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SOLUBLE GROUPS ISOMORPHIC TO THEIR NON-NILPOTENT SUBGROUPS

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Dedicated to Mike (M. F.) Newman on the occasion of his 65th birthday

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Abstract

A group G belongs to the class $\mathscr{W}$ if G has non-nilpotent proper subgroups and is isomorphic to all of them. The main objects of study are the soluble groups in $\mathscr{W}$ that are not finitely generated. It is proved that there are no torsion-free groups of this sort, and a reasonable classification is given in the finite rank case.

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1. Introduction

This paper continues our investigation of groups isomorphic to all their non-nilpotent subgroups. In [7] we considered groups isomorphic to their non-Abelian subgroups, and in [8] finitely generated groups isomorphic to their non-nilpotent subgroups. The special case of groups with all proper subgroups nilpotent has been the object of much study, for instance in [3, 4, 6]. The first of these papers contains a description of the celebrated Heineken-Mohamed groups; these are non-nilpotent p-groups with all proper subgroups nilpotent and subnormal. A far more elementary example of a non-nilpotent group with all proper subgroups nilpotent is the extension of C_{2^∞} by an inverting automorphism. A complete description is given in [4] of non-nilpotent groups with maximal subgroups that have all proper subgroups nilpotent: they are

Chernikov p-groups with additional properties, and thus satisfy the minimum condition for subgroups. One of these groups plays a key role in Theorem 1.3 below. Here we adopt the same approach as in [8], in that we restrict attention to groups that are isomorphic to all their non-nilpotent subgroups but do not have all their proper subgroups nilpotent, and denote the class of all such groups by $\mathscr{W}$. Thus $\mathscr{W}$-groups do not satisfy the minimum condition, whereas we shall see that they do have maximal subgroups.

In both [7] and [8] we were concerned with finitely generated groups—no restriction in the 'non-Abelian case', of course; groups isomorphic to all their non-Abelian subgroups were referred to as $\mathscr{X}$-groups. In each case the discussion split into consideration of the soluble and insoluble cases. A reasonable classification was given in [7] for soluble $\mathscr{X}$-groups, and modulo this classification, the same may be said for [8] with regard to finitely generated soluble $\mathscr{W}$-groups. In the insoluble case things are much less clear, as may be expected when one considers the existence of Tarski p-groups and other such monsters. Theorem 1 of [7] and Theorem 2 of [8] say a little about the (finitely generated) insoluble cases.

Turning now to groups that are not finitely generated, we observe that a $\mathscr{W}$-group fails to be finitely generated if and only if it is locally nilpotent (this fact will be used a number of times without further comment). It seems reasonable to tackle the soluble case first. Indeed, the general case appears to us to represent an extremely difficult challenge, and it may be worth noting here that it is not known even whether a non-trivial locally finite p-group with all proper subgroups nilpotent can be perfect. For soluble groups we have met with some limited success, and the object of the present article is to present the results obtained so far. The torsion-free case is very satisfactorily dealt with by Theorem 1.1: there are no torsion-free groups in $\mathscr{W}$ that are soluble and locally nilpotent. Theorem 1.3 gives a complete description of $\mathscr{W}$-groups of finite rank. The infinite rank case is much harder, and the best we can say is that $\mathscr{W}$-groups of infinite rank are Fitting groups (every element has nilpotent normal closure). That is the content of Theorem 1.2.

It is possible that some of the techniques illustrated here may be of some use in the case of soluble p-groups in $\mathscr{W}$, but more likely is that different methods and deeper insights will be necessary. We conjecture that there are no soluble p-groups in $\mathscr{W}$, despite the ingenious examples of locally nilpotent groups of infinite rank that have been constructed in the past. Some quite strong-looking reduction results are provided in the final Section 5 of the paper, though there is little of a decisive nature to report.

We conclude this section by stating our main results.

THEOREM 1.1. *Let G be a soluble group that is not finitely generated, and suppose that G is isomorphic to each of its non-nilpotent subgroups. If G is torsion-free then G is nilpotent.*

THEOREM 1.2. *Let G be a soluble group in $\mathscr{W}$ that is not finitely generated. If G is of infinite rank then it is a Fitting group.*

THEOREM 1.3. *Let G be a soluble group in $\mathscr{W}$ that is not finitely generated. The following three conditions are equivalent*:

(a) *G has finite rank.*

(b) *G is not a Fitting group.*

(c) *$G = P]\langle x\rangle$ for some divisible Abelian p-group P (p a prime) and some element x of infinite order such that*

(i) *$[P, x^p] = 1$, so that $x^p \in Z(G)$;*

(ii) *P has no infinite proper $\langle x\rangle$-invariant subgroups;*

(iii) *$G/\langle x^p\rangle$ has all proper subgroups nilpotent but is not itself nilpotent.*

The easiest example of a group satisfying the conditions of Theorem 1.3 is provided by the split extension of a C_{2^∞} by an infinite cyclic group inducing the inverting automorphism.

The lay-out is as follows. We establish some auxiliary results in Section 2, prove Theorem 1.1 and Theorem 1.3 in Section 3, and Theorem 1.2 in Section 4. This despite the fact that Theorem 1.2 is used in the proof of Theorem 1.3: our reason is that Theorem 1.2 takes far longer to prove and thus is left as late as possible. As stated already, Section 5 contains some reduction results for soluble p-groups in $\mathscr{W}$. Notation is standard except where otherwise stated.

We thank the referee for some very useful observations.

2. Some preliminary results

Our first preliminary appears in [8, Lemma 1].

LEMMA 1. *Let G be a group, N a normal nilpotent subgroup of G and suppose that $G = N\langle x\rangle$ for some element x. If M is a G-invariant subgroup of N such that $M\langle x\rangle$ is nilpotent, then $M \leq Z_n(G)$ for some integer n.*

For our second result, we recall from [2] the definition of *isolator*. If G is a locally nilpotent group and H a subgroup of G, then the isolator $I_G(H)$ of H in G is given by $I_G(H) = \{g \in G : g^n \in H$ for some non-zero integer $n\}$, and is a subgroup of G. Although isolators are easily avoided in the case of Lemma 2 (and its proof), nevertheless they are really useful for the proofs of Lemma 3 and Theorem 1.1.

LEMMA 2. *Let $G = A]\langle g\rangle$ be a countable torsion-free locally nilpotent group with A Abelian, and suppose that H is isomorphic to G whenever H is a subgroup of G with $I_G(H) = G$. Then G is free Abelian.*

PROOF. Let $S = \{b_i : i \in \mathbb{N}\}$ or $\{b_1, b_2, \ldots, b_k\}$ be a maximal $\mathbb{Z}$-independent subset of A. Since $A_1 := \langle b_1 \rangle^G$ is contained in $\langle b_1, x \rangle$, it is finitely generated by local nilpotency. Write $I_1 = I_A(A_1)$ and $a_1 = b_1$. Now choose i_2 least such that $b_{i_2} \in S \backslash I_1$ and write $B_2 = \langle b_{i_2} \rangle^G$. Then $(B_2 \cap I_1)/A_1$ is finitely generated periodic Abelian and is therefore finite, of order n_1 say, so that $(B_2 \cap I_1)^{n_1} \leq A_1$. Set $a_2 = b_{i_2}^{n_1}$; then $\langle a_2 \rangle^G \cap I_1 = B_2^{n_1} \cap I_1 = (B_2 \cap I_1)^{n_1} \leq A_1$. Since $\langle a_1, a_2 \rangle^G / A_1$ is free Abelian, it follows that $\langle a_1, a_2 \rangle^G = A_1 \times A_2$ for some finitely generated subgroup A_2 of A. Write $I_2 = I_A(A_1 A_2)$ and continue in the obvious manner to obtain in the end a $\langle g \rangle$-invariant free Abelian subgroup $A^* = A_1 \times A_2 \times \cdots$ of A whose isolator is A. By hypothesis, $G \simeq A^* \langle g \rangle$, and thus we assume that A is free Abelian.

By local nilpotency again, every element of A is in some term $Z_i(G)$ of the upper central series of G, so that A is generated by all the $A \cap Z_i(G)$. We claim that we may write $A = C_1 \times C_2 \times \cdots$ (possibly, with finitely many factors), in such a way that $C_1 \times C_2 \times \cdots \times C_i = A \cap Z_i(G)$ for each i. To see this, recall that $G = \langle A, g \rangle$ and that A is Abelian, so that the map

$$d \to [d, {}_i g]$$

is a homomorphism from $A \cap Z_{i+1}(G)$ into A with kernel $A \cap Z_i(G)$. Since A is free Abelian, $A \cap Z_i(G)$ is a direct factor of $A \cap Z_{i+1}(G)$ and the claim follows.

Suppose that G is not Abelian and choose a prime p such that $[C_1 C_2, \langle g \rangle] (= [C_2, \langle g \rangle])$ is not contained in C_1^p. Such exists because C_1 is free Abelian. Set $B = C_1 A^p = C_1 \times C_2^p \times C_3^p \times \cdots$, so that $G \cong B\langle g \rangle$ and thus $[Z_2(B\langle g \rangle), \langle g \rangle] \not\leq (Z_1(B\langle g \rangle))^p = C_1^p$. But $Z_2(B\langle g \rangle) = C_1 C_2^p$ and thus we have $[C_1 C_2^p, \langle g \rangle] \leq C_1^p$, a contradiction that proves Lemma 2. □

LEMMA 3. *Let G be a soluble locally nilpotent group that is isomorphic to each of its non-nilpotent subgroups. If G is torsion-free it is a Fitting group.*

PROOF. Let G be a group that satisfies these hypotheses and suppose for a contradiction that G is not a Fitting group. Since G is not nilpotent it has a countable non-nilpotent subgroup and so is itself countable. Choose g in G such that $\langle g \rangle^G$ is not nilpotent, so that $G' \langle g \rangle$ is not nilpotent. But G' is nilpotent since it has smaller derived length than G and so is not isomorphic to G. Since G is isomorphic to $G'\langle g \rangle$, we deduce that G has a normal nilpotent subgroup N with $G = N\langle x \rangle$ for some element x. From [5, Lemma 6.33] we see that $N \cap \langle x \rangle = 1$ and that $N = \operatorname{Fitt} G$. It follows easily that $N'\langle x \rangle$ is not isomorphic to G and is therefore nilpotent, so that $N' \leq Z_a(G)$ for some integer a, by Lemma 1. Let H be an arbitrary subgroup of G containing $Z_a(G)$ such that $I_G(H) = G$; again, H is not nilpotent. Now in any torsion-free locally nilpotent group centralizers are isolated; thus, if $g \in Z(H)$ then $G \leq C_G(g)$, that is, $g \in Z(G)$ and so $Z(G) = Z(H)$. Furthermore, $G/Z(G)$ is torsion-free and so an easy induction shows that $Z_a(G) = Z_a(H)$, where a is as above. It follows that $Z_a(G)$ is invariant under every isomorphism from G to H; thus, as $N' \leq Z_a(G)$, we see that $G/Z_a(G)$ is Abelian-by-cyclic. It is also torsion-free, so that Lemma 2 applies to give that $G/Z_a(G)$ is Abelian and hence that $G = Z_{a+1}(G)$, a contradiction that establishes the result. □

LEMMA 4. *Let $G = A\langle x\rangle$ be a locally nilpotent, residually nilpotent group, where A is a normal Abelian p-subgroup of G for some prime p. If G is not nilpotent, then it has a non-nilpotent subgroup $B\langle x\rangle$ for some subgroup B of A that is the direct product of finite G-invariant subgroups.*

PROOF. Firstly, there must exist a finite G-invariant subgroup B_1 of A such that $[B_1, x] \neq 1$, else G would be nilpotent. Thus there is a normal subgroup N_1 of G such that G/N_1 is nilpotent and $N_1 \cap B_1 = 1$. Next, since $G/(N_1 \cap A)$ is nilpotent and G not nilpotent, G does not act nilpotently on $N_1 \cap A$; thus , there is a finite G-invariant subgroup B_2 of $N_1 \cap A$ such that $[B_2, x, x] \neq 1$ (that is, B_2 is not second central) and a G-invariant subgroup £N_2 of N_1 such that G/N_2 is nilpotent and $N_2 \cap B_2 = 1$. Continuing in the obvious way, we obtain a subgroup B of A of the form $B = B_1 \times B_2 \times \cdots$, where, for each i, B_i is finite and $[B_{i,\,i}\, x] \neq 1$. Clearly, $B\langle x\rangle$ is a subgroup of the required kind. □

3. Proofs of Theorem 1.1 and Theorem 1.3

We begin by proving Theorem 1.1. Suppose for a contradiction that G is a non-nilpotent group satisfying the hypotheses of that theorem. Since G' is nilpotent, so is its isolator I, [5, Lemma 6.33]. There exists a free Abelian subgroup K/I of G/I such that $I_G(K) = G$, and K is not nilpotent by the same lemma. Therefore, K is isomorphic to G, which means that G has a normal nilpotent subgroup N with G/N free Abelian. Now if G is n-Engel for some integer n then it is nilpotent, by [5, Theorem 7.36] and the fact that G is torsion-free.

Thus G is not n-Engel for any n, whereas it is a Fitting group by Lemma 3. Let c be the nilpotency class of N. Since G is not $(c+2)$-Engel, we may choose $g_0 \in G$ such that $\langle g_0\rangle^G$ has nilpotency class $n_0 > c$. Let $I_0 = I_G(N\langle g_0\rangle)$, so that $G/N = I_0/N \times U_0/N$ for some subgroup U_0, since G/N is free Abelian. Since $[G, \langle g_0\rangle]$ is nilpotent, so is I_0. But $G = I_0 U_0$ and so U_0 is not nilpotent since I_0 and U_0 are normal. Thus, we may choose $g_1 \in U_0$ such that the nilpotency class n_1 of $\langle g_1\rangle^G$ is strictly greater than $c + n_0$. Clearly, $\{g_0, g_1\}$ is $\mathbb{Z}$-independent modulo N. As so often, we iterate. Suppose that for some $k \geq 1$ we have constructed a subset $\{g_0, g_1, \dots, g_k\}$ of G that is $\mathbb{Z}$-independent modulo N and such that the nilpotency class n_j of $\langle g_j\rangle^G$ is greater than $c + n_0 + \cdots + n_{j-1}$, for each $j = 1, \dots, k$. Write

$I_k = I_G(N\langle g_0, \ldots, g_k\rangle)$ and $G/N = I_k/N \times U_k/N$. As before, U_k is non-nilpotent and therefore contains an element g_{k+1} whose normal closure in G has nilpotency class n_{k+1} greater than $c + n_0 + \cdots + n_k$. We have defined inductively an infinite subset $\{g_0, g_1, \ldots\}$ that is $\mathbb{Z}$-independent modulo N and such that the classes n_i of $\langle g_i\rangle^G$ satisfy $n_{k+1} > c + n_0 + \cdots + n_k$ for all $k \geq 1$.

Now set $H = N\langle g_1, g_2, \ldots\rangle$. Since $\langle g_k\rangle^G \leq \langle g_k\rangle G'$ and $G' \leq N$, we see that H is non-nilpotent, and so there is an isomorphism θ from G onto H. Write $y = g_0\theta$; then $\langle y\rangle^H$ has nilpotency class n_0. Certainly $y \notin N$ since $n_0 > c$, and so $y = g_{i_1}^{\alpha_1} \ldots g_{i_k}^{\alpha_k} h$ for integers $i_1, \ldots, i_k$ with each α_j non-zero, $0 < i_1 < \cdots < i_k$ and some element h of N. Put $K = \langle g_{i_1}, \ldots, g_{i_k}, N\rangle$, so that $y \in K$. Since α_k is non-zero, the subgroup $L =: \langle y, g_{i_1}, \ldots, g_{i_k-1}, N\rangle$ has finite index in K; here we interpret L as $\langle y, N\rangle$ if $k = 1$. It follows that L has exactly the same nilpotency class as K. Now L is the product of the H-invariant subgroups N, $\langle y\rangle^H$, $\langle g_{i_1}\rangle^H, \ldots, \langle g_{i_k-1}\rangle^H$, so that it has class at most $d := c+n_0+n_i+\cdots+n_{i_{k-1}}$, by Fitting's Theorem. However, K contains the subgroup $\langle g_{i_k}\rangle^G$ of class n_{i_k}, which is certainly greater than d. This contradiction completes the proof of Theorem 1.1. □

Next we prove Theorem 1.3. It is easily seen that (c) implies (b), for if $\langle x\rangle^G$ is nilpotent we have the contradiction that G is nilpotent. Theorem 1.2 means that (b) implies (a), and thus it suffices to prove that (a) implies (c).

Suppose then that G is a soluble locally nilpotent $\mathscr{W}$-group of finite rank. As in the proof of Lemma 3, we have $G = N\langle x\rangle$ for some normal nilpotent subgroup N and element x; without loss we may choose N to be the Fitting subgroup of G. Each primary component of the torsion subgroup T of G has finite rank and is therefore Chernikov. Since a $\mathscr{W}$-group does not satisfy the minimum condition for subgroups, we deduce that G is not a p-group for any prime p. Furthermore, G is not the direct product of a non-trivial p-group and a non-trivial p'-group (one of them would be non-nilpotent!), and it follows that G is not a torsion group. Thus T is nilpotent and therefore contained in N. We claim that $T\langle x\rangle$ is not nilpotent. If it is, then $T \leq Z_a(G)$ for some integer a, by Lemma 1. But G/T is a torsion-free locally nilpotent group of finite rank and therefore nilpotent ([5, Theorem 6.36]) and we have the contradiction that G is nilpotent, thus establishing our claim. Thus G is isomorphic to $T\langle x\rangle$ and, since $T \leq N$, we may as well assume that $G = T\langle x\rangle = T]\langle x\rangle$ since x has infinite order. The next claim is that T is a p-group for some prime p. Otherwise $T = P \times Q$, where P is a non-trivial p-group and Q a non-trivial p'-group, and we have that each of $P\langle x\rangle$ and $Q\langle x\rangle$ is nilpotent (neither is isomorphic to G). Two applications of Lemma 1 yield the contradiction that PQ is in some term of the upper central series.

What we have now is that $G = P]\langle x\rangle$ for some nilpotent p-group P, in the right direction for establishing (c). Indeed, we show that this P and this x have all the properties required by (c). Let D denote the divisible part of P. We shall show that $D = P$, so that P is divisible Abelian. If $D < P$, then $D\langle x\rangle$ is nilpotent; however,

G/D is finite-by-cyclic since G has finite rank, and so G/D is nilpotent. Lemma 1 then supplies the contradiction that G is nilpotent, and so P is a divisible Abelian group, as claimed. Since P is certainly not central there is an integer k such that $[\Omega_k(P), x] \neq 1$. Since $\Omega_k(P)$ is finite and $\langle \Omega_k(P), x\rangle$ is nilpotent, there is a positive integer n such that $[\Omega_k(P), x^n] = 1$. For this n, $P\langle x^n\rangle$ is not isomorphic to G and is therefore nilpotent. Let m be the least positive integer such that $P\langle x^m\rangle$ is nilpotent. Write $m = qr$ where q is prime-we see that m is itself prime; indeed that $m = p$. For $P\langle x^r\rangle$ is not nilpotent and hence isomorphic to G, while $P\langle x^{rq}\rangle$ is nilpotent, and so m is prime. There is a finite G-invariant series of P centralized by x^m, and we can choose a factor A of it such that $[A, x] \neq 1$. Then x acts on A as an element of prime order m, and from local nilpotency it follows that the only value possible for m is p. Thus $P\langle x^p\rangle$ is nilpotent.

We claim that $P\langle x^p\rangle$ is in fact Abelian, that is, that $[P, x^p] = 1$. Assume not. Certainly $[P, x^p] < P$ since $P\langle x^p\rangle$ is nilpotent, and it follows without difficulty that the homomorphism $a \to [a, x^p]$ from P to itself has infinite kernel K, say. If H is the divisible part of K, then H is normal in G and has smaller rank than P, so that $\langle H, x\rangle$ is nilpotent and therefore Abelian, [5, Lemma 3.13]. It follows that $C_P(x)$ is infinite, and that the homomorphism $a \to [a, x]$ from P to itself is not onto. Since $[P, x]$ is divisible it has smaller rank than P, so that $[P, x]$ is not isomorphic to P. Thus $\langle [P, x], x\rangle$ is nilpotent and so $P\langle x\rangle$ is nilpotent. This contradiction shows that $[P, x^p] = 1$ and hence that $x^p \in Z(G)$. Part (i) of (c) is thereby established.

Now let H be a non-nilpotent subgroup of G. Then $H \cap P$ is isomorphic to P, being the 'periodic part' of H. It follows that $H \cap P = P$ by rank considerations, and so H contains P. If now H contains x^p then, since $P\langle x^p\rangle$ is Abelian, H must equal G and this confirms part (iii) of (c) : all proper subgroups of G containing x^p are nilpotent.

Finally, we prove part (ii). As we saw above, $[P, x]$ is isomorphic to P and therefore is equal to it. It follows that $P = G'$, and hence that the factor-group $B =: G/\langle x^p\rangle$ is isomorphic to the group $B(p, 1, 0)$ see [4, 4.5]. But then [4, 4.6] applies to give that all proper G-invariant subgroups of P are finite, as required to conclude the proof of Theorem 1.3. □

4. Proof of Theorem 1.2

Let G be an arbitrary soluble locally nilpotent $\mathscr{W}$-group of infinite rank, and suppose for a contradiction that G is not a Fitting group. As in the proof of Lemma 3, we have $G = N\langle x\rangle$ for some element x, where N is the Fitting subgroup of G and is itself nilpotent.

Suppose that T is the torsion subgroup of G, and H any non-nilpotent subgroup of

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G containing T. Then T is the torsion subgroup of H, so that T is invariant under any isomorphism from G to H and it follows that G/T is a torsion-free group isomorphic to each of its non-nilpotent subgroups. If G/T is finitely generated, it is nilpotent because G is locally nilpotent; if not, it is nilpotent by Theorem 1.1. If $G = T$, then G is a p-group for some prime p. If $G \neq T$, then G and T are not isomorphic, so that T is nilpotent and therefore contained in N, and so $T\langle x\rangle$ is not nilpotent, by Lemma 1. Also in this case, T is a p-group: see the proof of Theorem 1.3.

Summarising, we have that $G = N\langle x\rangle$ for some x, where $N = \mathrm{Fitt}(G)$, and either G is a p-group or $G = T\langle x\rangle$ where T is a p-group and x has infinite order. Further, $N = \mathrm{Fitt}\, H$ for every subgroup H containing N, and thus G/N is infinite cyclic or of prime order.

The proof proceeds in several stages.

(a) *G is not isomorphic to $N'\langle x\rangle$, so that $N' \leq Z_a(G)$ for some integer a.*

PROOF. This is easy if G/N is infinite: just consider the derived length of $\mathrm{Fitt}(G)$. Suppose then that G/N is of prime order p, and assume for a contradiction that $G \cong N'\langle x\rangle$.

We show by induction on i that for each $i \geq 2$, there exists y in G (depending on i) such that $G \cong \langle y\rangle\gamma_i(N)$: since N is nilpotent, this gives the contradiction that G is cyclic. For $i = 2$, this is the assumption that $G \cong N'\langle x\rangle$ that we are contradicting. Suppose that $G \cong \langle y\rangle\gamma_i(N)$ and write $K = \langle y\rangle\gamma_i(N)$. Since $K/(K \cap N)$ has order p, it is clear that $K \cap N = \mathrm{Fitt}\, K$ and hence that $K = \langle y\rangle(K \cap N)$. Since $G \cong K$, we may write $K = \langle z\rangle(K \cap N)$ for some z such that $\langle z\rangle(K \cap N)' \cong K$. Thus $K \cong \langle z\rangle(\langle y\rangle\gamma_i(N) \cap N)' = \langle z\rangle(\gamma_i(N)(\langle y\rangle \cap N))' \leq \langle z\rangle\gamma_{i+1}(N)$. But K is non-nilpotent, and therefore so is $\langle z\rangle\gamma_{i+1}(N)$, and the induction is complete. Thus $N'\langle x\rangle$ is not isomorphic to G, is therefore nilpotent and so $N' \leq Z_a(G)$ for some a, by Lemma 1. □

(b) *The nilpotent residual A of G is central in N and therefore Abelian.*

PROOF. Since $[Z_i(G), \gamma_i(G)] = 1$ for every positive integer i, we see that A centralizes $Z_\omega(G)$, the 'ω - hypercentre' of G. Let h be an arbitrary element of N. Then $\langle h, x\rangle$ is nilpotent, so that $\langle h\rangle^G N'/N'$ is finitely generated (just note that $G/N' = N/N'.\langle xN'\rangle$), and therefore it is contained in $Z_\omega(G/N')$. It now follows from (a) that $N \leq Z_\omega(G)$, and hence that $[N, A] = 1$. Since $A \leq G' \leq N$, this completes the proof of (b). □

(c) *$\langle x\rangle A$ is not nilpotent.*

PROOF. Suppose false. Then $A \leq Z_s(G)$ for some integer s by Lemma 1, and hence G/A is not nilpotent. If G/N has prime order p, then $\langle x^p\rangle N' \leq Z_b(G)$ for some integer $b \geq a$, again by Lemma 1. Thus, in either case (that where G/N is infinite and that where it is of order p), if $U/A := Z_b(G/A)$, then U is nilpotent and therefore contained in N, so that N/U is an Abelian p-group: note that $N' \leq Z_b(G) \leq U$. Next, G/U is residually nilpotent, since $X/Z_i(X)$ is residually nilpotent whenever X is residually nilpotent and i a positive integer. It follows that $U \leq Z_{s+b}(G)$. Now $G/U = N\langle x\rangle/U$, a group satisfying the requirements of Lemma 4 in the obvious way. Thus, there is a subgroup B/U of N/U with $B\langle x\rangle/U$ non-nilpotent and B/U a direct product of finite G/U-invariant subgroups. Let bU be any non-trivial element of B/U. Now finitely generated nilpotent groups are residually finite, so the structure of B/U just described yields the existence of a normal subgroup of finite index in G/U that does not contain bU.

We now choose an element v_1 of B such that $[v_{1,t}x] \notin U$ but $[v_{1,(t+1)}x] \epsilon U$, for suitable t. Such a choice is possible since B/U is Abelian and $B\langle x\rangle/U$ is not nilpotent. Write $V_1 = \langle v_1\rangle^G U$, so that V_1/U is finite. By the remark in the preceding paragraph, there is a normal subgroup N_1 of G such that $U \leq N_1 \leq B$, B/N_1 is finite and $N_1 \cap V_1 = U$. Now $N_1\langle x\rangle/U$ is not nilpotent, for otherwise we have $N_1\langle x\rangle$ nilpotent by definition of U, so that $N_1 \leq Z_n(G)$ for some integer n (Lemma 1 yet again!) and hence B is nilpotent, a contradiction. It is now straightforward to construct a G-invariant subgroup V/U of B/U such that V/U is the direct product of subgroups V_i/U, $i = 1, 2, \ldots$, where $V_i = \langle v_i\rangle^G U$ for some v_i such that $[v_{i,(2t+i)}x] \notin U$ for each $i > 1$.

Set $W = \langle V_i : i \geq 2\rangle$, so that $V/U = V_1/U \times W/U$. Since neither $\langle x\rangle V$ nor $\langle x\rangle W$ is nilpotent, there is an isomorphism θ from $\langle x\rangle V$ to $\langle x\rangle W$. Note that $\mathrm{Fitt}(\langle x\rangle V) = V$ and $\mathrm{Fitt}(\langle x\rangle W) = W$: if G/N is infinite this is clear, while for G/N of order p we recall that $x^p \epsilon U \leq W \leq V$. It follows that $V\theta = W$, and we write $v_1\theta = w$. Thus $[v_{1,(t+1)}x] \epsilon U$ implies $[w,_{(t+1)} x\theta] \epsilon U\theta = Y$, say. Now $x\theta = x^\lambda w'$ for some $w' \epsilon W$ and some integer λ such that $\langle x^\lambda\rangle W = \langle x\rangle W$. Since $W' \leq Y$, we have $[w,_{(t+1)} x\theta] = [w,_{(t+1)} x^\lambda w'] \equiv [w,_{(t+1)} x^\lambda] \mod Y$, and that means that $[w,_{(t+1)} x^\lambda] \epsilon Y$. If G/N is infinite then $\lambda = \pm 1$, while $(\lambda, p) = 1$ if G/N is finite; in either case we deduce that $[w,_{(t+1)} x] \epsilon Y$. Since $[U,_t G] = 1$ we have $[U,_t \langle x\rangle V] = 1$ and hence $[Y,_t \langle x\rangle W] = 1$. From all this it follows that $[w,_{(2t+1)} x] = 1$. A fortiori, we have $[w,_{2t+1)} x] \epsilon U$. But $w = z_2 \cdots z_r u$ for some r, where $u \epsilon U$ and $z_j \epsilon V_j$ for each j. From the structure of W/U we see that $[z_{j,(2t+1)}x] \epsilon U$ for each j, and by the choice of the v_j for $j \geq 2$ it follows that $z_j \epsilon [V_j, \langle x\rangle]U$ and hence $w \epsilon [W, \langle x\rangle]U$.

Next, we note that Y is $\langle x\rangle$-invariant, for $U \lhd V\langle x\rangle$ implies $Y = U\theta \lhd (V\langle x\rangle)\theta = W\langle x\rangle$. Thus, since $U \leq W$, $W' \leq Y$ and because U and W are also $\langle x\rangle$-invariant, we deduce that the element $[w,_t x]$ of $[[W, \langle x\rangle]U,_t x]$ lies in $[W,_{(t+1)} G][U,_t G]Y$. However, $[U,_t G] = 1$ and thus $[w,_t x] \epsilon [W,_{(t+1)} G]Y$. Applying θ^{-1}, we deduce that $[v_{1,t}x^{\theta^{-1}}]$ lies in $[V,_{(t+1)} G]U \cap V_1 = U([V,_{(t+1)} G] \cap V_1) = U[V_{1,t+1}G] = U$. Since $V' \leq U$ and $x\theta^{-1} = x^\mu v'$ for some $v' \epsilon V$ and integer μ such that $\langle x^\mu\rangle V = \langle x\rangle V$, we deduce as before that $[v_{1,t}x]\epsilon U$. This contradiction establishes (c). □

Where does this leave us? It means that $\langle x\rangle A \cong G$ and G is Abelian-by-cyclic, so that $G = B\langle y\rangle$ for some Abelian normal subgroup B and element y. It is easy to see that we may assume that B is a p-group and that $y = x$. Further, with A in its previous meaning, we have $A \leq B$ since $G' \leq B$.

(d) *G/A is not nilpotent.*

PROOF. Assume this to be false. Then A is some finite term of the lower central series, so $A = [A, G] = [A, \langle x\rangle] = [A, x]$. By Lemma 1, $A\langle x\rangle$ is not nilpotent so we may suppose that $G = A\langle x\rangle$ in this case. Choose a non-trivial element a_1 of A with $[a_1, x] = 1$, which is possible by local nilpotency. Since $A = [A, x]$, we can construct a sequence $a_1, a_2, \ldots$ of elements of A with $a_i = [a_{i+1}, x]$ for each $i \geq 1$. The subgroup $C = \langle a_i : i \geq 1\rangle$ is a G-invariant subgroup of A and $C\langle x\rangle$ is not nilpotent. There are two cases to consider.

Firstly, suppose that G/A is finite. Here G/A^p is nilpotent, since the normal closures in G/A^p of elements of A/A^p have bounded orders. Thus $A^p = A$ and A is divisible. Let U be an arbitrary Prüfer p-subgroup of A; since $[U, x^p] = 1$ we see that $\langle U, x\rangle$ has finite rank and is therefore not isomorphic to G. If D denotes the divisible radical of $U^{\langle x\rangle}$, then $D\langle x\rangle$ is nilpotent and it follows from [5, Lemma 3.13] that $[D, x] = 1$. Thus $[U, x] = 1$ and we have the contradiction that $[A, x] = 1$.

Now suppose that G/A is infinite, so that this time $G = A]\langle x\rangle$ and $\langle x\rangle$ is infinite cyclic. Let F be an arbitrary proper G-invariant subgroup of A and let D/F be a finite non-trivial G-invariant subgroup of A/F, which must exist by local nilpotency. If $F\langle x\rangle$ is non-nilpotent, then so is $D\langle x\rangle$, which is therefore isomorphic to G. But D has a non-trivial finite image and thus $[D, \langle x\rangle] < D$, contradicting the fact that $D = \gamma_\omega(D\langle x\rangle)$. Thus $F\langle x\rangle$ must be nilpotent for every proper G-invariant subgroup F of A, and it follows that $A = C$, the group constructed at the beginning of the proof of this section (d).

Suppose next that G/N is infinite, so that $N = A = C$. Then $Z(G) = \langle a_1\rangle$. Now $\langle a_1, a_2\rangle \lhd G$ and so $[\langle a_1, a_2\rangle, x^{p^k}] = 1$ for some positive integer k; however, $A\langle x^{p^k}\rangle$ is isomorphic to G and has centre of order greater than that of a_1. This contradiction shows that G/N must be of order p. If $A^p < A$ then, since $N = A\langle x^p\rangle$, which is therefore nilpotent, there is a positive integer r such that $[A,_r x^p] = 1$. Choosing l so that $p^l \geq r$, we have $[A,_{p^l} x^p] = 1$ and hence $[A, x^{p^{l+1}}] \leq A^p$. Thus, for each $a\epsilon A$, $\langle a\rangle^G$ has bounded order modulo A^p and it again follows that G/A^p is nilpotent, contradicting the fact that $A = \gamma_\omega(G)$. Thus $A = A^p$ after all, and A is divisible. Set $D = \Omega_1(A)$. Since $D\langle x\rangle \ncong G$, it is nilpotent and we have $D \leq Z_d(G)$ for some integer d by Lemma 1. From the structure of A we have $Z_d(G) \cap A = \langle a_1, \ldots, a_d\rangle$, so that D is finite. But then A is of finite rank and thus G has finite rank, a contradiction. This completes the proof of (d). □

(e) *The final step.*

Since G/A is non-nilpotent by (d) and $A\langle x\rangle$ is isomorphic to G by (c), we have $[A, \langle x\rangle] < A$ and we may choose $b_0 \epsilon A\backslash[A, \langle x\rangle]$. Recall that $G = B\langle x\rangle$, where B is a normal Abelian p-subgroup; we have $A = \bigcap_{i=1}^{\infty}[B,_i G] = \bigcap_{i=1}^{\infty}[B,_i x]$, so that $b_0 = [b_1, x]$ for some $b_1 \epsilon B\backslash A$. Set $B_1 = \langle b_1\rangle^G A$; by residual nilpotency of G/A there is an integer α_1 such that $B_1 \cap \gamma_{\alpha_1}(G) = A$, since B_1 is of course finite. Next, $A = \bigcap_{i=1}^{\infty}[\gamma_{\alpha_1}(G),_i x]$ and so we may choose b_2 in $\gamma_{\alpha_1}(G)$, which is contained in B, such that $b_0 = [b_2, x, x]$. Writing $B_2 = \langle b_2\rangle^G A$, we see that B_2/A is finite, so there is a positive integer α_2 such that $B_2 \cap \gamma_{\alpha_2}(G) = A$. Iterating, as so often, we obtain a subgroup H of B such that $A \leq H$ and H/A is the direct product of subgroups B_i/A, $i = 1, 2, \ldots$, with the following property. For each i we have $B_i = \langle b_i\rangle^G A$ for some element b_i with $[b_i,_i x] = b_0$. Clearly, $H\langle x\rangle$ is not nilpotent and so $H\langle x\rangle \cong G$. However, $\gamma_\omega(H\langle x\rangle) = \langle b_0\rangle^G$, which is finite, in contradiction to (b). This concludes the proof of Theorem 1.2. □

5. Concluding remarks

In this final section we are concerned for the most part with soluble p-groups in the class $\mathscr{W}$. There probably are none, and we give here some reduction results that could perhaps lead to a proof of this fact. If there exists such a group, then there must exist one that satisfies either (a) or (b) of 5.1 below, and we have a few words to say about case (a) in particular.

Before restricting our attention to p-groups, we record the following easy but useful result about maximal subgroups.

LEMMA 5. *Let G be a group that is isomorphic to each of its non-nilpotent subgroups. If G has no maximal subgroups then every proper subgroup of G is nilpotent.*

PROOF. Suppose the result false, and let H be a proper non-nilpotent subgroup of G. Choosing x in $G\backslash H$, we see that $\langle H, x\rangle$ is non-nilpotent and therefore isomorphic to G. But H is contained in a maximal subgroup of $\langle H, x\rangle$, and we have a contradiction.

Now suppose that G is a periodic soluble group in $\mathscr{W}$. By local nilpotency, G is therefore a p-group for some prime p and of course G' is nilpotent. Let B/G' be a basic subgroup of G/G' (see [1, Section 33]). If B is not nilpotent, then G has a normal nilpotent subgroup N such that G/N is a direct product of cyclic groups. If B is nilpotent, then G has a normal nilpotent subgroup N such that G/N is divisible Abelian. In this latter case, suppose that G/N has infinite rank and suppose further that every subgroup M containing N with G/M of finite rank is non-nilpotent. Then there is a finite subgroup F_1 of G with NF_1 of class at least one (note that NF is nilpotent for every finite subgroup F of G since G is a Fitting group by Theorem 1.2 and soluble p-groups of finite rank are Chernikov and thus do not lie in $\mathscr{W}$). Next, there exists a subgroup M_1 of G with $NF_1 \leq M_1$ and $G/N = M_1/N \times G_1/N$ for some G_1, where M_1/N has finite rank. Since G/G_1 has finite rank, G_1 is not nilpotent, so there exists a finite subgroup F_2 of G_1 with NF_2 of class at least 2. And so on: there is a subgroup H of G such that $N \leq H$ and $H/N = NF_1/N \times NF_2/N \times \cdots$, where the nilpotency of class of NF_i is at least i. Thus H is not nilpotent and isomorphic to G. On the other hand, if G/N is divisible and of finite rank r, an easy induction on r shows that N may be chosen so that G/N has rank 1. □

Summarising, we have:

(5.1) *If there exists a soluble p-group in $\mathscr{W}$ then there exists such a group G that has a normal nilpotent subgroup N such that one of the following holds*:

(a) $G/N \cong C_{p^\infty}$;

(b) *G/N is a direct product of cyclic groups.*

From this point we assume that G is a group as in (5.1). We record the following observations about such G.

(5.2) *If* (a) *holds then G is not hypercentral.*

We omit the proof, but note that the argument shows that the existence of a group satisfying 5.1(a) yields one that is reduced. The following property is reminiscent of the Heineken-Mohamed examples [3]. Our proof is lengthy and again we omit it.

(5.3) *If* (a) *holds, there is no nilpotent subgroup H of G such that $G = NH$.*

Finally on this topic:

(5.4) *If* (a) *holds then N has infinite exponent.*

This time the proof is short enough to record. Suppose for a contradiction that $G/N \cong C_{p^\infty}$ and that N is nilpotent and of finite exponent p^k. If K is an arbitrary normal subgroup of G with G/K of finite exponent, then $G = NK$ and so G/K has exponent dividing p^k. Let L be the intersection of all such subgroups, so that G/L also has exponent p^k. Since $L/\Phi(L)$ has exponent dividing p, it follows that $L = \Phi(L)$. But G does have maximal subgroups by Lemma 5, and it follows that L is nilpotent. Thus NL is nilpotent and, as G/NL has finite exponent and is divisible, we have the contradiction that $G = NL$. This establishes 5.4. □

Finally, we mention a result concerning case (b) of 5.1. Its proof is much like that of Theorem 1.1 and we omit it. Note first that if there exists a $\mathscr{W}$-group in which G/N has finite exponent, there is one in which G/N has exponent p.

(5.5) *Suppose that G satisfies* 5.1(b) *and that G/N has exponent p. Then there is a bound for the nilpotency classes of the subgroups $\langle x\rangle^G$, for $x \in G$.*

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