FUNCTIONS WHICH HAVE GENERALIZED RIEMANN DERIVATIVES

C. KASSIMATIS

1. Introduction. Let $f(x)$ be a measurable function defined in the interval (a, b), and let

$$\Delta_n(x, 2h; f) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} f(x + 2jh - nh) \qquad (h > 0; n = 1, 2, \ldots).$$

If the limit of $(2h)^{-n}\,\Delta_n(x, 2h; f)$ exists and is finite at the point x, as $h \to 0$, it is called the nth generalized Riemann derivative of $f(x)$ at the point x, $D^n f(x)$. Considering the upper and lower limits of the above expression we can similarly define the upper and lower nth generalized Riemann derivates, $\bar{D}^n f(x)$ and $\underline{D}^n f(x)$ respectively. If $\bar{D}^n f(x) = \underline{D}^n f(x)$, their common value is the nth generalized Riemann derivative $D^n f(x)$.

If two functions $F(x)$ and $G(x)$ are such that the nth ordinary derivative of $F(x) - G(x)$ is equal to zero then $F(x)$ and $G(x)$ differ by a polynomial of degree at most $n - 1$. The main purpose of this paper is to study the relations between two functions $F(x)$ and $G(x)$ where the nth generalized Riemann derivative of the continuous function $F(x) - G(x)$ is equal to zero, first for derivatives of second order and later for derivatives of higher order.

In the case $n = 2$, if $D^2(F - G) = 0$ then $F(x) - G(x)$ is linear. This follows from Denjoy's work. In order to form a background for a study of the cases in which $n > 2$ we first give a proof for $n = 2$ in conformity with our notations and methods. It turns out that for $n > 2$ additional conditions must be imposed on $F(x) - G(x)$ to ensure that $D^n(F - G) = 0$ makes $F(x) - G(x)$ a polynomial of degree at most $n - 1$. These conditions are considered in §4. Our main result is Theorem 4.2.

2. Definition of the operators H_2 and H_3. Let $F(x)$ be a single valued function defined over a given domain. Then

$$H_2(F\colon \alpha, \beta, \gamma) = F(\gamma) - \frac{\gamma - \alpha}{\beta - \alpha} F(\beta) - \frac{\gamma - \beta}{\alpha - \beta} F(\alpha), \tag{2.1}$$

$$H_3(F\colon \alpha, \beta, \gamma, \delta) = F(\delta) - \frac{(\delta - \alpha)(\delta - \beta)}{(\gamma - \alpha)(\gamma - \beta)} F(\gamma) - \frac{(\delta - \alpha)(\delta - \gamma)}{(\beta - \alpha)(\beta - \gamma)} F(\beta) - \frac{(\delta - \beta)(\delta - \gamma)}{(\alpha - \beta)(\alpha - \gamma)} F(\alpha), \tag{2.2}$$

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where $\alpha, \beta, \gamma, \delta$ are on the domain of $F(x)$, and α, β, γ are distinct points except in the case of H_2 when γ may coincide with α or with β. The operator H_n is defined in §4 (Definition 4.1).

3. The fundamental theorem for $n = 2$. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that $F(x) - G(x)$ is continuous on $[a, b]$, and $D^2(F - G) = 0$ at all points of (a, b), then*

$$H_2(F: x_1, x_2, x_3) = H_2(G: x_1, x_2, x_3)$$

for every three points of $[a, b]$ with $x_1 \neq x_2$.

This theorem has been proved for the case where $F(x)$ and $G(x)$ are both continuous by James **(3)** and by Jeffery **(4)** where use is made of convex functions. In our proof no use is made of convex functions.

In order to prove the fundamental theorem for $n = 2$ we need the following result due to Denjoy **(2**, pp. 18–19**)**. We give a proof in conformity with the notations and methods which we shall use for $n > 2$.

THEOREM 3.1. *Let $\bar{D}^2 f(x)$ and $\underline{D}^2 f(x)$ be the upper and lower second generalized Riemann derivates of $f(x)$ which is continuous on $[a, b]$. Then, for every three distinct points of $[a, b]$, x_1, x_2, x_3,*

$$\inf_{a<x<b} \bar{D}^2 f(x) \leqslant \frac{2\, H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)} \leqslant \sup_{a<x<b} \underline{D}^2 f(x). \tag{3.1}$$

To establish this theorem we consider the function

$$g(x) = H_3(f: x_1, x_2, x_3, x) \qquad (a < x < b). \tag{3.2}$$

According to (2.2), $g(x_1) = g(x_2) = g(x_3) = 0$. Let us assume $x_1 < x_3 < x_2$. Then, the continuous function $g(x)$ attains a non-negative maximum at some point q of the interval (x_1, x_2); this is obvious if $g(x) > 0$ at one point of (x_1, x_2). The point q may coincide with x_3, as it happens when $g(x) \leqslant 0$ at all points of (x_1, x_2). Consequently

$$[g(q + 2h) - g(q)] - [g(q) - g(q - 2h)] \leqslant 0,$$

whence, according to (2.1)

$$(2h)^{-2} H_2(g: q - 2h, q, q + 2h) \leqslant 0 \tag{3.3}$$

for any h, $0 < 2h < \min\,(q - x_1, x_2 - q)$.

Returning now to (3.2) we can obtain by simple computation

$$\frac{H_2(g: p_1, p_2, p_3)}{(p_3 - p_1)(p_3 - p_2)} = \frac{H_2(f: p_1, p_2, p_3)}{(p_3 - p_1)(p_3 - p_2)} - \frac{H_2(f: x_1, x_2, x_3)}{(x_3 - x_1)(x_3 - x_2)} \tag{3.4}$$

where p_1, p_2, p_3 are three arbitrary distinct points. Thus, setting $p_1 = q - 2h$, $p_2 = q$, $p_3 = q + 2h$:

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$$(3.5)\quad \frac{H_2(g: q-2h, q, q+2h)}{(2h)^2} = \frac{H_2(f: q-2h, q, q+2h)}{(2h)^2} - \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}.$$

Relations (3.3) and (3.5) combine to give

$$(3.6)\quad \frac{H_2(f: q-2h, q, q+2h)}{(2h)^2} \leqslant \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}.$$

Considering a sequence of h for which the left side of (3.6) tends to the upper second generalized Riemann derivate of $f(x)$ at the point q, $\bar{D}^2 f(q)$, we have

$$\bar{D}^2 f(q) \leqslant \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}$$

and consequently

$$(3.7)\quad \inf_{a<x<b} \bar{D}^2 f(x) \leqslant \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}.$$

By a similar argument dealing with the minimum attained by the continuous function $g(x)$ on the interval (x_1, x_2), we arrive at the relation

$$(3.8)\quad \sup_{a<x<b} \underline{D}^2 f(x) \geqslant \frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}.$$

Relations (3.7) and (3.8) establish Theorem 3.1.

In proving relation (3.1) we assumed $x_1 < x_3 < x_2$. However, (3.1) holds for x_1, x_2, x_3 arbitrary but distinct since the expression

$$\frac{2\,H_2(f: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)}$$

remains invariant under all permutations of x_1, x_2, x_3.

In order to prove the fundamental theorem for $n = 2$ we consider the functions $F(x)$ and $G(x)$, where $F(x) - G(x)$ is continuous on $[a, b]$ and $D^2(F - G) = 0$ at all points of (a, b). Then, by (3.1)

$$\frac{2\,H_2(F-G: x_1, x_2, x_3)}{(x_3-x_1)(x_3-x_2)} = 0$$

for every three distinct points of $[a, b]$, x_1, x_2, x_3. It follows that

$$H_2(F-G: x_1, x_2, x_3) = 0$$

whence, according to (2.1)

$$H_2(F: x_1, x_2, x_3) = H_2(G: x_1, x_2, x_3).$$

4. The fundamental theorem for $n > 2$. The fundamental theorem fails for $n = 3$ as we can easily show by considering two functions $F(x)$ and $G(x)$ that are defined on the interval $[-2, +3]$ and are such that $F(x)$

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$-G(x) = |x|$. The function $|x|$ is continuous on $[-2, +3]$ and is such that $D^3|x| = 0$ at all points of $(-2, +3)$. Yet, $H_3(|x|: -1, 0, 1, 2)$ is not zero as can be seen by applying (2.2). Similarly, relation (3.1) fails, because $H_3(|x|: -1, 0, 1, 2) \neq 0$.

As we have mentioned in §1, in order to generalize the fundamental theorem we must impose additional conditions on the difference $F(x) - G(x)$. One procedure is to impose conditions on the generalized Riemann derivatives of $F(x) - G(x)$; in this way, we can show the following:

THEOREM 4.1. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that at every point of (a, b), $D^3(F - G) = 0$, $D^2(F - G)$ exists, and $D^1(F - G)$ exists, then*

$$H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4)$$

for every four points x_1, x_2, x_3, x_4 of $[a, b]$, the first three being distinct.

Remark 4.1. We can make Theorem 4.1 stronger by replacing the existence of $D^2(F - G)$ with the weaker condition that

$$\lim_{h\to 0} (2h)^{-1} H_2(F - G: x - 2h, x, x + 2h) = 0$$

at every point x of (a, b).

LEMMA 4.1. *If $(2h)^{-1} H_2(f: x - 2h, x, x + 2h)$ tends to zero with h for every $x \in (a, b)$, and if $D^1f(x)$ exists at every point x of (a, b), then the derivative exists at every point x of (a, b): in fact*

$$\frac{d}{dx} f(x) = D^1 f(x).$$

It follows that $f(x)$ is continuous on $[a, b]$.

The truth of this lemma follows from the identities

$$+ (4h)^{-1}[f(x + 2h) - 2f(x) + f(x - 2h)] = (4h)^{-1}[f(x + 2h) - f(x - 2h)] - (-2h)^{-1}[f(x - 2h) - f(x)],$$

$$-(4h)^{-1}[f(x + 2h) - 2f(x) + f(x - 2h)] = (4h)^{-1}[f(x + 2h) - f(x - 2h)] - (2h)^{-1}[f(x + 2h) - f(x)].$$

Taking limits as $h \to 0$ $(h > 0)$, we get

$$0 = D^1f(x) - \frac{d}{dx} f(x).$$

In order to prove Theorem 4.1 we observe that the function $F(x) - G(x)$ satisfies the conditions of Lemma 4.1, and consequently its derivative exists everywhere on (a, b). Then, according to a theorem of Verblunsky (**6**, p. 393), together with the condition that $D^3(F - G) = 0$ everywhere on (a, b), we conclude that $F(x) - G(x)$ is a polynomial of degree at most 2 on $[a, b]$. We have by direct application of (2.2)

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$$H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4).$$

We now determine a set of conditions, different from those of Theorem 4.1, under which $H_3(F: x_1, x_2, x_3, x_4) = H_3(G: x_1, x_2, x_3, x_4)$.

DEFINITION 4.1. Let $F(x)$ be any single valued function defined over a given domain. Then we define

$$H_n(F: x_1, x_2, \ldots, x_n, x_{n+1}) = w_{n+1}(x_{n+1}) \cdot \sum_{j=1}^{n+1} \frac{F(x_j)}{w'_{n+2}(x_j)} \qquad (n = 0, 1, 2, \ldots),$$

where

$$w_{n+1}(y) = \prod_{i=1}^{n}(y - x_i),$$

and the "primes" denote ordinary differentiations. For $n = 1$, the above relation reads

$$H_1(F: x_1, x_2) = F(x_2) - F(x_1);$$

for $n = 0$, we have $H_0(F: x_1) = F(x_1)$.

Now, let $f(x)$ be defined and continuous on $[a, b]$, and suppose that

$$\inf_{a<x<b} \underline{D}^n f(x) \text{ and } \sup_{a<x<b} \bar{D}^n f(x)$$

are finite.

Set

$$(4.1) \qquad g(x) = H_{n+1}(f: x_1, x_2, \ldots, x_{n+1}, x) \qquad (a \leqslant x \leqslant b),$$

$$(4.2) \qquad y(x, h) = H_{n-2}(g: x - nh + 2h, x - nh + 4h, \ldots, x + nh - 2h) \qquad (h > 0)$$

where $x_1, x_2, \ldots, x_{n+1}$ are $n + 1$ arbitrary points of $[a, b]$ such that $x_1 < x_2 < \ldots < x_n < x_{n+1}$.

It follows directly from (4.1) that $g(x_i) = 0$, where $i = 1, 2, \ldots, n + 1$. Consequently, the continuous function $g(x)$ attains n extrema, each of which is an absolute extremum over one of the intervals (x_j, x_{j+1}), $(j = 1, 2, \ldots, n)$.

Let q be the point of (x_1, x_2) at which $g(x)$ attains its absolute extremum over the interval (x_1, x_2). Then, for h fixed and small, we can find two points x' and x'' of the intervals (x_1, q) and (q, x_2) respectively, such that

$$[g(x' + h) - g(x' - h)]\,[g(x'' + h) - g(x'' - h)] \leqslant 0$$

where $x' + h = q = x'' - h$. The function $u_1(x, h) = g(x + h) - g(x - h)$ is continuous in x for h fixed. It follows that $u_1(x, h)$ vanishes at some point of the interval $[x', x'']$ $(x_1 < x' < x'' < x_2)$, because $u_1(x, h)$ changes sign between x' and x''.

Dealing in a similar way with the absolute extrema of $g(x)$ over the remaining $n - 1$ intervals (x_s, x_{s+1}) $(s = 2, 3, \ldots, n)$, we conclude that the function $u_1(x, h)$, for h fixed and small, vanishes at n points of the interval (x_1, x_{n+1}).

Applying successively the same argument to the functions

$$u_t(x, h) = u_{t-1}(x + h, h) - u_{t-1}(x - h, h) \qquad (t = 2, 3, \ldots, n - 2),$$

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we conclude that the function $y(x, h) \equiv u_{n-2}(x, h)$, for h fixed and small, vanishes at three distinct points of the interval (x_1, x_{n+1}), L_h, M_h, and N_h with $L_h < M_h < N_h$. It follows then that the function $y(x, h)$, which is continuous in x, attains an absolute non-negative maximum at some point Q_h of the interval (L_h, N_h) if $y(x, h) > 0$ at one point of (L_h, N_h). The point Q_h may coincide with M_h if $y(x, h) \leqslant 0$ at all points of (L_h, N_h). Similarly, $y(x, h)$ attains an absolute non-positive minimum at some point R_h of the interval (L_h, N_h) and $Q_h \neq R_h$. Consequently

$$y(Q_h + 2h, h) - y(Q_h, h) \leqslant 0, \quad y(Q_h - 2h, h) - y(Q_h, h) \leqslant 0$$

whence

$$y(Q_h + 2h, h) - 2y(Q_h, h) + y(Q_h - 2h, h) \leqslant 0 \tag{4.3}$$

for any h, $0 < 2h < \min\,(Q_h - L_h, N_h - Q_h)$.

Relations (4.2) and (4.3) combine to give

$$(2h)^{-n} H_n(g\colon Q_h - nh, Q_h - nh + 2h, \ldots, Q_h + nh) \leqslant 0 \qquad (h > 0). \tag{4.4}$$

Dealing in a similar way with the point R_h, we obtain

$$(2h)^{-n} H_n(g\colon R_h - nh, R_h - nh + 2h, \ldots, R_h + nh) \geqslant 0 \qquad (h > 0) \tag{4.5}$$

for any h, $0 < 2h < \min\,(R_h - L_h, N_h - R_h)$.

Definition 4.2. The continuous function $f(x)$ belongs to the class K_n of continuous functions if for arbitrary ϵ there exist h' and h'', satisfying (4.4) and (4.5) respectively, such that the expressions

$$\begin{array}{l} (2h')^{-n} H_n(f\colon Q_{h'} - nh', Q_{h'} - nh' + 2h', \ldots, Q_{h'} + nh') \\ (2h'')^{-n} H_n(f\colon R_{h''} - nh'', R_{h''} - nh'' + 2h'', \ldots, R_{h''} + nh'') \end{array} \tag{4.6}$$

lie in the interval $[\inf \underline{D}^n f(x) - \epsilon, \sup \bar{D}^n f(x) + \epsilon]$ $(a < x < b)$.

Theorem 4.2. *If the continuous function $f(x)$ belongs to the class K_n of continuous functions on $[a, b]$, then*

$$\inf_{a<x<b} \underline{D}^n f(x) \leqslant \frac{n!\, H_n(f\colon x_1, x_2, \ldots, x_{n+1})}{(x_{n+1} - x_1)(x_{n+1} - x_2)\ldots(x_{n+1} - x_n)} \leqslant \sup_{a<x<b} \bar{D}^n f(x) \tag{4.7}$$

for every $n + 1$ distinct points of $[a, b]$, $x_1, x_2, \ldots, x_{n+1}$.

Consider the identity

$$\begin{aligned} \frac{H_n(g\colon p_1, \ldots, p_{n+1})}{(p_{n+1} - p_1)\ldots(p_{n+1} - p_n)} &= \frac{H_n(f\colon p_1, \ldots, p_{n+1})}{(p_{n+1} - p_1)\ldots(p_{n+1} - p_n)} \\ &\quad - \frac{H_n(f\colon x_1, \ldots, x_{n+1})}{(x_{n+1} - x_1)\ldots(x_{n+1} - x_n)} \end{aligned} \tag{4.8}$$

where $p_1, p_2, \ldots, p_{n+1}$ are $n + 1$ arbitrary distinct points. We substitute $p_1 = Q_{h'} - nh'$, $p_2 = Q_{h}' - (n - 2)h', \ldots, p_{n+1} = Q_{h'} + nh'$ and thus we obtain

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$$(4.9)\quad \frac{H_n(g: Q_{h'} - nh', \ldots, Q_{h'} + nh')}{(2h')^n} = \frac{H_n(f: Q_{h'} - nh', \ldots, Q_{h'} + nh')}{(2h')^n} - \frac{n!\,H_n(f: x_1, \ldots, x_{n+1})}{(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)}.$$

Similarly, we substitute in (4.8) $p_1 = R_{h''} - nh''$, $p_2 = R_{h''} - (n-2)h'', \ldots,$ $p_{n+1} = R_{h''} + nh''$, and thus we obtain

$$(4.10)\quad \frac{H_n(g: R_{h''} - nh'', \ldots, R_{h''} + nh'')}{(2h'')^n} = \frac{H_n(f: R_{h''} - nh'', \ldots, R_{h''} + nh'')}{(2h'')^n} - \frac{n!\,H_n(f: x_1, \ldots, x_{n+1})}{(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)}.$$

Relations (4.4), (4.5), (4.9), (4.10) combine to give

$$\frac{H_n(f: Q_{h'} - nh', \ldots, Q_{h'} + nh')}{(2h')^n} \leqslant \frac{n!\,H_n(f: x_1, \ldots, x_{n+1})}{(x_{n+1} - x_1) \ldots (x_{n+1} - x_n)} \leqslant \frac{H_n(f: R_{h''} - nh'', \ldots, R_{h''} + nh'')}{(2h'')^n},$$

whence Theorem 4.2 follows because the expressions (4.6) lie in the interval $[\inf \underline{D}^n f(x) - \epsilon, \sup \bar{D}^n f(x) + \epsilon]$ $(a < x < b)$.

THEOREM 4.3. *If $F(x)$ and $G(x)$ are defined on $[a, b]$ and are such that $F(x) - G(x)$ belongs to the class K_n, and $D^n(F - G) = 0$ at all points of (a, b), then*

$$H_n(F: x_1, \ldots, x_{n+1}) = H_n(G: x_1, \ldots, x_{n+1})$$

for every $n + 1$ points of $[a, b]$, $x_1, \ldots, x_{n+1}$, where $x_1, \ldots, x_n$ are distinct points.

To prove this theorem we consider the functions $F(x)$ and $G(x)$ where $F(x) - G(x)$ belongs to the class K_n and is such that $D^n(F - G) = 0$ for $a < x < b$. Then according to (4.7) $H_n(F - G: x_1, \ldots, x_{n+1}) = 0$. It then follows from the definition 4.1 that

$$H_n(F: x_1, \ldots, x_{n+1}) = H_n(G: x_1, \ldots, x_{n+1}).$$

5. Additional remarks. Theorem 4.2 reduces to Denjoy's theorem 3.1 for $n = 2$. Indeed, let $f(x)$ be defined and continuous on $[a, b]$ and suppose that

$$\inf_{a<x<b} \underline{D}^2 f(x) \text{ and } \sup_{a<x<b} \bar{D}^2 f(x)$$

are finite. Putting $n = 2$ in relations (4.1) and (4.2) we obtain

$$g(x) = H_3(f: x_1, x_2, x_3, x) \qquad (a \leqslant x \leqslant b)$$

$$y(x, h) = H_0(g: x)$$

whence, by the definition 4.1, we have $y(x, h) = g(x)$. Consequently, the roots and the extrema of $y(x, h)$ are independent of h and are identical with those of the function $g(x)$, respectively. The expressions that correspond to (4.6) are obtained by setting $n = 2$, and thus we get

$$(5.1) \qquad \begin{aligned} &(2h)^{-2} H_2(f\colon Q - 2h, Q, Q + 2h) \\ &(2h)^{-2} H_2(f\colon R - 2h, R, R + 2h). \end{aligned}$$

Due to the fact that $\bar{D}^2 f(Q)$, $\underline{D}^2 f(Q)$, $\bar{D}^2 f(R)$, $\underline{D}^2 f(R)$, lie in the interval $[\inf \underline{D}^2 f(x), \sup \bar{D}^2 f(x)]$ $(a < x < b)$, it follows that for arbitrary ϵ there exist values of h for which the expressions (5.1) lie in the interval $[\inf \underline{D}^2 f(x) - \epsilon, \sup \bar{D}^2 f(x) + \epsilon]$ $(a < x < b)$. Consequently, the arbitrary continuous function $f(x)$ belongs to the class K_2 and we conclude that the class K_2 is identical with the class of all continuous functions of x. Thus, Denjoy's theorem 3.1 as well as the fundamental theorem for $n = 2$ are particular cases of the general theorems 4.2 and 4.3, respectively.

Further, it is easy to show that if the function $f(x)$ possesses an nth ordinary derivative or a de La Vallée Poussin derivative of order n, $f_{(n)}(x)$ (**5**, p. 1), at every point x of (a, b), then $f(x)$ belongs to the class K_n of continuous functions. Indeed, in both these cases the function $f(x)$ possesses an nth generalized Riemann derivative $D^n f(x)$ equal to the nth ordinary derivative, or to $f_{(n)}(x)$, respectively, at every point x of (a, b). Moreover, it is known (**1**, p. 207) that in either of these cases the expression $(2h)^{-n} H_n(f\colon x - nh, x - nh + 2h, \ldots, x + nh)$, where $a < x - nh < x + nh < b$, lies in the interval $[\inf D^n f(x), \sup D^n f(x)]$ $(a < x < b)$. It then follows that the function $f(x)$ belongs to the class K_n.

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Queen's University
and
Cornell University

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