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ON MINIMALLY FREE ALGEBRAS

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1. Introduction. For us an "algebra" is a finitary "universal algebra" in the sense of G. Birkhoff [**9**]. We are concerned in this paper with algebras whose endomorphisms are determined by small subsets. For example, an algebra A is *rigid* (in the strong sense) if the only endomorphism on A is the identity id_A. In this case, the empty set determines the endomorphism set $E(A)$. We place the property of rigidity at the bottom rung of a cardinal-indexed ladder of properties as follows. Given a cardinal number κ, an algebra A is *minimally free over a set of cardinality* κ (κ*-free* for short) if there is a subset $X \subseteq A$ of cardinality κ such that every function $f:X \to A$ extends to a unique endomorphism $\phi \in E(A)$. (It is clear that A is rigid if and only if A is 0-free.) Members of X will be called *counters*; and we will be interested in how badly counters can fail to generate the algebra.

The property "κ-free" is a solipsistic version of "free on κ generators" in the sense that A has this property if and only if A is free over a set of cardinality κ relative to the concrete category whose sole object is A and whose morphisms are the endomorphisms of A (thus explaining the use of the adverb "minimally" above). In view of this we see that the free algebras on κ generators relative to any variety give us examples of "small" κ-free algebras. We wish, however, to focus on constructing examples which are "exotic" in the following sense. Define a κ-free algebra A to be *large* if the cardinality of A is infinite, exceeding both κ and the number of distinguished operations in the algebraic type $\tau = \tau_A$ of A. In such an algebra, the counters fail miserably to generate the algebra, but still "determine" it in a natural sense.

The underlying problem in the present study is to specify, given a cardinal κ and a class $\mathscr{K}$ of algebras of the same type, which are the κ-free algebras in $\mathscr{K}$.

1.1. *Examples.* (i) Let $\mathscr{K}$ be the class of fields. There are no κ-free fields for $\kappa > 0$; however there are rigid fields, e.g. the rational field **Q** and the real field **R**. (Here, **R** is a large 0-free algebra.)

(ii) Let $\mathscr{K}$ be the class of torsion-free divisible abelian groups. Each such group can be regarded as a vector space over **Q**; hence $A \in \mathscr{K}$ is κ-free if and only if A is the direct sum of κ copies of **Q**. (In particular there are no large κ-free algebras in $\mathscr{K}$ for any κ.)

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(iii) Let $\mathcal{K}$ be any variety of algebras such that $\mathcal{K}$ has rigid algebras of arbitrarily high cardinality. Then $\mathcal{K}$ contains arbitrarily large κ-free algebras for any fixed κ. (This is a recent result, due to I. Kříž and A. Pultr [**12**]. They actually prove a more general theorem, but this is the version of most interest here.) Examples of such classes $\mathcal{K}$ include 2-unary algebras, semigroups, and commutative unital rings (see [**9**]).

1.2. *Remark*. The question of the existence of classes $\mathcal{K}$ with arbitrarily large κ-free algebras was originally posed by P. Bankston at the Denver A.M.S. Annual Meeting in 1983. B. Jónsson [**10**] came up with an example for $\kappa = 1$ within a day. The problem got around; and J. Sichler remarked to A. Pultr [**16**] that if $\mathcal{K}$ is the variety of all 2-unary algebras then one can form a κ-free algebra in $\mathcal{K}$ simply by taking a coproduct of a suitably large rigid algebra and the free $\mathcal{K}$-algebra with κ generators. This is the genesis ž-Pultr result.

This paper is organized as follows. In Section 2 we consider 1-free semigroups, concentrating our energies on the problem of constructing large 1-free groups. Theorems (2.3) and (2.7), along with examples (2.8) and (2.9) (due to R. Schutt), are the principal results. (Only the trivial group is rigid, so the variety of groups does not fall under the Kříž-Pultr theorem.) In Section 3 we look at the problem of constructing large κ-free algebras of continuous functions. We regard Theorems (3.5), (3.7), (3.10), (3.12), and (3.18) (due to P. Bankston) as the main results of this section.

Throughout the paper we adopt a standard notation for ordinal and cardinal numbers (see [**13**]). In particular: each ordinal is the set of its predecessors; $\omega = \{0, 1, 2, \dots\}$ is the first infinite ordinal (and cardinal); c is the cardinality of the continuum, $c = 2^{\omega} = \exp(\omega)$ (the notation A^B can denote either a cartesian power or a cardinal, depending on context); arbitrary ordinals will be denoted by lower case Greek letters (the letters κ, λ, μ will usually be reserved for cardinals); and the cardinal successor of κ is denoted $\kappa^+ (\omega_1 = \omega^+)$.

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2. Semigroups. In this section we focus on κ-free semigroups for $\kappa = 1$; and, except for a few preliminary remarks, concern ourselves only with 1-free groups.

2.1. PROPOSITION. *Every* 1-*free group is abelian.*

Proof. Let G be 1-free with counter g. Then the inner automorphism I_g with respect to g fixes g, so must be the identity map. Thus g is in the

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center of G. If $a \in G$ is arbitrary then

$$I_a(g) = a^{-1}ga = a^{-1}ag = g,$$

hence must also be the identity map. Thus G is abelian.

2.2. *Remark*. Not every 1-free semigroup is abelian. M. Petrich [**14**] has observed that the free inverse semigroup on one generator is a noncommutative 1-free semigroup. Without going into excessive detail (the reader is referred to [**15**]), here are the reasons why.

An *inverse semigroup* is a semigroup with the property that to each x there is a unique y such that $x = xyx$, $y = yxy$. Clearly every semigroup homomorphism also preserves this "inverse".

An alternative description is to add a distinguished unary operation $(\ )^{-1}$ to the semigroup and to write down the equational axioms:

$$(xy)^{-1} = y^{-1}x^{-1}$$

$$(x^{-1})^{-1} = x$$

$$xx^{-1}x^{-1}x = x^{-1}xxx^{-1}$$

Given such, x^{-1} is uniquely determined by $x^{-1} = x^{-1}xx^{-1}$, $x = xx^{-1}x$.

The free algebra on one generator in this variety is the free inverse semigroup S on one generator. S is clearly 1-free, however it so happens that $gg^{-1} \neq g^{-1}g$ for the counter g.

In the sequel, G will denote a commutative semigroup and additive notation will be used. The set $E(G)$ of endomorphisms is a "semiring" (i.e., addition is merely a commutative semigroup operation) under the operations

$$(\phi + \psi)(x) = \phi(x) + \psi(x) \quad \text{and} \quad (\phi \cdot \psi)(x) = \phi(\psi(x)).$$

(Of course, $E(G)$ is a ring if G is an abelian group.) If R is any semiring let R^+ denote the additive semigroup of R. In the spirit of P. Schultz [**17**], R is an *E-semiring* (Schultz is concerned with rings, by the way) if for $\theta \in E(R^+)$ there is an $r \in R$ such that $\theta(x) = r \cdot x$ identically. Our first result characterizes 1-free commutative semigroups.

2.3. THEOREM. *Let G be a commutative semigroup and let $E = E(G)$. The following are equivalent.*

(i) *G is 1-free.*

(ii) *$G \cong E^+$ and E is an E-semiring.*

(iii) *There is a $g \in G$ such that*

$$G = Eg = \{\phi(g):\phi \in E\}$$

and E is commutative.

Proof. (i) $\Rightarrow$ (ii). Let G be 1-free with counter g, and consider the map $\theta: E^+ \to G$ defined by $\theta(\phi) = \phi(g)$. θ is clearly a homomorphism, θ is one-one because two members of E which agree at g agree everywhere, and θ is onto because for each $a \in G$ there is always an endomorphism taking g to a. Thus θ is an isomorphism. Now suppose f is an endomorphism of E^+. Define the map $\bar{f}: G \to G$ by

$$\bar{f}(a) = \bar{f}(\phi_a(g)) = f(\phi_a)(g)$$

(ϕ_a is, as before, the unique endomorphism on G sending g to a). Clearly $\bar{f} \in E$; and $f(\phi) = \bar{f} \cdot \phi$ for any $\phi \in E$, since members of E are determined by where they send g. Thus E is an E-semiring.

(ii) $\Rightarrow$ (iii). Let $\phi \in E$. The map $\psi \mapsto \psi \cdot \phi$ is an endomorphism of E^+, hence $\psi \cdot \phi$ is identically $\theta \cdot \psi$ for some $\theta \in E$. Letting $\psi = \mathrm{id}_G$, we get $\phi = \theta$. Thus E is commutative. Now let η be an isomorphism from E^+ to G. For each $\phi \in E$ define $l_\phi: E^+ \to E^+$ by $l_\phi(\psi) = \phi \cdot \psi$. Then

$$\eta \circ l_\phi \circ \eta^{-1} \in E \quad \text{and} \quad (\eta \circ l_\phi \circ \eta^{-1})(\eta(\mathrm{id}_G)) = \eta(\phi).$$

Thus we have $G = Eg$ where $g = \eta(\mathrm{id}_G)$.

(iii) $\Rightarrow$ (i). Let $g \in G$ be such that $G = Eg$ and suppose E is commutative. If $a \in G$ then $a = \phi(g)$ for some $\phi \in E$; we have to show that ϕ is unique. So let $a = \psi(g)$ for $\psi \in E$. For arbitrary $b \in G$, find $\theta \in E$ such that $b = \theta(g)$. Then

$$\phi(b) = \phi(\theta(g)) = \theta(\phi(g)) = \theta(\psi(g)) = \psi(\theta(g)) = \psi(b).$$

Thus $\phi = \psi$.

2.4. *Examples.* (i) Schultz [**17**] and Bowshell-Schultz [**4**] pointed out that the epimorphs of the ring $\mathbf{Z}$ of integers, the unital subrings of the field $\mathbf{Q}$ of rationals, and the pure subrings of the ring $\mathbf{R}_p$ of p-adic integers (p prime) are E-rings. Moreover, in [**17**], Schultz showed that the unital pure subrings of $\prod_{p\text{prime}} \mathbf{Z}_p$ (where, for any positive integer n, $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$) are E-rings.

(ii) One can show the "semifield" of positive elements of any Archimedean field is an E-semiring, as is any unital subsemiring of $\mathbf{Q}$.

The rest of this section is devoted to the construction of large (i.e., uncountable) E-rings. Among the examples above, $\prod_p \mathbf{Z}_p$, $\mathbf{R}_p$, and the semifield of positive real numbers are of particular interest since they have cardinality c.

We still do not know whether there are 1-free abelian groups, or even 1-free commutative semigroups of arbitrarily large cardinality; but we will put a dent in the problem by constructing E-rings of cardinalities $\exp(c)$ and $\exp^2(c)$.

In the sequel all groups are abelian. Before we give our examples we will need the following material from [**7**]. Let $\mathbf{H} = \prod_{i=1}^{\infty} \langle e_n \rangle$, where each

$\langle e_n \rangle$ is an infinite cyclic group with generator e_n. A torsion-free group G is *slender* if, for every homomorphism $\eta:\mathbf{H} \to G$, $\{n:\eta(e_n) \neq 0\}$ is finite (e_n being identified in $\mathbf{H}$ in the obvious way).

2.5. THEOREM [7, p. 165]. *A torsion-free group G is slender if and only if G does not contain a copy of $\mathbf{Q}^+$, $\mathbf{H}$, or $\mathbf{R}_p^+$ for any prime p.*

Recall that a cardinal number μ is *Ulam-measurable* if there is a countably complete nonprincipal ultrafilter on μ (see [5], [7], [8], [13], [18]). It is well known that the class of non-Ulam-measurable cardinals constitutes an infinite interval of cardinals which is closed under chain suprema and cardinal exponentiation.

2.6. THEOREM [7, p. 161]. *Let $\langle G_i:i \in I\rangle$ be a family of torsion-free abelian groups where the cardinality of I is non-Ulam-measurable, and let G be slender. If η: $\prod_{i\in I} G_i \to G$ is any homomorphism whose kernel includes the set of elements $\vec{a}$ of "finite support"* (i.e., $\{i \in I:a_i \neq 0\}$ *is finite*), *then $\eta = 0$.*

Define a family $\langle R_i:i \in I\rangle$ of rings to be *incomparable* if

$$\mathrm{Hom}(R_i^+, R_j^+) = 0 \quad \text{whenever } i \neq j.$$

2.7. THEOREM. *Let $\langle R_i:i \in I\rangle$ be an incomparable family of E-rings, where each R_i^+ is slender and $|I|$ is non-Ulam-measurable. Then $\prod_{i\in I} R_i$ is an E-ring.*

Proof. Let $R = \prod_{i\in I} R_i$. For each $i \in I$, identify R_i with $\{\vec{a} \in R:a_j = 0 \text{ for } j \neq i\}$, and let $S = \sum_{i\in I} R_i$. Then S, the (internal) direct sum of the R_i's, is just the elements of R of finite support. Let $\theta \in E(R^+)$. Since the R_i are incomparable, they are "absolutely invariant", i.e.,

$$\theta[R_i] \subseteq R_i \text{ for each } i \in I.$$

Now for each i there is some $r_i \in R_i$ with $\theta(x) = r_i \cdot x$ holding for $x \in R_i$. Letting $\vec{r} \in R$ have coordinates r_i, we then have $\theta(\vec{x}) = \vec{r} \cdot \vec{x}$ true for $\vec{x} \in S$ (because S is a direct sum). So let $\phi = \theta - \vec{r}$ (i.e., $\phi(\vec{x})$ is $\theta(\vec{x}) - \vec{r}\cdot\vec{x}$). Then $\phi \restriction S = 0$. For each $i \in I$ let $\pi:R \to R_i$ be projection. Then

$$(\pi_i \circ \phi) \restriction S = 0.$$

Since R_i^+ is slender, we can apply (2.6) and conclude that

$$\pi_i \circ \phi = 0 \quad \text{for each } i \in I.$$

Thus $\phi = 0$, and $\theta(\vec{x}) = \vec{r}\cdot\vec{x}$ holds for all $\vec{x} \in R$.

In our examples we exhibit two incomparable families of slender E-rings; one of cardinality c, the other of cardinality $\exp(c)$. The products, of cardinality $\exp(c)$ and $\exp^2(c)$ respectively, will be E-rings by (2.7).

2.8. *Example.* A 1-free group of cardinality $\exp(c)$.

Construction. For each set $X \subseteq \mathbf{P}$ (= the set of positive prime numbers), let

$$\mathbf{Q}_X = \left\{ \frac{a}{b} \in \mathbf{Q}: b \text{ is not a multiple of any } p \in X \right\}.$$

Thus $\mathbf{Q}_\mathbf{P} = \mathbf{Z}$, $\mathbf{Q}_\emptyset = \mathbf{Q}$, and each $\mathbf{Q}_X$ is an E-ring. Moreover, by (2.5), each $\mathbf{Q}_X^+$ for $X \neq \emptyset$ is slender. ($\mathbf{Q}_X^+$ is countable and reduced.) Suppose $X \not\subseteq Y$ and choose $p \in X \backslash Y$. Then $p\mathbf{Q}_Y^+ = \mathbf{Q}_Y^+$ (i.e., $\mathbf{Q}_Y^+$ is p-divisible). But

$$\bigcap_{n=1}^{\infty} p^n \mathbf{Q}_X^+ = 0;$$

hence $\mathbf{Q}_X^+$ contains no p-divisible subgroup. Now homomorphic images of p-divisible groups are p-divisible; therefore,

$$\mathrm{Hom}(\mathbf{Q}_Y^+, \mathbf{Q}_X^+) = 0.$$

Thus if X and Y are incomparable with respect to inclusion then $\mathbf{Q}_X^+$ and $\mathbf{Q}_Y^+$ are incomparable with respect to homomorphisms. Now in any set of cardinality ω there is a set I of c pairwise incomparable subsets. (This is a special case of a more general fact: replace ω by $\kappa \geqq \omega$ and c by $\exp(\kappa)$. In the case $\kappa = \omega$; identify ω with $\mathbf{Q} \subseteq \mathbf{R}$ and count pairwise incomparable open intervals.) Thus $\langle \mathbf{Q}_X : X \in I \rangle$ satisfies the hypotheses of (2.7) and the resulting direct product is an E-ring of cardinality $\exp(c)$.

2.9. *Example.* A 1-free group of cardinality $\exp^2(c)$.

Construction. Let p be a fixed prime, let $\mathbf{R}_p$ denote the ring of p-adic integers, and let $\mathbf{F}_p$ denote the field of p-adic numbers,

$$\mathbf{F}_p = \left\{ \frac{a}{p^n} : a \in \mathbf{R}_p,\ n \text{ a nonnegative integer} \right\}.$$

Recall that $\mathbf{R}_p$ is a local ring with unique maximal ideal $p\mathbf{R}_p$, hence any $x \in \mathbf{R}_p \backslash p\mathbf{R}_p$ is a unit.

Moreover, $\mathbf{R}_p^+$ is reduced (in fact $\cap_{n=1}^{\infty} p^n \mathbf{R}_p = 0$) but $n\mathbf{R}_p = \mathbf{R}_p$ if $(n, p) = 1$ (i.e., n is prime to p). Also

$$\mathbf{R}_p / p^n \mathbf{R}_p \cong \mathbf{Z}_{p^n} \quad \text{for every } n \geqq 0.$$

If $F \subseteq \mathbf{F}_p$ is any subfield, let $R_F = F \cap \mathbf{R}_p$. We will produce a family of $\exp(c)$ such subfields F such that the rings R_F are E-rings, the groups R_F^+ are slender, and the family of rings R_F is incomparable.

2.9.1. LEMMA. *R_F is a pure local subring of $\mathbf{R}_p$ and is hence an E-ring. Moreover, if F' is another subfield of $\mathbf{F}_p$ then $R_F \subseteq R_{F'}$ if and only if $F \subseteq F'$.*

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Proof. Fix F and let $R = R_F$. For any positive integer n, $nF = F$; so

$$nR = n(F \cap \mathbf{R}_p) = F \cap n\mathbf{R}_p = F \cap \mathbf{R}_p \cap n\mathbf{R}_p = R \cap n\mathbf{R}_p.$$

Hence R is pure in $\mathbf{R}_p$. Now

$$pR = R \cap p\mathbf{R}_p \neq R.$$

For otherwise choose n with $(n, p) = 1$. Then

$$nR = R \cap n\mathbf{R}_p = R \cap \mathbf{R}_p = R;$$

hence if $pR = R$ then R would be divisible and nonzero (since $1 \in R$). This is impossible since $\mathbf{R}_p$ is reduced. Since pR is a proper ideal of R, let $x \in R \backslash pR$. Then $x \notin p\mathbf{R}_p$, so x is a unit of $\mathbf{R}_p$. But $x^{-1} \in F$. Therefore

$$x^{-1} \in F \cap \mathbf{R}_p = R,$$

so x is a unit of R. This means pR is the unique maximal ideal of R, i.e., R is local. Now suppose F' is another subfield of $\mathbf{F}_p$ and let $R' = R_{F'}$. If $R \subseteq R'$ and $x \in F$ then $x = a/p^n$ for some $a \in \mathbf{R}_p$ and $n \geqq 0$. Thus

$$p^n \cdot x = a \in R \subseteq R' \subseteq F',$$

whence $x \in F'$.

The following result is well known (see, e.g. [**1**]).

2.9.2. Lemma. *Let G be a pure subgroup of $\mathbf{R}_p^+$ and let $\theta \in \mathrm{Hom}(G, \mathbf{R}_p^+)$. Then there is a fixed $a \in \mathbf{R}_p$ such that $\theta(x) = a \cdot x$ for all $x \in G$. In particular, the rings R_F above are E-rings.*

2.9.3. Lemma. *Let G be a pure subgroup of $\mathbf{R}_p^+$ containing 1 and let θ be a nonzero homomorphism of G into a pure local subring H of $\mathbf{R}_p$. Then $G \subseteq H$.*

Proof. By (2.9.2), there exists $a \in \mathbf{R}_p$ such that $\theta(x) = a \cdot x$ holds. Thus $aG \subseteq H$. Suppose $a = p^n \cdot u$ where u is a unit of $\mathbf{R}_p$. Now

$$p^nG = p^n\mathbf{R}_p \cap G \quad \text{and} \quad u\mathbf{R}_p = \mathbf{R}_p.$$

Thus,

$$\begin{aligned} aG &= p^n \cdot uG = u \cdot p^nG = u(p^n\mathbf{R}_p \cap G) \\ &= p^n\mathbf{R}_p \cap uG \cap H \subseteq p^nH. \end{aligned}$$

Therefore $uG \subseteq H$. Since H is pure in $\mathbf{R}_p$, the map

$$\phi: H/pH \to \mathbf{R}_p/p\mathbf{R}_p,$$

given by

$$\phi(h + pH) = h + p\mathbf{R}_p,$$

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is injective. And since

$$\mathbf{R}_p/p\mathbf{R}_p \cong \mathbf{Z}_p,$$

it follows that pH is a maximal ideal; hence the maximal ideal of H since H is local. Since $1 \in G$, we have $u \in H$. But

$$u \notin H \cap p\mathbf{R}_p = pH.$$

Thus u is a unit of H. So $G \subseteq u^{-1}H \subseteq H$ as desired.

By (2.9.3) we can immediately infer that two pure local subrings of $\mathbf{R}_p$ are comparable under additive homomorphisms if and only if they are comparable under inclusion. Now let $B \subseteq \mathbf{F}_p$ be an algebraically independent subset of cardinality c (i.e., no element of B is algebraic over the subfield of $\mathbf{F}_p$ generated by the remaining elements of B). Let I be a collection of $\exp(c)$ pairwise incomparable subsets of B. For each $i \in I$, let F_i be the subfield of $\mathbf{F}_p$ generated by i and let $R_i = F_i \cap \mathbf{R}_p$. If i and j are distinct members of I then F_i and F_j are incomparable under inclusion. Thus, in light of Lemmas (2.9.1) and (2.9.3), we conclude that $\langle R_i : i \in I\rangle$ is an incomparable family of E-rings. We can use (2.7) to conclude that $\prod_{i \in I} R_i$ is an E-ring of cardinality $\exp^2(c)$ once we show each R_i^+ is slender. By (2.5) we need only show R_i^+ contains no copy of $\mathbf{Q}$, $\mathbf{H}$, or $\mathbf{R}_q^+$ for any prime q. Now $R_i \subseteq \mathbf{R}_p$. Since $\mathbf{R}_p^+$ is reduced, R_i^+ contains no copy of $\mathbf{Q}$.

2.9.4. LEMMA ([7, p. 166, ex. 6]). *$\mathbf{R}_p^+$ contains no copy of $\mathbf{H}$; hence neither does R_i^+.*

If q is a prime distinct from p then $p\mathbf{R}_q = \mathbf{R}_q$ (i.e., $\mathbf{R}_q$ is p-divisible). But

$$\bigcap_{n=1}^{\infty} p^n\mathbf{R}_p = 0.$$

Since p-divisibility is preserved by homomorphisms, we conclude that $\mathbf{R}_p^+$ (and hence R_i^+) contains no copy of $\mathbf{R}_q^+$.

Now R_i is a pure and proper subring of $\mathbf{R}_p$ and $p\mathbf{R}_p$ is a maximal subgroup of $\mathbf{R}_p^+$. Thus

$$R_i \not\subseteq p\mathbf{R}_p \quad \text{and} \quad R_p^+ = R_i^+ + p\mathbf{R}_p^+.$$

Thus

$$p(\mathbf{R}_p^+/R_i^+) = \frac{p\mathbf{R}_p^+ + R_i^+}{R_i^+} = \mathbf{R}_p^+/R_i.$$

If $(n, p) = 1$ then

$$n(\mathbf{R}_p^+/R_i^+) = \frac{n\mathbf{R}_p^+ + R_i^+}{R_i^+} = \frac{\mathbf{R}_p^+ + R_i}{R_i^+} = \mathbf{R}_p^+/R_i^+.$$

Thus $\mathbf{R}_p^+/R_i^+$ is a nonzero divisible group, and hence infinite (in fact torsion-free since R_i^+ is pure). But

$$\mathbf{R}_p^+/p^n\mathbf{R}_p \cong \mathbf{Z}_{p^n}$$

which is finite. It follows that R_i^+ cannot contain any copy of $p^n\mathbf{R}_p$. It is easily seen that these are the only copies of $\mathbf{R}_p$ in $\mathbf{R}_p$. Thus R_i^+ contains no copy of $\mathbf{R}_p^+$ and is hence slender. This completes the construction.

2.10. *Remark*. Of course the construction in (2.9) can be used to obtain E-rings of cardinality $\exp(c)$. We included (2.8) because of its much greater simplicity.

3. Algebras of continuous functions. This last section is devoted to a study of κ-free algebras of continuous functions. Let E be a topological algebra of finitary algebraic type τ, and let $\mathcal{X}$ be a full subcategory of the category of topological spaces and continuous maps such that the contravariant functor $X \mapsto C(X, E)$ which takes a space $X \in \mathcal{X}$ to the τ-algebra of continuous E-valued functions (with operations defined pointwise) is a category duality from $\mathcal{X}$ to the category $C[\mathcal{X}, E]$ of algebras $C(X, E)$ and τ-homomorphisms, for $X \in \mathcal{X}$. (If $f: Y \to X$ is continuous for $X, Y \in \mathcal{X}$ then $C(f)(g) = g \circ f$ defines the induced homomorphism from $C(X, E)$ to $C(Y, E)$.)

3.1. *Examples*. (i) Let E be the two-element discrete Boolean algebra, and let $\mathcal{X}$ be the class of totally disconnected compact Hausdorff spaces. Then $C[\mathcal{X}, E]$ is the variety of Boolean algebras by the Stone duality theorem [**18**].

(ii) Let $\mathbf{I}$ denote the closed unit interval in the real line. We consider $\mathbf{I}$ as a topological algebra with the lattice operations for distinguished binary operations and with all elements of $\mathbf{I}$ as distinguished nullary operations (constants). Thus we let E be

$$\langle \mathbf{I}, \vee, \wedge, \{c_t : t \in \mathbf{I}\} \rangle.$$

Let $\mathcal{X}$ be the class of compact Hausdorff spaces. Then $X \mapsto C(X, E)$ defines a duality [**2**]. (In Banaschewski's parlance, $C[\mathcal{X}, E]$ is the class of "separated, functionally complete **I**-lattices".)

(iii) Let E be the topological ring $\mathbf{R}$ of real numbers, and let $\mathcal{X}$ again be the class of compact Hausdorff spaces. Then $C[\mathcal{X}, E]$ is dual to $\mathcal{X}$ by the Gel'fand-Kolmogorov duality theorem [**8**].

(iv) Let E be as in (iii), but let $\mathcal{X}$ consist of all realcompact Tichonov spaces. Then $C[\mathcal{X}, E]$ is dual to $\mathcal{X}$, again by Gel'fand-Kolmogorov duality.

Let us now look at what it means for an algebra $C(X, E)$ to be κ-free.

We can easily dispense with the case $\kappa = 0$; for $C(X, E)$ is rigid if and only if $|X| \leqq 1$ if and only if $C(X, E)$ is either degenerate or the τ-algebra E. (Note that in the examples (3.1) all of the classes $C[\mathscr{X}, E]$ have rigid algebras.) To analyze the case $\kappa > 0$, define a continuous map $f:X \to Y$ to be a *coreflection map* if for each continuous $g:X \to Y$ there is a unique continuous $h:X \to X$ such that $g = f \circ h$.

3.2. *Remark*. Let $\mathscr{A}$, $\mathscr{B}$ be categories, with $\mathscr{A}$ a subcategory of $\mathscr{B}$. A functor $F:\mathscr{B} \to \mathscr{A}$ is a *coreflection* if F is right-adjoint to the inclusion functor (see, e.g., [**19**]). The canonical morphism from $F(X)$ back to X is always a "coreflection map" in the above sense.

3.3. PROPOSITION. *Let $f:X \to Y$ be a coreflection map. Then f is a bijection.*

Proof. Let $y \in Y$ and let g be $x \mapsto y$. The existence of h forces $y \in f[X]$, so f is onto. Suppose $y = f(x_1) = f(x_2)$, and let g again be $x \mapsto y$. The maps h_1, h_2 defined by $x \mapsto x_1$, $x \mapsto x_2$ respectively, both satisfy $g = f \circ h$. Thus $h_1 = h_2$ and f is one-one.

Thus we may view a coreflection map $f:X \to Y$ as a "uniformly defined" enrichment of the topology on Y. Typical coreflection functors in topology are specified in just this way.

3.4. *Examples*. The following topology-enriching operations give rise to topological coreflections.

(i) $F(X) = D(X)$ is the discrete topology on X.

(ii) $F(X = k(X)$ is the "k-modification" of X, i.e., $A \subseteq X$ is closed in $k(X)$ if and only if $A \cap K$ is closed in K for each compact subspace K of X (see [**5**]).

(iii) For a given cardinal λ, $F(X) = (X)_\lambda$ is the "λ-modification" of X, i.e., basic open sets in $(X)_\lambda$ are intersections of fewer than λ open sets in X (see (**15**]).

3.5. THEOREM. *Let $\mathscr{X}$ be a class of spaces such that the functor $X \mapsto C(X, E)$ defines a duality between $\mathscr{X}$ and $C[\mathscr{X}, E]$. The algebra $C(X, E)$ is κ-free if and only if there is a coreflection map $f:X \to E^\kappa$, where E^κ is the usual cartesian power with the (Tichonov) product topology. Moreover, if the underlying space E^κ is in $\mathscr{X}$, then $C(E^\kappa, E)$ is not only κ-free, but the free $C[\mathscr{X}, E]$-algebra over a κ-element set.*

Proof. Assume $f:X \to E^\kappa$ is a coreflection map, and let

$$\pi_\xi:E^\kappa \to E$$

be the ξ^{th} projection map, $\xi < \kappa$. It is straightforward to show that $C(X, E)$ is κ-free with counters $f_\xi = \pi_\xi \circ f$. First of all, the f_ξ's are all distinct since f is onto. Suppose $\langle g_\xi:\xi < \kappa\rangle$ is a κ-sequence of elements of $C(X, E)$. Then the map $g:X \to E^\kappa$, defined by $\pi_\xi \circ g = g_\xi$, is continuous and there is a unique continuous $h:X \to X$ with $g = f \circ h$. Thus $C(h)$ is a homomorphism taking f_ξ to g_ξ, $\xi < \kappa$. The uniqueness of $C(h)$ is guaranteed by the uniqueness of h and the duality between $\mathscr{X}$ and $C[\mathscr{X}, E]$. A minuscule amount of extra work shows that $C(E^\kappa, E)$ is the free $C[\mathscr{X}, E]$-algebra over $\langle \pi_\xi:\xi < \kappa \rangle$ whenever $E^\kappa \in \mathscr{X}$ (see also [**3**]).

Now suppose $C(X, E)$ is κ-free with counters $\langle f_\xi:\xi < \kappa\rangle$. Defining $f:X \to E^\kappa$ by the conditions $\pi_\xi \circ f = f_\xi$, we show f is a coreflection map. Indeed, let $g:X \to E^\kappa$ be given, and let

$$\psi:C(X, E) \to C(X, E)$$

be an endomorphism taking f_ξ to g_ξ, $\xi < \kappa$. By duality there is a unique $h:X \to X$ such that $\psi = C(h)$. Thus h is unique such that $f \circ h = g$.

3.6. *Remark*. We can now specify the κ-free algebras in $C[\mathscr{X}, E]$ for the first three examples in (3.1): (i) A Boolean algebra is κ-free if and only if it is free on κ generators.

(ii) A separated, functionally complete **I**-lattice is κ-free if and only if it is of the form $C(\mathbf{I}^\kappa, \mathbf{I})$.

(iii) there are no κ-free unital rings $C(X) = C(X, \mathbf{R})$ for X compact Hausdorff and $\kappa > 0$.

From the above remark, we know that there can be no large κ-free Boolean algebras for any κ; but the story is less clear for the **I**-lattices $C(X, \mathbf{I})$. Since there are c distinguished operations in this type, we need to address the question of when

$$|C(\mathbf{I}^\kappa, \mathbf{I})| > c \cdot \kappa.$$

3.7. THEOREM. *Let κ be any cardinal. Then*

$$|C(\mathbf{I}^\kappa, \mathbf{I})| = c \cdot \kappa^\omega.$$

Proof. If $\kappa < \omega$ then the equality is well known, so assume κ is infinite and let $A \subseteq \mathbf{I}^\kappa$ be

$$\{\vec{x} \in 2^\kappa:x_\xi = 1 \text{ for exactly one } \xi < \kappa\}.$$

Then A is discrete and its closure $\bar{A}$ in $\mathbf{I}^\kappa$ is easily seen to be $A \cup \{\vec{0}\}$. Thus $\bar{A}$ is the one-point compactification of the κ-element discrete space. Let $S \subseteq \mathbf{I}$ be the set

$$\{0\} \cup \left\{\frac{1}{n}:n > 0\right\}.$$

Now every continuous surjection $f:\bar{A} \to S$ can be specified by: (i) choosing a countable $B \subseteq A$; (ii) sending $\bar{A}\setminus B$ to 0 ($f(\vec{O}) = 0$ since f is onto); and (iii) sending B to $S\setminus\{0\}$ in such a way that point-inverses are finite. This can be accomplished in κ^ω different ways, so

$$\kappa^\omega \leqq |C(\bar{A}, S)|.$$

Since we can extend **I**-valued maps from $\bar{A}$ to the full cube, we have

$$c \cdot \kappa^{\omega} \leqq |C(\mathbf{I}^{\kappa}, \mathbf{I})|.$$

To get the reverse inequality we invoke a theorem of R. Engelking (see [**11**, Theorem 4.9]) to the effect that every continuous f: $\mathbf{I}^{\kappa} \to \mathbf{I}$ depends on only countably many coordinates (i.e., there is a countable $J \subseteq \kappa$ and a continuous g: $\mathbf{I}^{J} \to \mathbf{I}$ such that $f = g \circ \pi_J$). This immediately gives us

$$C(\mathbf{I}^{\kappa}, \mathbf{I}) \leqq c \cdot \kappa^{\omega}.$$

3.8. COROLLARY. *There is a large κ-free separated, functionally complete* **I**-*lattice if and only if* $c \cdot \kappa^{\omega} > c \cdot \kappa$ (*i.e.*, $\kappa^{\omega} > \kappa > c$).

3.9. *Remark*. In the absence of questionable axioms of set theory, it is impossible to pin down those cardinals κ such that $\kappa^{\omega} > \kappa$. Of course if κ is of the form 2^{λ}, then κ does not have this property. On the other hand, if κ has countable cofinality then König's lemma (see [**13**]) tells us that $\kappa^{\omega} > \kappa$. If κ has uncountable cofinality then

$$\kappa^{\omega} = \kappa \cdot \sup\{\lambda^{\omega}:\lambda < \kappa\}.$$

Thus, assuming the Generalized Continuum Hypothesis,

$$\kappa^{\omega} \leqq \kappa \cdot \sup\{\lambda^{+}:\lambda < \kappa\} = \kappa.$$

For the remainder of this section, RCF denotes the class of unital rings $C(X) = C(X, \mathbf{R})$ for X an arbitrary topological space. It is well known (see [**8**]) that if X is a space then there is a realcompact Tichonov space X' such that $C(X) \cong C(X')$. (X' is obtained by identifying points of X which cannot be separated by members of $C(X)$; by suitably topologizing the set of equivalence classes; and then by applying the Hewitt realcompactification.)

Combining this fact with (3.5) and the Gel'fand-Kolmogorov duality, we have the following.

3.10. THEOREM. $R \in$ RCF *is κ-free if and only if there is a realcompact Tichonov space X and a coreflection map f:$X \to \mathbf{R}^{\kappa}$ such that $R \cong C(X)$. Consequently the cardinality of any κ-free ring $R \in$* RCF *lies between* $c \cdot \kappa^{\omega}$ *and* $\exp^2(\kappa \cdot \omega)$.

Proof. We need only justify the second assertion. Clearly if $C(X)$ is κ-free then

$$|C(X)| \leqq \exp^2(\kappa \cdot \omega)$$

since $|X| = |\mathbf{R}^{\kappa}| = \exp(\kappa \cdot \omega)$ (by (3.3)). Also, since $\mathbf{R}^{\kappa}$ is realcompact, $C(\mathbf{R}^{\kappa})$ is κ-free and its cardinality is $c \cdot \kappa^{\omega}$. (Use the argument in (3.7) to get $|C(\mathbf{I}^{\kappa})| \geqq c \cdot \kappa^{\omega}$.) Since $\mathbf{I}^{\kappa}$ is a retract of $\mathbf{R}^{\kappa}$, we get

$$|C(\mathbf{R}^{\kappa})| \geqq c \cdot \kappa^{\omega}.$$

On the other hand, Engelking's theorem also applies to continuous maps from $\mathbf{R}^\kappa$ to $\mathbf{R}$. Thus $|C(\mathbf{R}^\kappa)| = c \cdot \kappa^\omega$.)

3.11. *Question.* Given κ, is there always a large κ-free unital ring in RCF?

Although we cannot answer (3.11) completely, we can give an affirmative answer "for all practical purposes". In particular, from (3.9) and (3.10) we know that

$$|C(\mathbf{R}^\kappa)| = c \cdot \kappa^\omega;$$

and hence that $C(\mathbf{R}^\kappa)$ is large provided $0 \leqq \kappa < c$ or κ is of countable cofinality. In further pursuit of an answer, let us consider $C(D(\mathbf{R}^\kappa))$. This ring is of cardinality $\exp^2(\kappa \cdot \omega)$, so is certainly large whenever it is κ-free.

3.12. THEOREM. *The following are equivalent.*

(i) $D(\mathbf{R}^\kappa)$ *is realcompact.*

(ii) $C(D(\mathbf{R}^\kappa))$ *is* κ*-free.*

(iii) $C(D(\mathbf{R}^\kappa))$ *is* λ*-free for some* λ.

(iv) κ *is not Ulam-measurable.*

Proof. (i) $\Rightarrow$ (ii). This follows by (3.4 (i)) and (3.5).

(ii) $\Rightarrow$ (iii). This is immediate.

(i) $\Leftrightarrow$ (iv). This is well known [**8**].

(iii) $\Rightarrow$ (i). Let X be the Hewitt realcompactification of $D(\mathbf{R}^\kappa)$. Then $C(X) \cong C(D(\mathbf{R}^\kappa))$ is λ-free, so there is a coreflection map $f:X \rightarrow \mathbf{R}^\lambda$. Now $\mathbf{R}^\lambda$ carries a topological group structure, and is hence (point-) homogeneous. We show X is also homogeneous. For let $x, y \in X$ and let k be a homeomorphism on $\mathbf{R}^\lambda$ taking $f(x)$ to $f(y)$. Since f is a coreflection map, there is a unique $h:X \rightarrow X$ such that $f \circ h = k \circ f$. Then

$$f(h(x)) = k(f(x)) = f(y),$$

so $h(x) = y$ (since f is one-one). Also there is a unique $h':X \rightarrow X$ such that $f \circ j = k^{-1} \circ f$. Thus

$$f \circ h' \circ h = k^{-1} \circ f \circ h = k^{-1} \circ k \circ f = f;$$

whence $h' \circ h = \mathrm{id}_X$. Similarly $h \circ h' = \mathrm{id}_X$, so h is a homeomorphism. Since X has isolated points, we must have that X is discrete. But the realcompactification of a discrete space is never homogeneous, unless the discrete space is realcompact to begin with.

3.13. COROLLARY. *There is always a large* κ*-free unital ring in* RCF, *provided* κ *is either non-Ulam-measurable or of countable cofinality.*

Another issue which has direct bearing on (3.11) is the following.

3.14. *Question.* How many κ-free unital rings in RCF are there?

By (3.10) we need look only at Tichonov topologies $\mathscr{T}$ on the set $\mathbf{R}^\kappa$ such that: (i) $\mathscr{T}$ extends the usual product topology; (ii) $\mathscr{T}$ is realcompact; and (iii) $\mathscr{T}$ is "coreflective" (with respect to the product topology), i.e., any function $f:\mathbf{R}^\kappa \to \mathbf{R}^\kappa$ which pulls usual open sets back to $\mathscr{T}$-open sets also pulls $\mathscr{T}$-open sets back to $\mathscr{T}$-open sets. Now two of the main results of Comfort-Retta [**6**] are: (1) if $\mathscr{T}$ is a realcompact topology and $\mathscr{T}'$ is any topology with $\mathscr{T} \subseteq \mathscr{T}' \subseteq (\mathscr{T})_{\omega_1}$ (= the ω_1-modification of $\mathscr{T}$) then $\mathscr{T}'$ is realcompact; and (2) if $\mathscr{T}$ is a realcompact topology then so is $(\mathscr{T})_\mu$, where μ is the first Ulam-measurable cardinal (should one exist). Williams [**20**] went on to extend the first result; in particular he showed that if $\mathscr{T}$ is a realcompact topology, α is a non-Ulam-measurable cardinal, and $\mathscr{T}'$ is a Tichonov topology with $\mathscr{T} \subseteq \mathscr{T}' \subseteq (\mathscr{T})_{\alpha^+}$ and $\mathscr{T}' = (\mathscr{T}')_{\mathrm{cf}(\alpha)}$ (cf(α) is the cofinality of α) then $\mathscr{T}'$ is realcompact. Since the topologies $(\mathscr{T})_\alpha$ are all coreflections of $\mathscr{T}$, we can state the following.

3.15. THEOREM. *The unital ring $C((\mathbf{R}^\kappa)_\alpha)$ is κ-free for any $\alpha \leqq \mu$.*

Although (3.15) sheds important light on question (3.14), it falls short of providing a complete answer. When $\kappa \leqq \omega$, (3.15) gives us just $C(\mathbf{R}^\kappa)$ and $C(D(\mathbf{R}^\kappa))$. The Comfort-Retta theorem tells us that any coreflective Tichonov topology will give another κ-free ring, however we do not know whether any such topologies exist. (What one can easily check out is that if $\mathscr{T}$ is such a topology on $\mathbf{R}$, nondiscrete and distinct from the usual topology, then $\mathscr{T}$ is point-homogeneous; moreover any new $\mathscr{T}$-open set must inherit a dense ordering without endpoints from the usual ordering on $\mathbf{R}$.)

Presumably we could also extend (3.13) to cover the case $\kappa = \mu$ using the second Comfort-Retta result. While it is true that $C((\mathbf{R}^\mu)_\mu)$ is μ-free, we do not know whether it is large. (What complicates matters is that μ is strongly inaccessible.)

That said, let us return to smaller cardinals and consider $\kappa = \omega_1$.

3.16. PROPOSITION. *There are at least three ω_1-free unital rings in* RCF:$C(\mathbf{R}^{\omega_1})$, *of cardinality* c; $C((\mathbf{R}^{\omega_1})_{\omega_1})$, *of cardinality* $\exp(c)$; *and* $C(D(\mathbf{R}^{\omega_1}))$, *of cardinality* $\exp^2(\omega_1)$.

Proof. We need only show

$$|C((\mathbf{R}^{\omega_1})_{\omega_1})| = \exp(c).$$

Indeed 2^ω is closed in $\mathbf{R}^{\omega_1}$, hence $(2^\omega)_{\omega_1} = D(2^\omega)$ is closed discrete in $(\mathbf{R}^{\omega_1})_{\omega_1}$. Now $(\mathbf{R}^{\omega_1})_{\omega_1}$ is well known to be normal (even hereditarily ultra-paracompact), so every map

$$f:D(2^\omega) \to \mathbf{R}$$

extends to $(\mathbf{R}^{\omega_1})_{\omega_1}$. Thus

$$|C((\mathbf{R}^{\omega_1})_{\omega_1}| \geqq \exp(c).$$

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To continue, $\mathbf{R}^{\omega_1}$ has weight ω_1. Thus the weight of $(\mathbf{R}^{\omega_1})_{\omega_1}$ is $\leqq \omega_1^{\omega} = c$; hence

$$|C((\mathbf{R}^{\omega_1})_{\omega_1})| \leqq \exp(c).$$

We close this section by considering the question of when a ring $R \in$ RCF is κ-free for more than one cardinal κ. The following is obvious.

3.17. PROPOSITION. *Let κ be non-Ulam-measurable. Then $C(D(\mathbf{R}^{\kappa}))$ is λ-free if and only if $c^{\lambda} = c^{\kappa}$. In particular, $C(D(\mathbf{R}))$ is λ-free for $1 \leqq \lambda \leqq \omega$.*

By contrast we have the following result.

3.18. THEOREM. *$C(\mathbf{R}^{\kappa})$ is λ-free if and only if $\lambda = \kappa$.*

Proof. All we need establish is that there can be no continuous bijection $f:\mathbf{R}^{\kappa} \to \mathbf{R}^{\lambda}$ for $\kappa \neq \lambda$. We have two cases.

Case (i). $\kappa > \lambda$. If $\lambda > \omega$, f embeds $\mathbf{I}^{\kappa}$ into $\mathbf{R}^{\lambda}$, an impossibility for reasons of dimension. If $\lambda \geqq \omega$ then the dyadic space $2^{\kappa} \subseteq \mathbf{R}^{\kappa}$, which has weight κ, cannot possibly be embedded in a space of lower weight.

Case (ii). $\kappa < \lambda$. If $\lambda < \omega$ we can write $\mathbf{R}^{\kappa}$ as a countable union of the cubes $[-n, n]^{\kappa}$, $n = 1, 2, \dots$. Let A_n be the image of $[-n, n]^{\kappa}$ under f. Then

$$\mathbf{R}^{\lambda} = \bigcup_{n=1}^{\infty} A_n,$$

and it is easy to see that each A_n is nowhere dense. This contradicts the Baire category theorem. Now suppose $\lambda \geqq \omega$, and let $A \subseteq \mathbf{R}^{\lambda}$ be

$$\{\vec{x} \in 2^{\lambda}: x_{\xi} = 1 \text{ for exactly one } \xi < \lambda\}.$$

Then A is discrete of cardinality λ. But $f^{-1}[A] \subseteq \mathbf{R}^{\kappa}$ has a dense subset of cardinality $\kappa \cdot \omega < \lambda$, hence so does A. This cannot be.

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