $f \in \mathcal{S}$, then it has the series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

The *logarithmic coefficients* γ_n of $f \in \mathcal{S}$ are defined by

$$F_f(z) := \log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

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The logarithmic coefficients γ_n play a central role in the theory of univalent functions. A very few exact upper bounds for γ_n seem to have been established. Milin [17] highlighted the importance of the logarithmic coefficients within the framework of the Bieberbach conjecture. For $f \in \mathcal{S}$ and $n \geq 2$, Milin conjectured that

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0.$$

De Branges [9] proved the Bieberbach conjecture by proving Milin's conjecture. The logarithmic coefficients for the Koebe function $k(z) = z/(1-z)^2$ are $\gamma_n = 1/n$. For many extremal problems in the class $\mathcal{S}$, the Koebe function $k(z)$ serves as the extremal function; thus, $|\gamma_n| \leq 1/n$ was predicted to hold for functions in $\mathcal{S}$. However, this is not always the case, not even in order of magnitude. Indeed, there exists a bounded function f in the class $\mathcal{S}$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ (see [10, Theorem 8.4]). By differentiating (1.2) and then equating coefficients,

$$\begin{aligned}
\gamma_1 &= \tfrac{1}{2}a_2,\\
\gamma_2 &= \tfrac{1}{2}(a_3 - \tfrac{1}{2}a_2^2),\\
\gamma_3 &= \tfrac{1}{2}(a_4 - a_2a_3 + \tfrac{1}{3}a_2^3),\\
\gamma_4 &= \tfrac{1}{4}(a_5 - a_2a_4 + a_2^2a_3 - \tfrac{1}{2}a_3^2 - \tfrac{1}{4}a_2^2).
\end{aligned} \tag{1.3}$$

If $f \in \mathcal{S}$, it is easy to see that $|\gamma_1| \leq 1$, because $|a_2| \leq 2$. Using the Fekete–Szegö inequality [10, Theorem 3.8] for functions in $\mathcal{S}$ in (1.3), we obtain the sharp estimate

$$|\gamma_2| \leq \tfrac{1}{2}(1 + 2e^{-2}) = 0.635\ldots.$$

For $n \geq 3$, the problem seems much harder, and no significant bounds for $|\gamma_n|$ when $f \in \mathcal{S}$ appear to be known. In 2017, Ali and Allu [1] obtained bounds for the initial logarithmic coefficients for close-to-convex functions. The problem of computing bounds for the logarithmic coefficients is considered in [2, 8, 20, 21] for several subclasses of close-to-convex functions.

Given $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of the Taylor coefficients of a function $f \in \mathcal{A}$ of the form (1.1) is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

Hankel determinants of various order have been studied recently by several authors (see [5, 18, 19]). One can easily observe that the Fekete–Szegö functional is the second Hankel determinant $H_{2,1}(f)$. Fekete and Szegö generalised the estimate to $|a_3 - \mu a_2^2|$ with μ real for f given by (1.1) (see [10, Theorem 3.8]).

Kowalczyk and Lecko [13] studied the Hankel determinant whose entries are logarithmic coefficients of $f \in \mathcal{S}$,

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

They obtained sharp bounds for $|H_{2,1}(F_f/2)|$ for the classes of convex and starlike functions. In [14], they gave sharp bounds for $|H_{2,1}(F_f/2)|$ for the classes of starlike and convex functions of order α $(0 \le \alpha < 1)$ and, in [15], they gave sharp bounds for $|H_{2,1}(F_f/2)|$ for the classes of strongly starlike and strongly convex functions. Allu and Arora [3] obtained sharp bounds for $|H_{2,1}(F_f/2)|$ for various subclasses of univalent functions. Allu *et al.* [4] obtained sharp bounds for $|H_{2,1}(F_f/2)|$ for the classes of starlike and convex functions with respect to symmetric points. Recently, Allu and Shaji [6] obtained the sharp bound for the second Hankel determinant for inverse logarithmic coefficients for the classes of convex and starlike functions. Also, Eker *et al.* [12] obtained sharp bounds for $|H_{2,1}(F_f/2)|$ for the classes of strongly Ozaki close-to-convex functions and their inverse functions.

In this paper, we consider the second Hankel determinant for logarithmic coefficients of order 2, that is,

$$H_{2,2}(F_f/2) = \begin{vmatrix} \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_4 \end{vmatrix} = \gamma_2\gamma_4 - \gamma_3^2.$$

From (1.3),

$$\begin{aligned} H_{2,2}(F_f/2) = \tfrac{1}{288}(a_2^6 - 6a_2^4a_3 + 18a_2^2a_3^2 - 36a_3^3 - 12a_2^3a_4 + 72a_2a_3a_4 \\ - 72a_4^2 - 36a_2^2a_5 + 72a_3a_5). \end{aligned} \tag{1.4}$$

It is now appropriate to remark that $|H_{2,2}(F_{f^{-1}}/2)|$ is invariant under rotation, since for $f_\theta(z) := e^{-i\theta}f(e^{i\theta}z), \theta \in \mathbb{R}$ and $f \in \mathcal{S}$,

$$\begin{aligned} H_{2,2}(F_{f_\theta}/2) = \frac{e^{i6\theta}}{288}(a_2^6 - 6a_2^4a_3 + 18a_2^2a_3^2 - 36a_3^3 - 12a_2^3a_4 + 72a_2a_3a_4 \\ - 72a_4^2 - 36a_2^2a_5 + 72a_3a_5) = e^{i6\theta}H_{2,2}(F_f/2). \end{aligned}$$

The main aim of this paper is to a find sharp upper bound for $|H_{2,2}(F_f/2)|$ when f belongs to the class of starlike and convex functions. A domain $\Omega \subseteq \mathbb{C}$ is said to be starlike with respect to a point $z_0 \in \Omega$ if the line segment joining z_0 to any point in Ω lies entirely in Ω. If z_0 is the origin, then we say that Ω is a starlike domain. A function $f \in \mathcal{A}$ is said to be starlike if $f(\mathbb{D})$ is a starlike domain. We denote by $\mathcal{S}^*$ the class of starlike functions f in $\mathcal{S}$. It is well known that a function $f \in \mathcal{A}$ is in $\mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{1.5}$$

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Further, a domain $\Omega \subseteq \mathbb{C}$ is called convex if the line segment joining any two points of Ω lies entirely in Ω. A function $f \in \mathcal{A}$ is called convex if $f(\mathbb{D})$ is a convex domain. We denote by C the class of convex functions in $\mathcal{S}$. A function $f \in \mathcal{A}$ is in C if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{1.6}$$

2. Preliminary results

In this section, we present the key lemmas which will be used to prove the main results of this paper. Let $\mathcal{P}$ denote the class of all analytic functions p having positive real part in $\mathbb{D}$, with the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots. \tag{2.1}$$

Members of $\mathcal{P}$ are called Carathéodory functions. To prove our Theorem 3.1, we need the following lemmas.

LEMMA 2.1 [10]. *For a function $p \in \mathcal{P}$ of the form (2.1), the sharp inequality $|p_n| \leq 2$ holds for each $n \geq 1$. Equality holds for the function $p(z) = (1 + z)/(1 - z)$.*

LEMMA 2.2 [16]. *For a function $p \in \mathcal{P}$ of the form (2.1), the sharp inequality $|p_n - p_k p_{n-k}| \leq 2$ holds for $n \geq 2$ and $k \geq 1$.*

In view of Lemma 2.2, it is easy to see that $|p_{n+1}p_{n-1} - p_n^2| \leq 4$ for a function $p \in \mathcal{P}$ of the form (2.1) and $n \geq 2$. In particular, if $n = 3$,

$$|p_2 p_4 - p_3^2| \leq 4. \tag{2.2}$$

Let $\mathcal{B}_0$ denote the class of analytic functions $w : \mathbb{D} \to \mathbb{D}$ such that $w(0) = 0$. Functions in $\mathcal{B}_0$ are known as the Schwarz functions. A function $w \in \mathcal{B}_0$ can be written as a power series

$$w(z) = \sum_{n=1}^{\infty} c_n z^n. \tag{2.3}$$

It is clear that if

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)},$$

then $p \in \mathcal{P}$ if and only if $\omega \in \mathcal{B}_0$.

Due to the evident connection between Carathéodory functions and Schwarz functions, the results applicable to the coefficients of Schwarz functions are useful in solving coefficient problems associated with starlike and convex functions. To prove Theorem 3.3, we need the following lemmas for the Schwarz functions.

LEMMA 2.3 [7]. *Let $w(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then,*

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2 \quad \textit{and} \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}.$$

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LEMMA 2.4 [11]. *If $\omega \in \mathcal{B}_0$ of the form (2.3) and $\lambda \in \mathbb{C}$, then*

$$|c_4 + 2c_1c_3 + \lambda c_2^2 + (1 + 2\lambda)c_1^2c_2 + \lambda c_1^4| \leq \max\{1, |\lambda|\}.$$

LEMMA 2.5 [22]. *If $\omega \in \mathcal{B}_0$ of the form (2.3), then*

$$|c_2c_4 - c_3^2 + c_1^2c_4 - 2c_1c_2c_3 + c_2^3| \leq 1 - |c_1|^2.$$

3. Main results

THEOREM 3.1. *If $f \in \mathcal{S}^*$, then*

$$|H_{2,2}(F_f/2)| \leq \tfrac{1}{8}.$$

The bound is sharp.

PROOF. Let $f \in \mathcal{S}^*$ be of the form (1.1). By (1.5),

$$\frac{zf'(z)}{f(z)} = p(z) \tag{3.1}$$

for some $p \in \mathcal{P}$ of the form (2.1). By comparing the coefficients on both sides of (3.1),

$$\begin{aligned} a_2 &= p_1, \\ a_3 &= \tfrac{1}{2}(p_1^2 + p_2), \\ a_4 &= \tfrac{1}{6}(p_1^3 + 3p_1p_2 + 2p_3), \\ a_5 &= \tfrac{1}{24}(p_1^4 + 6p_1^2p_2 + 3p_2^2 + 8p_1p_3 + 6p_4). \end{aligned} \tag{3.2}$$

Hence, by (1.4),

$$H_{2,2}(F_f/2) = \tfrac{1}{288}(9p_2p_4 - 8p_3^2).$$

By taking the modulus on both sides and applying the triangle inequality,

$$|H_{2,2}(F_f/2)| \leq \tfrac{1}{288}(8|p_2p_4 - p_3^2|) + |p_2||p_4| \leq \tfrac{36}{288} = \tfrac{1}{8}.$$

Equality holds for the function $f \in \mathcal{A}$ given by (3.1), where

$$p(z) = \frac{1 + z^2}{1 - z^2}.$$

Then, $c_1 = c_3 = 0$ and $c_2 = c_4 = 2$. So by (3.2), $a_2 = 0, a_3 = 1, a_4 = 0$ and $a_5 = 1$. Therefore, by (1.4),

$$|\gamma_2\gamma_4 - \gamma_3^2| = \tfrac{1}{8},$$

which completes the proof of the theorem. □

EXAMPLE 3.2. The Koebe function,

$$k(z) = \frac{z}{(1 - z)^2} = z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + \cdots,$$

has logarithmic coefficients $\gamma_n = 1/n$ and $|\gamma_2\gamma_4 - \gamma_3^2| = 1/72$.

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THEOREM 3.3. *If $f \in \mathcal{C}$, then*

$$|H_{2,2}(F_f/2)| \leq \tfrac{13}{1080}. \tag{3.3}$$

The bound is sharp.

PROOF. Let $f \in \mathcal{C}$ be of the form (1.1). By (1.6),

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1+\omega(z)}{1-\omega(z)} \tag{3.4}$$

for some $\omega \in \mathcal{B}_0$ of the form (2.3). By comparing the coefficients of powers of z on both sides of (3.4),

$$\begin{aligned} a_2 &= c_1, \\ a_3 &= \tfrac{1}{3}(3c_1^2 + c_2), \\ a_4 &= \tfrac{1}{6}(6c_1^3 + 5c_1c_2 + c_3), \\ a_5 &= \tfrac{1}{30}(30c_1^4 + 43c_1^2c_2 + 6c_2^2 + 14c_1c_3 + 3c_4). \end{aligned} \tag{3.5}$$

Hence, by (1.4),

$$\begin{aligned} H_{2,2}(F_f/2) = \tfrac{1}{4320}(&15c_1^6 + 54c_1^4c_2 + 24c_1^2c_2^2 + 52c_2^3 + 42c_1^3c_3 \\ &- 72c_1c_2c_3 - 30c_3^2 + 54c_1^2c_4 + 36c_2c_4). \end{aligned} \tag{3.6}$$

By rearranging the terms in (3.6), we can write the right-hand side as

$$\begin{aligned} \tfrac{1}{4320}(36(c_2c_4 - c_3^2 + c_1^2c_4 - 2c_1c_2c_3 + c_2^3) + 16c_1^2(c_4 + 2c_1c_3 + c_2^2 + 3c_1^2c_2 + c_1^4) \\ + 3c_1^2(c_4 + 2c_1c_3 + c_1^2c_2) + c_1^2(6c_1^2c_2 + 9c_2^2 + 6c_1c_3) + 16c_2^2 + 6c_3^2). \end{aligned}$$

By applying the triangle inequality and using Lemmas 2.3, 2.4 and 2.5,

$$|H_{2,2}(F_f/2)| \leq \tfrac{1}{4320}\, g(|c_1|, |c_2|),$$

where

$$\begin{aligned} g(x, y) = 18x^2 + 36(1 - x^2) + 6x^4y + 16y^3 + 9x^2y^2 \\ + 6x^3\Big(1 - x^2 - \frac{y^2}{1+x}\Big) + 6\Big(1 - x^2 - \frac{y^2}{1+x}\Big)^2. \end{aligned}$$

Since the class $\mathcal{C}$ and the functional $|H_{2,2}(F_f/2)|$ are rotationally invariant, we can assume that $c \in [0, 1]$. Further, in view of Lemma 2.1, the region of variability of the pair (x, y) coincides with the set

$$\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2\}.$$

Since y varies between 0 and 1, it is clear that

$$g(x, y) \leq G(x, y),$$

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where

$$G(x,y) = 18x^2 + 36(1-x^2) + 6x^4y + 16y^2 + 9x^2y^2 \\ + 6x^3\left(1 - x^2 - \frac{y^2}{1+x}\right) + 6\left(1 - x^2 - \frac{y^2}{1+x}\right)^2.$$

Therefore,

$$|H_{2,2}(F_f/2)| \leq \tfrac{1}{4320}\, G(x,y). \tag{3.7}$$

We need to find the maximum value of $G(x,y)$ over the region Ω. The critical points of G satisfy the conditions

$$\frac{\partial G}{\partial x} = \frac{6}{(1+x)^2}(10x + 27x^2 + 17x^3 - 6x^4 + 11x^6 + 5x^7 - 4x^3y - 12x^4y \\ - 12x^5y - 4x^6y - 2y^2 - 9xy^2 - 12x^2y^2 - 6x^3y^2 - x^4y^2 + 2y^4) = 0$$

$$\frac{\partial G}{\partial y} = \frac{12x^5 + 6x^6 + 40xy + 74x^2y + 48x^3y + 6x^4(1+y) + 8(y+3y^3)}{(1+x)^2} = 0.$$

It is clear that, in the interior of Ω, $\partial G/\partial y > 0$. So the function $G(x,y)$ has no critical point in the interior of Ω and cannot have a maximum in the interior of Ω. Since G is continuous in the compact set Ω, it attains its maximum on the boundary of Ω. Considering the boundary of Ω leads to the following three cases.

Case (i): if $x = 0$ and $0 \leq y \leq 1$, then $G(0,y) = 42 + 4y^2 + 6y^4 \leq 52$.

Case (ii): if $y = 0$ and $0 \leq x \leq 1$, then

$$G(x,0) = 42 - 30x^2 + 6x^3 + 6x^4 - 6x^5 = h_1(x).$$

Since $h_1'(x) = -60x + 18x^2 + 24x^3 - 30x^4 < 0$ for $0 < x \leq 1$, the function h_1 is decreasing. So $G(x,0) \leq h_1(0) = 42$.

Case (iii): if $y = 1 - x^2$ and $0 \leq x \leq 1$, then

$$G(x, 1-x^2) = 52 - 35x^2 - 2x^4 + 3x^6 = h_2(x).$$

Clearly, $h_2'(x) = -70x - 8x^3 + 18x^5 < 0$ for $0 < x < 1$. Hence, the function h_2 is decreasing and $G(x, 1-x^2) \leq h_2(0) = 52$.

Thus, combining the three cases,

$$\max_{(x,y)\in\Omega} G(x,y) = 52$$

and by (3.7),

$$|H_{2,2}(F_f/2)| \leq \tfrac{13}{1080}. \tag{3.8}$$

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To prove the equality in (3.8), we consider the function

$$f(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots .$$

A simple computation shows that f belongs to the class C and $|H_{2,2}(F_f/2)| = 13/1080$. This completes the proof. □

EXAMPLE 3.4. For an example illustrating Theorem 3.3, consider the function

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots$$

It is easy to see that the function f belongs to the class C and $\gamma_n(f) = 1/2n$. Hence,

$$|H_{2,2}(F_f/2)| = |\gamma_2\gamma_4 - \gamma_3^2| = \tfrac{1}{288} \leq \tfrac{13}{1080}.$$

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