# TOPOLOGIES ON GENERALIZED INNER PRODUCT SPACES

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**1. Introduction.** In the present note we introduce a straightforward algebraic generalization of inner product spaces, which we appropriately name *generalized inner product* (GIP) *spaces*. In the same fashion in which different topologies can be introduced in inner product spaces, adequate topologies can be introduced in GIP spaces in such a manner that topological vector spaces are obtained. We enumerate and derive some fundamental properties of different topologies in GIP spaces, having primarily in mind their possible later application to quantum physics.

The desirability of having in quantum physics more general structures than Hilbert spaces (in which quantum mechanics is usually formulated) is suggested by Dirac's formalism (**2**), which deals with "unnormalizable" vectors. Unfortunately, although this formalism is very elegant from the point of view of the facility of dealing with its symbolism, it completely lacks in mathematical rigour. Recent attempts (cf. **1** for detailed references) have been made to treat Dirac's formalism in the context of rigged Hilbert spaces. However, we believe that the GIP spaces introduced here might offer some advantages, since their definition is purely algebraic (like that of inner product spaces) as opposed to the definition of rigged Hilbert spaces (**1**) which contains a topology inseparably embedded in it.

**2. Algebraic properties of generalized inner product spaces.** An inner product space is a linear space on which an inner (scalar) product $(x, y)$ is defined. When the linear space is complex, we adopt the convention that $(x, y)$ is anti-linear with respect to the first argument, and consequently linear with respect to the second argument.

*Definition* 2.1. A linear space $\mathscr{L}$ is a GIP space if and only if:

1. There is a subspace $\mathscr{N}$ of $\mathscr{L}$ which is an inner product space (which will be called the nucleus of the GIP space);

2. There is a set $\mathscr{A}$ of linear operators on $\mathscr{L}$ which is *adequate with respect to* $\mathscr{N}$, i.e., it has the following properties:

(a) Each element of $\mathscr{A}$ maps $\mathscr{L}$ into $\mathscr{N}$, i.e. $\mathscr{A}\,\mathscr{L} \subset \mathscr{N}$;

(b) The relation $Ax = 0$ is satisfied for all $A \in \mathscr{A}$ only by $x = 0$.

We denote such a GIP space by the triple $(\mathscr{L}, \mathscr{A}, \mathscr{N})$. Clearly, every inner product space is also a GIP space in a trivial sense, i.e. $\mathscr{N} = \mathscr{L}$ and $\mathscr{A} = \{\mathbf{1}\}$,

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where **1** denotes the identity operator on $\mathscr{L}$. A non-trivial example is the following.

*Example* 2.1. Take $\mathscr{L} = \mathscr{C}^0[(-\infty, +\infty)]$, i.e., $\mathscr{L}$ is the family of all real continuous functions on the real line. Choose the nucleus to consist of all square integrable functions in $\mathscr{C}^0[(-\infty, +\infty)]$, and adopt the inner product in $\mathscr{N}$ to be

$$(x, y) = \int_{-\infty}^{+\infty} x(t)y(t)\,dt.$$

Take $\mathscr{A}$ to be the family of all projectors $E(I)$,

$$(E(I)x)(t) = \chi_I(t)x(t)$$

($\chi_S(t)$ denotes the characteristic function of the set $S$) corresponding to all the finite non-degenerate intervals. It is straightforward to check that the present $(\mathscr{L}, \mathscr{A}, \mathscr{N})$ is a GIP space, which we denote by $\mathscr{C}_0$.

The concept of GIP space is not general enough to cover all the instances which could be of real interest in quantum physics; hence, a more general definition is also desirable.

*Definition* 2.2. A union $\mathscr{L} = \bigcup_{r \in R} \mathscr{L}_r$ of linear spaces $\mathscr{L}_r$ over the same field is a *composite* GIP space if each of the spaces $\mathscr{L}_r$ is a GIP space and all $\mathscr{L}_r$ have a common nucleus, i.e., $\mathscr{N} \subset \bigcap_{r \in R} \mathscr{L}_r$.

Such a GIP space will be denoted by $(\mathscr{L}_r, \mathscr{A}_r, \mathscr{N}), r \in R$, where $\mathscr{A}_r$ denotes the family of operators on $\mathscr{L}_r$ which is adequate with respect to $\mathscr{N}$, and write

$$\mathscr{A} = \bigcup_{r \in R} \mathscr{A}_r.$$

The following proposition follows immediately from Definitions 2.1 and 2.2.

PROPOSITION 2.1. *If* $Ax = 0$ *for all* $A \in \mathscr{A}$, *then* $x = 0$.

*Example* 2.2. Let $\mathscr{S}'$ denote the space of all Schwartz distributions on the space $\mathscr{S}$ of all infinitely differentiable functions on the real line of faster than polynomial decrease at infinity (**5**). Denote by $\mathscr{S}_p'$ the regular (in the Gel'fand (**3**) sense) distributions which can be represented by piecewise continuous bounded functions, and by $\mathscr{S}_q'$ those whose Fourier transforms can be represented by piecewise continuous bounded functions, i.e. $x \in \mathscr{S}_p'$ if

$$x(f) = \int_{-\infty}^{+\infty} x(q)f(q)\,dq, \qquad f \in \mathscr{S},$$

and $y \in \mathscr{S}_q'$ if

$$y(g) = \int_{-\infty}^{+\infty} y(p)\tilde{g}(p)\,dp, \qquad g \in \mathscr{S},$$

where

$$\tilde{g}(p) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{ipq} g(q)\,dq,$$

while $x(q)$ and $y(p)$ are bounded and piecewise continuous. Let $\mathscr{S}_{(p,q)}'$ be the union of $\mathscr{S}_p'$ and $\mathscr{S}_q'$. Take $\mathscr{L} = \mathscr{S}_{(p,q)}'$ and choose the nucleus $\mathscr{N}$ to consist of those regular distributions in $\mathscr{S}_{(p,q)}'$ which can be represented by square integrable functions. Let $\mathscr{A}_p$ and $\mathscr{A}_q$ denote, respectively, the sets of all the projectors $E(I)$ and $F(I)$ corresponding to finite non-degenerate intervals

$$\widetilde{(E(I)x)}(p) = \chi_I(p)\tilde{x}(p), \qquad (F(I)y)(q) = \chi_I(q)y(q).$$

The resulting structure is a composite GIP space.

Note that the eigenfunctions of the operators $Q$ and $P$,

$$(Qx)(q) = qx(q), \qquad (Py)(p) = py(p),$$

belong to $\mathscr{S}_{(p,q)}'$; we have that

$$Q\delta(q - q') = q'\delta(q - q') \quad \text{and} \quad P\delta(p - p') = p'\delta(p - p').$$

PROPOSITION 2.2. *If $\mathscr{L}$ is a GIP space, and for some $x \in \mathscr{L}$ we have that $(y, Ax) = 0$ for all $y \in \mathscr{N}$, $A \in \mathscr{A}$, then $x = 0$.*

*Proof*. For any given $A \in \mathscr{A}$ and given $x \in \mathscr{L}$ we have that $Ax \in \mathscr{N}$. Hence, $(y, Ax) = 0$ for all $y \in \mathscr{N}$ implies that $Ax = 0$. Since this is true for any $A \in \mathscr{A}$, we obtain, from Proposition 2.1, that $x = 0$.

PROPOSITION 2.3. *If $\mathscr{L}$ is a GIP space and for some $x \in \mathscr{L}$ we have that $(Ax, Ax) = 0$ for all $A \in \mathscr{A}$, then $x = 0$.*

*Proof*. If $(Ax, Ax) = 0$, then $Ax = 0$. As this is true for any $A \in \mathscr{A}$, it follows from Proposition 2.1 that $x = 0$.

**3. Strong topologies.** There are obviously many convenient ways to introduce a topology in a GIP space in order to obtain a topological vector space. We can discriminate among all the alternatives by choosing those topologies which could make a GIP space of use in quantum physics.

As far as composite GIP spaces are concerned, we can either introduce a topology which would make it into a topological space (in general not linear) or we can treat each of its component GIP spaces separately.

We shall introduce in GIP spaces strong topologies by constructing neighbourhood bases of some point $x \in \mathscr{L}$ from sets of the form

$$V(x; A_1, \ldots, A_n; \epsilon) = \{y: ||A_1(y - x)|| < \epsilon, \ldots, ||A_n(y - x)|| < \epsilon, \quad y \in \mathscr{L}\}$$

for all $\epsilon > 0$, $A_1, \ldots, A_n \in \mathscr{A}$ and $n = 1, 2, \ldots$. We deduce a few important features of these topologies by establishing some properties of the sets $V(x; A_1, \ldots, A_n; \epsilon)$ in the following two lemmas.

LEMMA 3.1. *Each $V(0; A_1, \ldots, A_n; \epsilon)$ is balanced and convex, and therefore absolutely convex.*

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*Proof.* If $x \in V(0; A_1, \ldots, A_n; \epsilon)$, then

$$||A_1 x|| < \epsilon, \ldots, ||A_n x|| < \epsilon,$$

and consequently, for $|\lambda| \leqq 1$,

$$||A_k(\lambda x)|| = |\lambda| \, ||A_k x|| < \epsilon, \qquad k = 1, \ldots, n,$$

i.e., $\lambda x \in V(0; A_1, \ldots, A_n; \epsilon)$. Thus, $V(0; A_1, \ldots, A_n; \epsilon)$ is balanced. Furthermore, it is convex since, if $x_1, x_2 \in V(0; A_1, \ldots, A_n; \epsilon)$ and $0 \leqq \lambda \leqq 1$, then

$$||A_k(\lambda x_1 + (1 - \lambda)x_2)|| \leqq \lambda ||A_k x_1|| + (1 - \lambda)||A_k x_2|| < \epsilon,$$

i.e.,

$$\lambda x_1 + (1 - \lambda)x_2 \in V(0; A_1, \ldots, A_n; \epsilon).$$

LEMMA 3.2. *If in a GIP space* $(\mathscr{L}, \mathscr{A}, \mathscr{N})$ *a topology is introduced in which the sets* $V(x; A; \epsilon)$ *are neighbourhoods of* $x$ *for all* $\epsilon > 0, A \in \mathscr{A}$, *then the resulting topological space is Hausdorff.*

*Proof.* If the topological space is not Hausdorff, then there are at least two elements $x_1, x_2 \in \mathscr{L}$, $x_1 \neq x_2$, for which any two neighbourhoods have common points. Thus, for any $V(x_1; A; 1/n)$ and $V(x_2; A; 1/n)$ there is at least one $y_n \in \mathscr{L}$ such that

$$y_n \in V\left(x_1; A; \frac{1}{n}\right) \cap V\left(x_2; A; \frac{1}{n}\right).$$

Therefore,

$$||A(x_1 - y_n)|| < \frac{1}{n}, \qquad ||A(x_2 - y_n)|| < \frac{1}{n},$$

and we have that

$$||A(x_1 - x_2)|| \leqq ||A(x_1 - y_n)|| + ||A(x_2 - y_n)|| < \frac{2}{n}.$$

Since the above is true for any positive integer $n$, it follows that $A(x_1 - x_2) = 0$. Since this conclusion is true for any $A \in \mathscr{A}$, we obtain from Definition 2.1 that $x_1 - x_2 = 0$, i.e. $x_1 = x_2$, contrary to the assumption.

THEOREM 3.1. *In the strong topology on the GIP space* $(\mathscr{L}, \mathscr{A}, \mathscr{N})$, *defined as the topology in which the family of all sets* $V(x; A_1, \ldots, A_n; \epsilon)$, $x \in \mathscr{L}$, $\epsilon > 0$, $A_1, \ldots, A_n \in \mathscr{A}$, $n = 1, 2, \ldots$, *constitute a neighbourhood basis,* the space* $\mathscr{L}$ *is a locally convex Hausdorff linear space.*

*Proof.* The above topology is compatible with the vector operations. For instance, the operation of vector summation is continuous since for any $V(x_1 + x_2; A_1, \ldots, A_n; \epsilon)$ we have that

$$y_1 + y_2 \in V(x_1 + x_2; A_1, \ldots, A_n; \epsilon)$$

*If the GIP space is an inner product space with $\mathscr{A} = \{\mathbf{1}\}$, this strong topology is the norm topology in $\mathscr{L}$.

whenever

$$y_1 \in V(x_1; A_1, \ldots, A_n; \tfrac{1}{2}\epsilon), \qquad y_2 \in V(x_2; A_1, \ldots, A_n; \tfrac{1}{2}\epsilon)$$

since

$$||A_k(y_1 + y_2 - x_1 - x_2)|| \leqq ||A_k(y_1 - x_1)|| + ||A_k(y_2 - x_2)|| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon, \quad k = 1, \ldots, n.$$

Similarly, it is easy to show that the operation of multiplication by a scalar is continuous.

In the resulting topology, $\mathscr{L}$ is Hausdorff according to Lemma 3.2, and is locally convex due to Lemma 3.1.

COROLLARY. *The sets $V(0; A; \epsilon)$ are absorbent.*

For the proof cf. (**2**, Chapter 1, Proposition 3(i)).

THEOREM 3.2. *The GIP space $(\mathscr{L}, \mathscr{A}, \mathscr{N})$ with the ultra-strong topology, in which the family of all sets*

$$V(x; A_1, A_2, \ldots; \epsilon) = \bigcap_{k=1}^{\infty} V(x; A_k; \epsilon), \qquad A_k \in \mathscr{A}, \epsilon > 0,$$

*form a neighbourhood basis, is a locally convex linear Hausdorff space.*

The proof of the above theorem is a slightly altered version of the proof of Theorem 3.1.

Clearly, the ultra-strong topology is finer than the strong topology.

In settling the important question of completion, it is very convenient when a topological vector space is metrizable. The following theorem covers a great number of practically important instances of GIP spaces.

THEOREM 3.3. *A GIP space with strong (ultra-strong) topology is metrizable if there is a countable subset $\mathscr{B}$ of $\mathscr{A}$ which has the property that for any $A \in \mathscr{A}$ there is a $B$ in the linear manifold $L_B$ generated by $\mathscr{B}$, such that*

$$||Bx|| \geqq ||Ax|| \tag{1}$$

*for all $x \in \mathscr{L}$.*

*Proof.* We shall show that the family

$$\left\{ V\left(0; B_1, \ldots, B_k; \frac{1}{n}\right); B_1, \ldots, B_k \in \mathscr{B}, k, n = 1, 2, \ldots \right\} \tag{2}$$

is a neighbourhood basis of the origin in the strong topology.

For every $A \in \mathscr{A}$ we can find, due to (1), a $B \in L_B$ for which

$$V(0; B; \epsilon) \subset V(0; A; \epsilon).$$

As $\mathscr{B}$ generates $L_B$, we have that

$$B = \lambda_1 B_1 + \ldots + \lambda_k B_k, \qquad B_1, \ldots, B_k \in \mathscr{B},$$

and consequently

$$||Bx|| \leqq |\lambda_1| \, ||B_1 x|| + \ldots + |\lambda_k| \, ||B_k x||$$

for all $x \in \mathscr{L}$. Thus, if we choose an integer $n$ such that

$$\frac{1}{n} \leqq \frac{\epsilon}{k|\lambda_1|}, \ldots, \frac{1}{n} \leqq \frac{\epsilon}{k|\lambda_k|},$$

we have that

$$V(0; B; \epsilon) \supset V\left(0; B_1; \frac{1}{n}\right) \cap \ldots \cap V\left(0; B_k; \frac{1}{n}\right)$$
$$= V\left(0; B_1, \ldots, B_n; \frac{1}{n}\right),$$

which shows that the family (2) is a neighbourhood basis of the origin. As the set (2) is obviously countable since $\mathscr{B}$ is countable, it follows (cf. **4**, Chapter **1**, Theorem 4) that $\mathscr{L}$ is metrizable in the strong topology.

The proof for the ultra-strong topology can be obtained in the same manner.

In general, a composite GIP space is not a linear space. When we introduce topologies separately in each of the GIP spaces constituting a composite GIP space, then a special case is very desirable.

*Definition* 3.1. The topologies on the GIP spaces constituting a composite GIP space are *compatible* if the corresponding topologies induced on the nucleus $\mathscr{N}$ of the GIP space are equivalent.

## 4. Strong topologies in special cases.

PROPOSITION 4.1. *The GIP space* $\mathscr{C}_0$ (Example 2.1) *is metrizable in the strong and ultra-strong topology.*

*Proof.* Select the countable family

$$\mathscr{B} = \{E([n, n + 1)) : n = 0, \pm 1, \pm 2, \ldots\}$$

of projectors from $\mathscr{A}$. We prove the proposition by showing that $\mathscr{B}$ fulfills condition (2) appearing in Theorem 3.3.

If $E(I) \in \mathscr{A}$, then $I$ is a finite interval, and consequently, integers $m_1$ and $m_2$, $m_2 > m_1$, can be found so that

$$I \subset \bigcup_{n=m_1}^{m_2} [n, n + 1).$$

Therefore, we obviously have that for any $x \in \mathscr{L}$

$$||E(I)x|| \leqq ||Bx||, \qquad B = \sum_{n=m_1}^{m_2} E([n, n + 1)),$$

where $B$ evidently belongs to the linear manifold generated by $\mathscr{B}$.

For the strong and ultra-strong topologies on the component GIP spaces $\mathscr{S}_p'$ and $\mathscr{S}_q'$ of $\mathscr{S}_{(p,q)}'$ in Example 2.2, we can prove the following results in a similar manner.

PROPOSITION 4.2. *The spaces $\mathscr{S}_p'$ and $\mathscr{S}_q'$ are metrizable in the strong and ultra-strong topology.*

PROPOSITION 4.3. *The ultra-strong topologies on $\mathscr{S}_q'$ and $\mathscr{S}_p'$ in the GIP space $\mathscr{S}_{(q,p)}'$ are compatible* (cf. Definition 3.1 at the end of § 3); *in $\mathscr{N}$ they are both equivalent to the norm topology.*

*Proof.* We prove the proposition by showing that the ultra-strong topology on $\mathscr{S}_p'$ induces in $\mathscr{N}$ a topology equivalent to the norm topology; the case of $\mathscr{S}_p'$ can be treated in a very similar manner.

In the norm topology of $\mathscr{N}$, the family of all sets $N(\epsilon) = \{x: ||x|| < \epsilon, x \in \mathscr{N}\}$ corresponding to all $\epsilon > 0$ constitutes a neighbourhood basis of the origin. Since $||E(I)x|| \leqq ||x||$ for any $A = E(I) \in \mathscr{A}_q$, it follows that

$$N(\epsilon) \subset V_0(0; A; \epsilon) \quad \text{for all } A \in \mathscr{A}_q, \qquad \epsilon > 0,$$

where

$$V_0(0; A; \epsilon) = \{x: ||Ax|| < \epsilon, x \in \mathscr{N}\}, \tag{3}$$

and consequently the norm topology is finer than the ultra-strong topology induced on $\mathscr{N}$.

On the other hand,

$$\lim_{n\to\infty} ||A_n x - x|| = 0, \qquad A_n = E([-n, n]),$$

and consequently

$$V_0(0; A_1, A_2, \ldots; \epsilon/2) = \bigcap_{n=1}^{\infty} V_0(0; A_n; \epsilon/2) \subset N(\epsilon),$$

i.e., the induced ultra-strong topology is finer than the norm topology. Thus, they are equivalent.

PROPOSITION 4.4. *The strong topologies of $\mathscr{S}_q'$ and $\mathscr{S}_p'$ in $\mathscr{S}_{(p,q)}'$ are not compatible, nor does one of these topologies induce on $\mathscr{N}$ a topology which is finer than the topology induced by the other.*

*Proof.* If the topologies of $\mathscr{S}_q'$ and $\mathscr{S}_p'$ were compatible, then they would induce in $\mathscr{N}$ equivalent topologies. Thus, if, using the notation (3), $V_0(0; E(I); \epsilon)$ is a neighbourhood of the origin of $\mathscr{N}$ in the topology induced in $\mathscr{N}$ by the topology on $\mathscr{S}_q'$, then it should contain some neighbourhood of the topology induced in $\mathscr{N}$ by the topology of $\mathscr{S}_p'$. Such a neighbourhood has the most general form

$$\bigcap_{k=1}^{n} V_0(0; F(I_k); \epsilon_k), \qquad F(I_k) \in \mathscr{A}_p.$$

But then we also have that

$$V_0(0; F(J); \delta) \subset V_0(0; E(I); \epsilon), \tag{4}$$

where

$$\delta = \min_{k=1,\ldots,n} \epsilon_k, \qquad J = \bigcup_{k=1}^{n} I_k.$$

In order to see that (4) is not true, note that any $x \in \mathscr{N}$ satisfying

$$||E(I)x||^2 = \int_I |\tilde{x}(p)|^2 \, dp < \epsilon \tag{5}$$

lies within $V_0(0; E(I); \epsilon)$. As $I$ is a finite interval, we can consider functions $\tilde{x}(p)$ which satisfy (5) but have a $\delta$-like behaviour with a sharp peak *outside* $I$. The more such a function resembles the $\delta$-function, the more will its inverse Fourier transform $x(q)$ behave almost like a constant. By choosing an $\tilde{x}_1(p)$ satisfying (5) with a sufficiently high and sharp peak outside $I$, we can satisfy the inequality

$$|x_1(q)| > (\delta/|J|)^{1/2} \quad \text{for all } q \in J,$$

where $|J|$ is the sum of the lengths of all disjoint intervals constituting $J$. Consequently, we have that

$$||F(J)x_1|| = \int_J |x_1(q)|^2 \, dq > \delta,$$

and therefore

$$x_1 \in V_0(0; E(I); \epsilon), \qquad x_1 \notin V_0(0; F(J); \delta),$$

proving that (4) is false. Thus, the strong topology induced in $\mathscr{N}$ from $\mathscr{S}_p'$ is not finer than the topology induced in $\mathscr{N}$ from $\mathscr{S}_p'$; the converse can be proved in precisely the same manner.

**5. Dual spaces.** If $(\mathscr{L}, \mathscr{A}, \mathscr{N})$ is a GIP space, we can assign to each $A \in \mathscr{A}$ and each $\xi \in \mathscr{N}$ a linear functional

$$\phi(x; A, \xi) = (\xi, Ax)$$

on $\mathscr{L}$. Denote by $\mathscr{M}_0$ the family of all such functionals. Note that, in general, $\mathscr{M}_0$ is not a linear space.

THEOREM 5.1. *The linear space $\mathscr{M}$ (over the same scalar field as $\mathscr{L}$) spanned by $\mathscr{M}_0$ and the linear space $\mathscr{L}$ constitute a dual pair* (cf. **4**, Chapter II, § 3).

*Proof.* If $\phi(x) = 0$ for all $\phi \in \mathscr{M}$, then

$$(\xi, Ax) = 0,$$

for all $\xi \in \mathscr{N}$ and all $A \in \mathscr{A}$. According to Proposition 2.2, the above implies that $x = 0$.

Vice versa, if for a given $\phi_0 \in \mathscr{M}$ we have that $\phi_0(x) = 0$ for all $x \in \mathscr{M}$, then, by definition, $\phi_0$ is the zero element of $\mathscr{M}$.

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We introduce the notation

$$\langle \phi, x \rangle = \phi(x), \qquad \phi \in \mathscr{M}, x \in \mathscr{L}.$$

Obviously we have the following results.

PROPOSITION 5.1. *$\langle \phi, x \rangle$ is a bilinear form on $\mathscr{M}$ and $\mathscr{L}$.*

THEOREM 5.2. *Each $\phi \in \mathscr{M}$ is continuous on the vector space $\mathscr{L}$ with the strong or ultra-strong topology.*

*Proof*. For arbitrary $\epsilon > 0$ we have that

$$|\phi(x; A, \xi) - \phi(x_0; A, \xi)| = |(\xi, A(x - x_0))| \leqq ||\xi|| \, ||A(x - x_0)|| < \epsilon$$

whenever

$$||A(x - x_0)|| < \epsilon/||\xi||,$$

i.e. for all

$$x \in V(x_0; A; \epsilon/||\xi||).$$

Thus, each element of $\mathscr{M}_0$ is a continuous functional on $\mathscr{L}$ when $\mathscr{L}$ is supplied with the strong or ultra-strong topology. Hence, the continuity of an arbitrary element of $\mathscr{M}$ follows.

The above proposition tells us that $\mathscr{M}$ is contained in the linear space conjugate to the space $\mathscr{L}$ with a strong topology. However, we can sometimes extend the above result, as in the following theorem.

THEOREM 5.3. *If the nucleus $\mathscr{N}$ of a GIP space $(\mathscr{L}, \mathscr{A}, \mathscr{N})$ is finite-dimensional, then $\mathscr{M}$ is isomorphic to the linear space conjugate to the space $\mathscr{L}$ with the strong topology.*

*Proof*. We have to show that if $f(x)$ is a linear functional continuous on $\mathscr{L}$ provided with the strong topology, then necessarily $f \in \mathscr{M}$.

Since $f(x)$ is continuous, for a given $\epsilon$, $0 < \epsilon < 1$, we can find a strong neighbourhood $V(0; A_1, \ldots, A_k; \delta)$ of the origin such that $|f(x)| < \epsilon$ for all $x$ from the above neighbourhood.

On the other hand, as $\mathscr{N}$ is finite-dimensional, there is a basis $\xi_1, \ldots, \xi_n \in \mathscr{N}$ spanning $\mathscr{N}$. Consider the finite set of continuous linear functionals

$$\phi_{ij}(x) = (\xi_i, A_j x), \qquad i = 1, \ldots, n, \qquad j = 1, \ldots, k. \tag{6}$$

If $f(x)$ were independent of the above family (6) of linear functionals, then there would be an element $x_1 \in \mathscr{L}$ for which (cf. **4**, p. 32, Lemma 5)

$$f(x_1) = 1, \qquad \phi_{11}(x_1) = \ldots = \phi_{nk}(x_1) = 0. \tag{7}$$

Since $\xi_1, \ldots, \xi_n$ is a basis in $\mathscr{N}$, the above would imply that

$$A_1 x_1 = \ldots = A_k x_1 = 0,$$

and consequently that $x_1$ belongs to the neighbourhood $V(0; A_1, \ldots, A_k; \delta)$. But if $x_1$ belongs to $V(0; A_1, \ldots, A_k; \delta)$, then $|f(x_1)| < 1$, which contradicts the first relation in (7).

It is easy to find non-trivial examples of GIP spaces with finite-dimensional nuclei, i.e. examples for which $\mathscr{N} \neq \mathscr{L}$. For instance, choose $\mathscr{L}$ to be the family of all one-row infinite matrices with real elements $(a_1, a_2, \ldots)$, and take $\mathscr{N}$ to be the one-dimensional space of all one-row real matrices $(a_1, 0, 0, \ldots)$ in which only the first element is non-vanishing. Adopt in $\mathscr{N}$ the customary inner product. If we choose

$$\mathscr{A} = \{P_1, P_2, \ldots\},$$

where $P_n$ is the linear operator

$$P_n(a_1, a_2, \ldots, a_n, \ldots) = (a_n, 0, \ldots, 0, \ldots),$$

then $(\mathscr{L}, \mathscr{A}, \mathscr{N})$ constitutes a GIP space with a one-dimensional nucleus.

**6. Weak topologies.** Following standard terminology, we call the coarsest topology on $\mathscr{L}$ in which all the linear functionals from $\mathscr{M}$ are continuous the *weak topology*. As is well-known (cf. **4**, Chapter II, § 3), the family of all subsets of $\mathscr{L}$

$$W(x_0; \phi_1, \ldots, \phi_n) = \{x: |\phi_1(x - x_0)| < 1, \ldots, |\phi_n(x - x_0)| < 1\}$$

corresponding to all $\phi_1, \ldots, \phi_n \in \mathscr{M}$, $n = 1, 2, \ldots$, is a neighbourhood basis of $x_0 \in \mathscr{L}$. As $\mathscr{M}_0$ generates $\mathscr{M}$, the family of all neighbourhoods

$$W(0; \xi_1, A_1, \ldots, \xi_n, A_n) = \{x: |(\xi_1, A_1 x)| < 1, \ldots, |(\xi_n, A_n x)| < 1\}$$

corresponding to all $\xi_1, \ldots, \xi_n \in \mathscr{N}$, $A_1, \ldots, A_n \in \mathscr{A}$, $n = 1, 2, \ldots$, is also a neighbourhood basis of the origin. As $\mathscr{L}$ and $\mathscr{M}$ are dual pairs (Proposition 5.1), we have that $\mathscr{L}$ is a Hausdorff topological space in the weak topology. Due to the general properties of weak topologies we have the following result.

PROPOSITION 6.1. *The space $\mathscr{L}$ provided with the weak topology is a locally convex Hausdorff vector space.*

We can define in a similar manner a topology on $\mathscr{L}$ given by the neighbourhood basis of each $x_0 \in \mathscr{L}$, where this neighbourhood basis is the family of all sets

$$W(x_1; \phi_1, \ldots, \phi_n, \ldots) = \{x: |\phi_1(x - x_0)| < 1, \ldots, |\phi_n(x - x_0)| < 1, \ldots\}$$

corresponding to all sequences $\phi_1, \ldots, \phi_n, \ldots \in \mathscr{M}$. We call the above topology the *infra-weak topology*.

It is very easy to check that the infra-weak topology is compatible with the vector operations on $\mathscr{L}$. The sets $W(x_0; \phi_1, \ldots, \phi_n, \ldots)$ are obviously convex. Furthermore, since the infra-weak topology is evidently finer than the weak topology, $\mathscr{L}$ is also separated under this topology. To summarize, we have the following result.

PROPOSITION 6.2. *The infra-weak topology on $\mathscr{L}$ is finer than the weak topology, and $\mathscr{L}$ is a locally convex Hausdorff vector space in this topology.*

A non-trivial example of an infra-weak topology is obtained when this topology is introduced in the space $(\mathcal{S}_q', \mathcal{A}_q, \mathcal{N})$ defined in Example 2.2. We then have the following result.

PROPOSITION 6.3. *The infra-weak topology on the GIP space* $(\mathcal{S}_q', \mathcal{A}_q, \mathcal{N})$ *is finer than the topology induced in* $\mathcal{S}_q'$ *by the topology on the Schwartz space* $\mathcal{S}'$ (see **5**).

*Proof.* For the topology induced in $\mathcal{S}_q'$ by the topology of $\mathcal{S}'$, the family of all sets

$$S(f_1, \ldots, f_n; \epsilon) = \{x: |x(f_1)| < \epsilon, \ldots, |x(f_n)| < \epsilon, x \in \mathcal{S}_q'\}$$

corresponding to all $\epsilon > 0, f_1, \ldots, f_n \in \mathcal{S}, n = 1, 2, \ldots,$ constitutes a neighbourhood basis of the origin. According to the definition of $\mathcal{S}_q'$, to each $x \in \mathcal{S}_q'$ corresponds a bounded piecewise continuous function $x(p)$ such that for any $f \in \mathcal{S}$

$$x(f) = \int_{-\infty}^{+\infty} x(p)\tilde{f}(p)\, dp,$$

where $\tilde{f}(p)$ is the Fourier transform of $f$. Thus, $x \in S(f_1, \ldots, f_n; \epsilon)$ if and only if

$$\left| \int_{-\infty}^{+\infty} x(p)\tilde{f}_1(p)\, dp \right| < \epsilon, \ldots, \left| \int_{-\infty}^{+\infty} x(p)\tilde{f}_n(p)\, dp \right| < \epsilon.$$

By taking the countable set of elements $\xi_{ik}$, $i = 1, \ldots, n$, $k = 0, \pm 1, \ldots,$ defined by the square integrable functions

$$\xi_{ik}(p) = \frac{2^{k+2}}{\epsilon} \chi_k(p)\tilde{f}_i(p), \qquad \chi_k(p) = \begin{cases} 0, & p < k, p \geqq k+1, \\ 1, & k \leqq p < k+1, \end{cases}$$

we see that

$$W(0; \phi_{11}, \phi_{21}, \ldots) \subset S(f_1, \ldots, f_n; \epsilon)$$

with

$$\phi_{ik}(x) = (\xi_{ik}, E(I_k)x);$$

namely, for any $x \in W(0; \phi_{11}, \phi_{21}, \ldots)$ we have that

$$\frac{2^{k+2}}{\epsilon} \left| \int_{I_k} x(p)\tilde{f}_i(p)\, dp \right| = |(\xi_{ik}, E(I_k)x)| < 1, \qquad i = 1, \ldots, n,$$

and consequently

$$\left| \int_{-\infty}^{+\infty} x(p)f_i(p)\, dp \right| \leqq \sum_{k=1}^{n} \left| \int_{I_k} x(p)\tilde{f}_i(p)\, dp \right| < \frac{\epsilon}{4} \sum_{k=-\infty}^{+\infty} \frac{1}{2^k} < \epsilon.$$

It is easy to establish, however, that the weak topology on $\mathcal{S}_q'$ is neither finer nor coaser than the topology induced in $\mathcal{S}_q'$ by the topology on $\mathcal{S}'$. On the other hand, we have the following result.

PROPOSITION 6.4. *The weak topology on* $(\mathscr{S}_q', \mathscr{A}_q, \mathscr{N})$ *is finer than the topology induced in* $\mathscr{S}_q'$ *by the topology of the Schwartz space* $\mathscr{D}'$ *(the space of distributions on the space of all infinitely differentiable functions, on the real line, with compact support* (**5**)).

The proof of this proposition can be carried out in a manner analogous to the way of proving Proposition 6.3.

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