BANACH ALGEBRAS OF POWER SERIES

Dedicated to the memory of Hanna Neumann

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Let $C[[t]]$ denote the algebra of all formal power series over the complex field C in a commutative indeterminate t with the weak topology determined by the projections $p_j: \Sigma\alpha_i t^i \mapsto \alpha_j$. A subalgebra A of $C[[t]]$ is a *Banach algebra of power series* if it contains the polynomials and is a Banach algebra under a norm such that the inclusion map $A \subset C[[t]]$ is continuous. Such algebras were first introduced in [13] when considering algebras with one generator, and studied, in a special case, in [23]. For a partial bibliography of their subsequent study and application see the references of [9] (note that the usage of the term Banach algebra of power series in [9] differs from that here), and also [2], [3], [11]. Indeed, an examination of their use in [11], under more general topological conditions than here, led the present author to the results of [14], [15], [16], [17].

In this paper we begin a study of the structure of Banach algebras of power series, especially concerning the properties of the spectrum of the indeterminate t. Section 1 is concerned with the question of homomorphisms of general Banach algebras onto Banach algebras of power series. Various known results are obtained as corollaries to Theorem 2, and the case of one generator is discussed in some detail. In Section 2 properties of $\sigma(t)$ are considered, the most definitive results being for the case when polynomials are dense. The final section uses a result of [10] to answer a question raised by the present author in [16].

To conclude this introduction we give some examples of Banach algebras of power series. These will also serve to set some notation. It should be noted that throughout this paper algebra will mean commutative algebra with identity unless otherwise specified.

(a) Let $\{c_n\}_{n\geq 0}$ be a sequence of positive reals, $1 \leqq p \leqq \infty$ and define

$$K^p\langle c_n\rangle = \{x = \Sigma x_n t^n : \|x\| = (\Sigma |x_n c_n|^p)^{1/p} < \infty\}$$

with the usual interpretation for $p = \infty$. If $p = 1$, $c_0 = 1$ and $c_{n+m} \leqq c_n c_m$ for $n, m \geqq 1$ then $K^1\langle c_n \rangle$ is well known to be a Banach algebra with $\| \cdot \|$ as norm. Under the weaker conditions $c_0 > 0$, $\{c_{n+m}/c_n c_m\}$ bounded then $K^1\langle c_n \rangle$ is a Banach algebra under a norm equivalent to $\| \cdot \|$. If $p > 1$ then there are various conditions on the $\{c_n\}$ which ensure that $K^p\langle c_n \rangle$ is an algebra, and hence a Banach algebra under a norm equivalent to $\| \cdot \|$ since multiplication is necessarily continuous by the closed graph theorem. See, for example [20], [9].

(b) Let Ω be an open set in C which is bounded, connected and contains 0. Define

$$H^\infty(\Omega) = \{f : f \text{ analytic on } \Omega : \|f\| = \sup_{z \in \Omega} |f(z)| < \infty\},$$
$$H(\bar{\Omega}) = \{f : f \text{ continuous on } \bar{\Omega}\} \cap H^\infty(\Omega).$$

These are Banach algebras under pointwise operations and $\| \cdot \|$ as norm. By identifying a function with its Taylor series expansion about 0 they are realized as Banach algebras of power series, the projections being continuous by the maximum modulus principle and Cauchy's inequalities.

In (a) polynomials in t are dense, in fact the series are norm convergent, provided $p < \infty$. If p $= \infty$ then polynomials cannot be dense since $K^\infty\langle c_n \rangle$ is non-separable. Denoting the closure of the polynomials by $K^\omega\langle c_n \rangle$ we have

$$K^\omega\langle <c^n \rangle = \{x = \sum x_n t^n \in K^\infty\langle c_n \rangle : x_n c_n \to 0\}$$

and it is easily seen that the series in $K^\omega\langle c_n \rangle$ are norm convergent. Similarly in (b) — if Δ is the open unit disc then polynomials fail to be dense in $H^\infty(\Delta)$, indeed the closure of the polynomials in $H^\infty(\Delta)$ is precisely $H(\bar{\Delta})$. Thus polynomials are certainly dense in $H(\bar{\Delta})$ though the series will not be norm convergent in general. In fact, denoting by $s_n(f)$ the n-th partial sum of the series expansion for f, the set

$$S = \{f \in H(\bar{\Delta}) : s_n(f) \to f \text{ boundedly}\}$$

is meagre in $H(\bar{\Delta})$. For otherwise the principle of uniform boundedness show there is a constant M such that $\| s_n(f) \| \leqq M \|f\|$, $n \geqq 0$, $f \in H(\bar{\Delta})$. But by [12] §3 there is $f \in H(\bar{\Delta})$ with $\| s_n(f) \| \geqq |s_n(f)(1)| = O(\log n)$, giving the desired contradiction.

1

Let A be a Banach algebra of power series. Then A is an integral domain, so in particular if $M = \ker p_0$ then M is a maximal ideal and is not nilpotent. Further $\overline{M^n} \subset \ker p_{n-1}$ for each $n \geqq 1$, so that $\cap \overline{M^n} = \{0\}$. Our first result is a generalization of this and creates a perspective for Theorem 2.

THEOREM 1. *Let A be a Banach algebra, $\phi : A \to B$ a homomorphism of A onto a Banach algebra of power series B. Then A contains a maximal ideal M*

such that

(i) M *is not nilpotent*

(ii) $\cap \overline{M^n} = \ker \phi$.

PROOF. By [16] ϕ is continuous so $J = \ker \phi$ is a closed ideal in A. Let θ denote the canonical map of A onto A/J and ψ the induced map of A/J onto B. By [16] again ψ is continuous, hence bicontinuous by the open mapping theorem. Let $N_0 = \ker p_0$ in B, $N = \psi^{-1}(N_0)$, so that N is a maximal ideal in A/J which is not nilpotent, and $\cap \overline{N^n} = \{0\}$. But then $M = \theta^{-1}(N)$ satisfies the theorem. For M is certainly maximal, and $M^n = \theta^{-1}(N^n) \neq \{0\}$. If $x \in \overline{M^n}$ then $\theta(x) \in \overline{N^n}$, so $x \in \cap \overline{M^n}$ implies $\theta(x) \in \cap \overline{N^n} = \{0\}$, that is, $x \in J$. Conversely if $x \in J$ then $\theta(x) = 0 \in \cap N^n$ so $\theta(x) \in N^n$ for all n, and hence $x \in \theta^{-1}(\theta(x)) \subset M^n$ for all n.

In the converse direction we have the following.

THEOREM 2. *Let A be a Banach algebra containing a maximal ideal M such that*

(i) M *is not nilpotent,*

(ii) $\cap \overline{M^n} = \{0\}$,

(iii) $\dim(M/\overline{M^2}) = 1$.

Then A is (isometrically isomorphic to) a Banach algebra of power series.

PROOF. The first two hypotheses show that $\overline{M^n} \neq \overline{M^{n+1}} \neq \{0\}$ for $n \geqq 1$, the third that $M = \overline{M^2} \oplus Ct$ for some $t \in M$. Indeed, by [22] we have $\overline{M^n} = \overline{M^{n+1}} \oplus Ct^n$ for $n \geqq 1$. Thus if $x \in A$ a simple induction shows that for $n \geqq 1$,

$$x = \sum_{i=0}^{n} \lambda_i t^i + y_n$$

where $y_n \in \overline{M^{n+1}}$ and the $\{\lambda_i\}$ are uniquely determined. Thus functionals $p_j \colon x \mapsto \lambda_j$ are uniquely defined, and clearly linear. If $x \in \ker p_j$ for all j then $x \in \cap \overline{M^j}$ whence $x = 0$. Thus the mapping $x \mapsto \sum p_j(x) t^j$ is an isomorphism of A onto an algebra of formal power series. Carrying over the topology via this isomorphism the result will follow once we show the $\{p_j\}$ are continuous. Now p_0 is continuous since it has closed kernel M, so suppose that p_i is continuous for $i < k$, and take $\{x_n\} \subset A$ with $x_n \to 0$. Then

$$x_n = \sum_{i=0}^{k} p_i(x_n) t^i + y_{n,k}$$

where $y_{n,k} \in \overline{M^{k+1}}$. It follows that $p_k(x_n)t^k + y_{n,k} \to 0$, so if $p_k(x_n) \not\to 0$ we deduce $t^k \in \overline{M^{k+1}}$, a contradiction.

By Theorem 1 conditions (i) and (ii) are necessary, and some condition such as (iii) is also necessary, since Banach algebras of power series in several indeterminates (defined analogously to the present case) satisfy (i) and (ii). Indeed, without our assumption that Banach algebras of power series contain polynomials $\dim(M/\overline{M^2})$ any finite value is possible with subalgebras of $H(\bar{\Delta})$.

If we delete (i) and (ii) and strenghten (iii) to $\dim(\overline{M^n}/\overline{M^{n+1}}) = 1$ for all n, then the same argument gives a homomorphism $x \mapsto \sum p_i(x)t^i$ of A onto an algebra of power series with kernel $\cap \ker p_n = \cap \overline{M^n}$. The inclusion $\subset$ is clear, for the other suppose $x \in \cap \overline{M^n}$ and let k be the least index, if there is one, such that $p_k(x) \neq 0$. Then $x = p_k(x)t^k + y_k$, whence $t^k \in \overline{M^{k+1}}$, which is impossible since $\overline{M^k} = \overline{M^{k+1}} \oplus Ct^k$. Thus we have proved the following.

THEOREM 3. *Suppose A has a maximal ideal M such that* $\dim(\overline{M^n}/\overline{M^{n+1}}) = 1$ *for all n. Then $A/\cap \overline{M^n}$ is a Banach algebra of power series.*

This result is essentially contained in Theorem 3.2 of [5]. As a corollary we have the following result stated in [13].

COROLLARY. *Suppose A has one generator t and that the ideals $\overline{At^n}$ are all distinct. Then $A/\cap \overline{At^n}$ is a Banach algebra of power series.*

PROOF. The hypothesis shows $\overline{At}$ is a maximal ideal with $\dim(\overline{At^n}/\overline{At^{n+1}}) = 1$ for all n.

A direct proof can be modelled on [24].

In this one generator case Lorch [13] states without proof some of the structure of the spectrum. Indeed, suppose $0 \in \operatorname{Int}\sigma(t)$ and let D be the component of $\operatorname{Int}\sigma(t)$ containing 0. Denote $\hat{x}|D$ by $\check{x}$, $x \in A$, and let j be identity function on D. Then $x \mapsto \check{x}$ is a continuous map of A into $H(\bar{D})$, $\widecheck{At^{n+1}} = j^{n+1}\check{A}$ which is closed in $\check{A}$. Thus $\widecheck{\overline{At^{n+1}}} = j^{n+1}\check{A}$ whence it follows that the hypothesis of the corollary holds. Suppose then that $\check{x}(\lambda) = \sum \alpha_k \lambda^k$ on D. Then $x = \sum_{i=0}^{n} \beta_i t^i + y_n$ where $y_n \in \overline{At^{n+1}}$, so $\check{x}(\lambda) = \sum_{i=0}^{n} \beta_i \lambda^i + \check{y}_n(\lambda)$ where $\check{y}_n \in j^{n+1}\check{A}$. The uniqueness of power series expansion gives $\alpha_n = \beta_n$ for all n. Thus if $J = \cap \overline{At^n}$ then $J = k(D)$ and A/J has spectrum $hk(D)$ (for details of the hull-kernel topology see [19], §15). Since $hk(D)$ may strictly contain $\bar{D}$ this answers the question raised in [13], p. 460. Indeed, if A is a uniform algebra then $hk(D)$ is clearly the polynomial convex hull of $\bar{D}$. Even in this case, however, the determination of the norm on A/J remains unresolved. Clearly $\|x + J\| \geqq \sup_{\lambda \in D} |\hat{x}(\lambda)| = \sup_{\lambda \in hk(D)} |\hat{x}(\lambda)|$, but when does equality hold? Certainly if $hk(\bar{D})$ has positive distance from the other components of $\operatorname{Int}\sigma(t)$. For taking open sets U, V containing $hk(\bar{D})$ but disjoint from the other components of $\operatorname{Int}\sigma(t)$, with $\bar{U}$ compact, $\bar{U} \subset V$, there is a continuous function y on C such that $0 \leqq y \leqq 1$, $y|U = 0$, $y|\sim V = 1$. By Mergelyan's theorem $y \in J$, whence $xy \in J$ for any $x \in A$, and for fixed x we may choose V such that $\|x - xy\|$ is a close as we please to $\sup_{\lambda \in D} |\hat{x}(\lambda)|$.

One way in which a homomorphism from a Banach algebra into a Banach

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algebra of power series arises is via analytic discs. For if A is a Banach algebra with spectrum Φ_A an analytic disc at $\phi \in \Phi_A$ is a continuous injection $\Gamma: \Delta \to \Phi_A$ such that $\Gamma(0) = \phi$ and $\hat{x} \circ \Gamma \in \mathrm{Hol}(\Delta)$ for each $x \in A$, and so $x \mapsto \hat{x} \circ \Gamma$ is a homomorphism of A into $\mathrm{Hol}(\Delta)$. For more details of the relation between such homomorphisms and analytic discs see [18]. We conclude this section with the following result on analytic structure.

THEOREM 4. *Let A be a Banach algebra which has a principal maximal ideal At, and denote by ϕ the point of Φ_A corresponding to At. Suppose ϕ is not isolated. Then*

(i) *$A/\cap At^n$ is a semisimple Banach algebra of power series,*

(ii) *there is an analytic disc at ϕ,*

(iii) *for any $x \in A$, $\hat{x}$ vanishes on a neighbourhood of ϕ if and only if $x \in \cap At^n$.*

REMARKS. 1. If ϕ were isolated then (ii) is clearly impossible, except in a trivial sense, and in the semisimple case $\cap At^n = At$, so that (i) is impossible and (iii) is trivially true. To see this observe that if ϕ were isolated then Silov's theorem gives an idempotent $f \in A$ such that $\hat{f}(\phi) = 0$, $\hat{f}(\psi) = 1$, $\psi \in \Phi_A \setminus \{\phi\}$. But then $At = Af$ so $At^n = (At)t^{n-1} = (Af)t^{n-1} = At^{n-1}f = \cdots = Af = At$.

2. The result (ii) is well known, being a special case of [6]. For detailed analysis of the semisimple case see [4]. (i) extends considerably Theorems 2.6, 2.7 of [7].

3. A more general result along the lines of (iii) has been announced in [21].

4. The author is indebted to a referee for pointing out an error in the original proof.

PROOF. Suppose $At^n = At^{n+1}$ for some $n \geqq 1$, so there is $x \in A$ with $t^n(e - xt) = 0$. By maximality of At, $\hat{t}$ vanishes only at ϕ, whence $\widehat{e-xt}$ must vanish off $\{\phi\}$ while $\phi(e - xt) = 1$. But this means ϕ is isolated, contrary to hypothesis. It follows that $At^n = At^{n+1} \oplus tC^n \underset{\neq}{\supset} At^{n+1}$ for each $n \geqq 1$.

A simple induction now shows that At^n has codimension n in A, and hence is closed for each n. But then Theorem 3 applies to show $B = A/\cap At^n$ is a Banach algebra of power series. For $x \in A$ let $\bar{x}$ denote the coset $x + \cap At^n$. Then $\bar{t}$ is certainly not a divisor of zero, $B\bar{t}$ is the image of At and so is maximal, and hence $\bar{t}$ is not a topological divisor of zero. Thus the map $T: \bar{x} \mapsto \bar{x}\bar{t}$ has a continuous inverse and a simple induction gives $\| p_k \| \leqq (2 \| T^{-1} \|)^k$ for $k \geqq 0$. If Ω is an open disc centre 0 and radius $\delta < (2 \| T^{-1} \|)^{-1}$ then the map $\bar{x} \mapsto \sum p_i(\bar{x}) z^i$ of B into $H(\bar{\Omega})$ is an algebraic isomorphism so that B must be semisimple. This proves (i).

For (ii) define functionals $\{\phi_\lambda : |\lambda| < \delta\}$ on A by $\phi_\lambda : x \mapsto \sum p_i(\bar{x})\lambda^i$. Then $\Gamma: \lambda \mapsto \phi_{\lambda\delta}$ is easily seen to be an analytic disc at ϕ.

To prove (iii) note firstly that if $T_\lambda : x \geqq (t - \lambda e)x$ then T_0 is a semi-Fredholm operator of deficiency 1. Thus there is $\eta > 0$ such that T_λ has deficiency $\leqq 1$ for $|\lambda| < \eta$. Let $\mathscr{E} = \min(\eta, \delta)$. Then if $|\lambda| < \varepsilon$, $T_\lambda(A) \subset \ker \phi_\lambda$ and codim $T_\lambda(A) \geqq \operatorname{codim} \ker \phi_\lambda$, so that $\ker \phi_\lambda = A(t - \lambda e)$.

Let $\Omega_1 = \{z : |z| < \varepsilon\, \delta^{-1}\}$, $U = \{\psi \in \Phi_A : |\psi(t)| < \varepsilon\}$. Then from what we have just shown $\Gamma : \Omega_1 \to U$ is a continuous bijection so if $x \in \cap At^n$ then $\hat{x} | U = 0$. Conversely, suppose $\hat{x} | V = 0$ for some neighbourhood V of ϕ. Then $U \cap V$ is infinite since ϕ is not isolated, and certainly $\hat{x} | U \cap V = 0$. But this means $\phi_\lambda(x) = 0$ for infinitely many points in Ω, whence $\phi_\lambda(x) \equiv 0$ on Ω since it is analytic on the larger open disc $\{\lambda : |\lambda| < (2\|T^{-1}\|)^{-1}\}$. Thus $\bar{x} = 0$, that is, $x \in \cap At^n$.

2

We now turn to consideration of the properties of $\sigma(t)$ in a Banach algebra of power series A.

LEMMA 1. *In any Banach algebra of power series we have*

$$\overline{\lim} \|p_n\|^{-1/n} \leqq \inf\{|\lambda| : \lambda \in \rho(t)\} \leqq \nu(t),$$

and so if $\overline{\lim} \|p_n\|^{-1/n} = M$ *then* $\sigma(t) \supseteq \{\lambda : |\lambda| \leqq M\}$.

PROOF. If $\lambda \in \rho(t)$ then equating coefficients shows that $R(\lambda, t) = (\lambda e - t)^{-1} = \Sigma t^i \lambda^{-i-1}$. Thus

$$|\lambda^{-n-1}| = |p_n(R(\lambda, t))| \leqq \|p_n\| \, \|R(\lambda, t)\|, \text{ whence } \overline{\lim} \|p_n\|^{-1/n} \leqq |\lambda|.$$

THEOREM 5. *Let A be a Banach algebra of power series in which polynomials are dense. Then the following are equivalent, and necessitate A semi-simple*:

(i) *there is $\delta > 0$ such that $\sigma(t) \supseteq \{\lambda : |\lambda| \leqq \delta\}$*,

(ii) $\overline{\lim} \|p_n\|^{1/n} < \infty$,

(iii) $\lim \|p_n\|^{1/n}$ *exists finitely*,

(iv) *there is a non-zero homomorphism $\Psi : A \to H(\bar{\Delta})$ with $\Psi(t)$ singular, but not a topological divisor of zero.*

PROOF. (i) $\Rightarrow$ (ii), (iv). Let $x = \Sigma \alpha_i t^i \in A$ and take $q_n = \Sigma_{i=1}^{i_n} \beta_{in} t^i$ a sequence of polynomials converging to x. Then by continuity of projections $\beta_{in} \to \alpha_i$ for each i. Also, $\hat{q}_n$ converges uniformly to $\hat{x}$ on $\sigma(t)$, so $\hat{x}$ is analytic on $\{\lambda : |\lambda| < \delta\}$ and continuous on the closure of this disc. If $\hat{x}(\lambda) = \Sigma \gamma_i \lambda^i$ for $|\lambda| < \delta$ then $\beta_{in} \to \gamma_i$ for each i by Cauchy's inequalities, whence $\gamma_i = \alpha_i$ for all i. Thus $|\alpha_i| \leqq \nu(x)\delta^{-i} \leqq \|x\|\delta^{-i}$, so $\|p_i\| \leqq \delta^{-i}$ and hence $\overline{\lim} \|p_n\|^{1/n} \leqq \delta^{-1}$, proving (ii). The map $\Psi : \Sigma \alpha_i t^i \mapsto \Sigma \alpha_i \delta^i z^i$ satisfies (iv).

(ii) $\Rightarrow$ (i), (iii). Let $\underline{\lim} \|p_n\|^{1/n} = M \leqq \overline{\lim} \|p_n\|^{1/n}$. By Lemma 1 $M > 0$ and so $\overline{\lim} \|p_n\|^{-1/n} \geqq M^{-1}$ whence $\sigma(t) \supseteq \{\lambda : |\lambda| \leqq M^{-1}\}$ proving (i). But then $\overline{\lim} \|p_n\|^{1/n} \leqq M$ by the above; the reverse inequality being true we have (ii).

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(iii) $\Rightarrow$ (ii) is trivial.

(iv) $\Rightarrow$ (i). Since $\Psi(t)$ is non-zero yet singular it must be non-constant, so $S = \{\Psi(t)(\lambda): |\lambda| < 1\}$ is open. If $0 \notin S$ then there must be λ_0, $|\lambda_0| = 1$ with $\Psi(t)(\lambda_0) = 0$. But then $\left(\frac{z+\lambda_0}{\lambda_0+\bar{\lambda}_0}\right)^n \Psi(t) \to 0$ showing $\Psi(t)$ is a topological divisor of zero. This contradicts the hypothesis and so $0 \in S$ and (i) follows.

Finally, (i) shows A must be semisimple.

The following result shows that semisimplicity can be proven without any supposition on the polynomials. The argument works for base algebras more general than C, whereas that of Theorem 5 soon runs into difficulties if the norm is not severely restricted.

THEOREM 6. *Let A be a Banach algebra of power series in which $\overline{\lim} \| p_n \|^{1/n} < \infty$. Then A is semisimple.*

PROOF. Suppose $x = \Sigma \alpha_i t^i$ is quasinilpotent, and let k be the least index, if one exists, such that $\alpha_k \neq 0$. Since $x \in \ker p_0$, $k \geqq 1$. But then

$$|\alpha_k| = |\alpha_k^n|^{1/n} = \| p_{nk}(x^n) \|^{1/n} \leqq (\| p_{nk} \|^{1/nk})^k \| x^n \|^{1/n} \to 0,$$

whence $\alpha_k = 0$.

It is not known whether or not the converse of Theorem 5 is true in general, but we have the following partial results.

THEOREM 7. *Let A be a Banach algebra of power series in which the series actually converge. Then $0 \in \operatorname{Int} \sigma(t)$ if and only if A is semisimple. Indeed, $\sigma(t) = \{\lambda: |\lambda| \leqq \nu(t)\}$.*

REMARK. If A were a Banach algebra whose elements were norm convergent power series then A would be a Banach algebra of power series, the projections necessarily being continuous.

PROOF. If $x = \Sigma \alpha_i t^i \in A$ then $\hat{x}(\lambda) = \Sigma \alpha_i \lambda^i$ for $\lambda \in \sigma(t)$, the series being convergent. In particular, if $\mu \in \rho(t)$, $\lambda \in \sigma(t)$ then $R(\mu, t)^\wedge(\lambda) = \Sigma \mu^{-i-1} \lambda^i$. Now if $\sigma(t) = \{0\}$ A is certainly not semisimple (A is actually a radical algebra with identity adjoined). If $\sigma(t) \neq \{0\}$ choose $\lambda \in \sigma(t) \setminus \{0\}$. Supposing $0 \in \partial\sigma(t)$ there is $\mu \in \rho(t)$ with $|\mu| < |\lambda|$. But then $\Sigma \mu^{-i-1} \lambda^i$ is divergent. Thus $0 \in \operatorname{Int} \sigma(t)$ if t is not quasinilpotent. Theorem 5 shows A is semisimple.

For the stronger result, define mappings $\{s_n\}$ by $s_n(\Sigma \alpha_i t^i) = \Sigma_{i=0}^n \alpha_i t^i$. Then by the principle of uniform boundedness $\{s_n\}$ are equicontinuous and hence the functionals $\{\| t^j \| p_j\}$ are equicontinuous. Thus there is a constant K such that $1 \leqq \| p_n \| \| t^n \| \leqq K$ whence $\lim(\| p_n \| \| t^n \|)^n = 1$. The desired result now follows as in [15] Theorem 3.6.

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THEOREM 8. *Let $X \subset C$ be compact, connected, with connected complement, $\mathscr{P}(X)$ the closure in the supremum norm over X of the polynomials. Suppose that $\mathscr{P}(X)$ is a Banach algebra of power series such that t is the identity function on X. Then $0 \in \operatorname{Int} X$.*

PROOF. We simply show that if $0 \in \partial X$ then a branch of the function $z \mapsto \sqrt{z}$ lies in $\mathscr{P}(X)$, and clearly there is no power series $\Sigma \alpha_i t^i$ with $(\Sigma \alpha_i t^i)^2 = t$. The result is not surprising but the argument is somewhat delicate.

Thus suppose $0 \in \partial X$. Let B be an open disc centred at the origin, so large that $X \subset B$, and let $\beta_0 \in \partial B$. Take a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ such that $U_n = \{z : |z| < \varepsilon_n\}$ are all contained in B. Since $0 \in \partial X$ and X has connected complement there is $\alpha \in \partial U_1 \setminus X$. Let $\{c_1(\tau) : 0 \leqq \tau \leqq 1\}$ be a simple polygonal arc in $B \cup \{\beta_0\} \setminus X$ joining α and β_0. Define $\tau_0 = \sup\{\tau : c_1(\tau) \in \partial U_1\}$ and $\beta_1 = c_1(\tau_0) \in \partial U_1 \setminus X$, and set $\Gamma_1 = \{c_1(\tau) : \tau_0 \leqq \tau \leqq 1\}$. Choose $\xi \in B \setminus (\bar{U}_1 \cup \Gamma_1)$ and let f_0 be a branch of $z \mapsto \sqrt{z}$ in a neighbourhood V of ξ, $V \subset B \setminus (\bar{U}_1 \cup \Gamma_1)$. Then $D_1 = B \subset (\bar{U}_1 \setminus \Gamma_1)$ is connected, simply connected, excludes 0, and so f_0 can be analytically continued throughout D_1, giving f_1.

Suppose now that β_i, D_i, f_i have been defined for $i < n$. As before there is $\alpha \in \partial U_n \setminus X$; let $\{c_n(\tau) : 0 \leqq \tau \leqq 1\}$ be a simple polygonal arc in $U_{n-1} \cup \{\beta_{n-1}\} \setminus X$ joining α to β_{n-1}, $\tau_0 = \sup\{\tau : c_n(\tau) \in \partial U_n\}$, $\beta_n = c_n(\tau_0)$, $\Gamma_n = \{c_n(\tau) : \tau_0 \leqq \tau \leqq 1\} \cup \Gamma_{n-1}$, $D_n = B \setminus (\bar{U}_n \cup \Gamma_n)$, f_n the analytic continuation of f_{n-1} to D_n. Then $D = \cup D_n \supset X \setminus \{0\}$ and f defined by $f(z) = f_n(z)$ if $z \in D_n$ is a branch of $\sqrt{z}$ on D, everywhere analytic on D. Finally $|f(z)| \to 0$ as $z \to 0$ so that f is continuous at 0 if we define $f(0) = 0$. But then $f \in \mathscr{P}(X)$ by Mergelyan's theorem.

REMARK. The hypothesis that X was connected was not used in the proof but necessarily holds if $\mathscr{P}(X)$ is a Banach algebra of power series with $\sigma(t) = X$, as do the requirements that X be compact with connected complement.

The question of necessary and sufficient conditions for $\mathscr{P}(X)$ to be a Banach algebra of power series runs into the problems we have encountered in Section 1 when considering the singly generated case. The conditions $0 \in \operatorname{Int}(X)$, $X = \overline{\operatorname{Int}(X)}$, $\operatorname{Int}(X)$ connected are certainly sufficient when taken together; the first two are necessary, the last is not.

We conclude this section with a result which extends a theorem of [9], namely that a Banach algebra of power series without identity is radical if and only if it is analytically closed. (See [8] for definition.)

THEOREM 9. *Let A be a Banach algebra of power series, $x = \Sigma \alpha_i t^i \in \ker p_0$. Let f be a function which is analytic on (a neighbourhood of) $\sigma(x)$, with $f(\lambda) = \Sigma \beta_i \lambda^i$ for $|\lambda|$ sufficiently small. Then denoting by $f(x)$ the formal power series $\Sigma \beta_i (\Sigma \alpha_j t^j)^i$, we have $f(x) \in A$ and $f(x)^\wedge(\phi) = f(\hat{x}(\phi))$ for $\phi \in \Phi_A$.*

PROOF. Let $\mathscr{C}$ be a contour surrounding $\sigma(x)$ once positively, and contained in the domain of analyticity of f. By the functional calculus

$$y = \frac{1}{2\pi i}\int_{\mathscr{C}} f(\lambda)R(\lambda, x)d\lambda$$

is an element of A with $\hat{y}(\phi) = f(\hat{x}(\phi))$ for $\phi \in \Phi_A$; we show $y = f(x)$.

Setting $f(x) = \Sigma\gamma_i t^i$ we have $\gamma_0 = \beta_0$, and for $k \geqq 1$

$$\gamma_k = \sum_{i=1}^{k} \beta_i\left(\sum_{n_1+\cdots+n_i=k} \alpha_{n_1}\cdots\alpha_{n_i}\right);$$

and setting $R(\lambda, x) = \Sigma\eta_i(\lambda)t^i$ we have $p_j(y) = 1/(2\pi i)\int_{\mathscr{C}} f(\lambda)\eta_j(\lambda)d\lambda$. Now $(\Sigma\eta_i(\lambda)t^i)(\lambda e - \Sigma\alpha_i t^i) = e$, so that $\eta_0 = 1/\lambda$ and $\eta_k = 1/\lambda\,\Sigma_{i=0}^{k-1}\eta_i\alpha_{k-i}$ for $k \geqq 1$. Suppose that

$$n_j = \sum_{i=1}^{j} \lambda^{-i-1}\left(\sum_{n_1+\cdots+n_i=j} \alpha_{n_1}\cdots\alpha_{n_i}\right)$$

for $j = 1, \cdots, n-1$; this is certainly true for $j = 1$. Then

$$\begin{aligned}\eta_n &= \lambda^{-1}\sum_{i=1}^{n-1}\left(\sum_{l=1}^{i}\lambda^{-l-1}\sum_{n_1+\cdots+n_l=i}\alpha_{n_1}\cdots\alpha_{nl}\right)\alpha_{n-i} + \alpha_n\lambda^{-2}\\ &= \lambda^{-1}\sum_{l=1}^{n-1}\lambda^{-l-1}\sum_{n_1+\cdots+n_{l+1}=n}\alpha_{n_1}\cdots\alpha_{n_{l+1}} + \alpha_n\lambda^{-2}\\ &= \sum_{l=1}^{n}\lambda^{-l-1}\sum_{n_1+\cdots+n_l=n}\alpha_{n_1}\cdots\alpha_{n_l}.\end{aligned}$$

Substitution of this for η_j in the integral for $p_j(y)$ yields $p_j(y) = \gamma_j$, and the result follows.

3

When proving [16] Theorem 1 the author overlooked the paper [10] which contains a more general result, to wit Theorem 9.1. The proof of that theorem is given for the Banach space case with indications of the modifications required in the Fréchet case (Notes (ii) and (iii)), and considers sequence spaces over C, though the more general case, with sequences in a Fréchet space follows in the same way. Using this result we now settle a question raised on page 4 of [16], at least for complex sequences. Thus let A be an algebra of formal power series over C which has a Fréchet space topology such that there is a sequence $\{\gamma_n\}$ of positive reals such that $\{\gamma_n^{-1} p_n\}$ is an equicontinuous family, and let B be a topological algebra over C with a Fréchet space topology.

THEOREM 10. *Let $\phi: B \to A$ be a homomorphism whose range is not one dimensional. Then ϕ is continuous.*

PROOF. Suppose $\phi(B)$ is not one dimensional, so there is $y \in B$ with $\phi(y) = x \notin Ce$. Since ϕ is a homomorphism $\phi L_y = L_x \phi$, where L_z denotes left multiplication by z in the appropriate algebra. To apply [10] Theorem 9.1 we first note that [16] Theorem 2 shows that the conditions of [10] Note (iii) are satisfied. Thus it suffices to note that L_x is triangular (for if $x = \sum \alpha_i t^i$ then $L_x(\sum \beta_i t^i) = \sum(\sum_{i+j=n} \alpha_i \beta_j) t^n$) and has empty point spectrum (since A is an integral domain and $x \notin Ce$). The result now follows from [10] Theorem 9.1.

If $\phi(B)$ is one dimensional, that is, $\phi(B) = Ce$, then ϕ is a multiplicative linear functional on B. The continuity of such functionals has been an open problem for many years and is yet to be settled.

Finally we remark that the uniqueness of the Fréchet space topology of the algebra of all complex sequences, considered in [16], is proved in [1], a side result on the way to proving the surprising result that this algebra is normable.

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