DECOMPOSITION ALGEBRAS OF RIESZ OPERATORS

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Let H be a Hilbert space and let $\mathbf{B}$ denote the Banach algebra of all bounded linear operators on H with $\mathbf{K}$ denoting the closed ideal of compact operators in $\mathbf{B}$. If $T \in \mathbf{B}$, $\sigma(T)$ and $r(T)$ will denote the spectrum and spectral radius of T, respectively, and π the canonical mapping of $\mathbf{B}$ onto the Calkin algebra $\mathbf{B}/\mathbf{K}$.

$R \in \mathbf{B}$ is called a Riesz operator if $\pi(R)$ is a quasinilpotent element of the Calkin algebra. The second author [**9**] has proved that every Riesz operator $R = C + Q$ where C is compact normal, Q is quasinilpotent and $\sigma(R) = \sigma(C)$. It follows from a theorem of Ruston [**8**] that this decomposition characterises Riesz operators on Hilbert spaces and it is an open problem whether such a decomposition is possible in Banach spaces. Similar decompositions have occurred in the work of Gohberg and Krein [**3**, p. 17] and Stampfli [**7**, Lemma 6]. As there may be many different decompositions of a Riesz operator into the sum of a compact and a quasinilpotent operator we shall call the decomposition as carried out in [**9**] and described below a *West decomposition.* There can be more than one West decomposition of the same Riesz operator.

Gillespie and West [**2**] have given an example of a Riesz operator on a Hilbert space which cannot be decomposed in any manner so that the compact and quasinilpotent parts commute. From now on we consider a fixed West decomposition $R = C + Q$. Chui, Smith and Ward [**1**] have proved that $N = CQ - QC$ is quasinilpotent and R. E. Harte has remarked that $p(C, Q)N$ is quasinilpotent for any polynomial p in two non-commuting variables. This prompted an investigation of the (in general non-unital) closed subalgebra of $\mathbf{B}$ generated by C and Q which we call the *decomposition algebra of R* associated with C and Q and denote by $\mathbf{D}(C, Q)$ or, when there is no possibility of ambiguity, by $\mathbf{D}$. We show that Q and N are both contained in $\operatorname{Rad} \mathbf{D}$ the radical of $\mathbf{D}$ which consists of all the quasinilpotent elements of $\mathbf{D}$, that $\mathbf{D}$ is commutative modulo its radical, that $\mathbf{D}$ consists entirely of Riesz operators and thus is a Riesz algebra in the sense of Smyth [**6**], that the quotient algebra $\mathbf{D}/\operatorname{Rad} \mathbf{D}$ is invariant for all West decompositions of R and that $\mathbf{D}$ is the direct sum of $\operatorname{Rad} \mathbf{D}$ and the closed subalgebra generated by C.

Smyth [**5**] has successfully put Riesz theory in an algebraic setting and has obtained a perfect analogue of the West decomposition for Riesz elements of C*-algebras. Legg [**4**] has shown that the result of Chui, Smith and Ward holds in this context; similarly all the results proved here extend easily to the C*-algebra setting.

We shall need the facts that the spectral radius is continuous relative to the norm topology for operators in a commutative algebra or if the spectrum of the limit operator is totally disconnected (as is the case for Riesz operators). Also if $A \in \mathbf{B}$ can be written as

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the operator matrix, relative to some decomposition of H,

$$A=\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

then $\sigma(A)\subseteq\sigma(A_{11})\cup\sigma(A_{22})$.

Let R be a Riesz operator on H. To avoid trivialities we assume that H is infinite dimensional and that $\sigma(R)=\{\lambda_j\}_1^\infty\cup\{0\}$, where $\{\lambda_j\}_1^\infty$ is the set of non-zero eigenvalues of R. It is well known that $\lambda_j\to 0(j\to\infty)$. Let P_j denote the spectral projection for R corresponding to the eigenvalue λ_j given by the formula

$$P_j=\frac{1}{2\pi i}\int_{\Gamma_j}(zI-R)^{-1}\,dz,$$

where Γ_j is a circle centre λ_j containing no other eigenvalue of R in its interior or on its boundary. Put $S_k=\sum_{j=1}^{k}P_j$, and F_k for the orthogonal projection whose range is the same as that of S_k. Write $E_j=F_j-F_{j-1}$ $(j=1,2,\ldots)$ with $F_0=0$. Then $C=\sum_{j=1}^{\infty}\lambda_jE_j$ is the compact normal operator in the West decomposition of R (relative to this arrangement of the eigenvalues) and the essence of the proof lies in showing that $Q=R-C$ is quasinilpotent. If $C_k=\sum_{j=1}^{k}\lambda_jE_j$ and we consider H as the orthogonal direct sum of the range and null-space of the projection F_k, C_k and Q may be written in matrix form

$$C_k=\left[\begin{array}{cccc|c} \lambda_1 & & & 0 & 0 \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ 0 & & & \lambda_k & \\ \hline & & 0 & & 0 \end{array}\right],\qquad Q=\left[\begin{array}{cccc|c} 0 & & & * & * \\ & 0 & & & \\ & & 0 & & \\ & & 0 & 0 & \\ \hline & & 0 & & * \end{array}\right],$$

where in the C_k matrix the eigenvalues are repeated according to multiplicity. Note that we have essentially chosen a basis of the range of each of the E_j's (up to $j=k$) which upper-triangularises the matrices for C, R and Q.

LEMMA 1 (Harte). *$p(C,Q)N$ is quasinilpotent, where $p(C,Q)$ is any polynomial in C and Q.*

Proof. Put $N_k=C_kQ-QC_k$. Then $C_k\to C$ $(k\to\infty)$ hence

$$p(C_k,Q)N_k\to p(C,Q)N \qquad (k\to\infty).$$

N is compact and therefore

$$r\{p(C_k,Q)N_k\}\to r\{p(C,Q)N\} \qquad (k\to\infty).$$

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It suffices to show that $p(C_k, Q)N_k$ is quasinilpotent for each k. Now

$$N_k = \left[\begin{array}{cc|c} \begin{matrix} 0 & & * \\ & 0 & \\ & 0 & 0 \end{matrix} & & * \\ \hline & 0 & 0 \end{array}\right], \qquad p(C_k, Q) = \left[\begin{array}{cc|c} \begin{matrix} \mu_1 & & * \\ & \mu_2 & \\ 0 & & \end{matrix} & & * \\ \hline & 0 & * \end{array}\right],$$

$$p(C_k, Q)N_k = \left[\begin{array}{cc|c} \begin{matrix} 0 & & \\ & 0 & * \\ & 0 & 0 \end{matrix} & & * \\ \hline & 0 & 0 \end{array}\right]$$

and the result follows as the matrix is nilpotent.

PROPOSITION 2. $N \in \operatorname{Rad} \mathbf{D}$.

Proof. It suffices to show that $r(TN) = 0$ for each $T \in \mathbf{D}$. Now there exists a sequence of polynomials $p_n(C, Q)$ converging to T as $n \to \infty$. Thus

$$p_n(C, Q)N \to TN \qquad (n \to \infty)$$

and N is compact; hence

$$r\{p_n(C, Q)N\} \to r(TN) \qquad (n \to \infty).$$

The result follows from Lemma 1.

COROLLARY 3. $\mathbf{D}' = \mathbf{D}/\operatorname{Rad} \mathbf{D}$ *is commutative.*

PROPOSITION 4. $\operatorname{Rad} \mathbf{D}$ *consists of the quasinilpotent elements of* $\mathbf{D}$.

Proof. Every operator in $\operatorname{Rad} \mathbf{D}$ is quasinilpotent. By Corollary 3 and the results of **[10]** the spectral radius is subadditive and submultiplicative on $\mathbf{D}$; hence the quasinilpotents form an ideal of $\mathbf{D}$ which is therefore contained in $\operatorname{Rad} \mathbf{D}$.

COROLLARY 5. $Q \in \operatorname{Rad} \mathbf{D}$.

PROPOSITION 6. *Every element of* $\mathbf{D}$ *is a Riesz operator.*

Proof. If $T \in \mathbf{D}$ there is a sequence of polynomials $p_n(C, Q)$ without constant term converging to T. Then

$$\pi(p_n(C, Q)) = p_n(0, \pi(Q)) \to \pi(T) \qquad (n \to \infty).$$

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$\pi(Q)$ is quasinilpotent and $p_n(0, \pi(Q))$ is a polynomial in $\pi(Q)$ without a constant term; thus $\pi(T)$ is contained in the commutative subalgebra generated by $\pi(Q)$; hence $r\{p_n(0, \pi(Q))\} \to r(\pi(T))$ $(n \to \infty)$ and $r(\pi(T)) = 0$.

PROPOSITION 7. $\mathbf{D}'$ *is isometrically isomorphic to the closed subalgebra* $\mathbf{A}(C)$ *of* $\mathbf{B}$ *generated by* C, *and is thus independent of the particular West decomposition of* R.

Proof. $\mathbf{D}'$ is a commutative semi-simple Banach algebra generated by $C' = C + \operatorname{Rad} \mathbf{D}$. Now $\sigma_{\mathbf{D}'}(T') = \sigma_{\mathbf{D}}(T)$ $(T \in \mathbf{D})$, and so $r(T') = r(T) = \|T\|$ if $T \in \mathbf{A}(C)$ (as C is normal); hence $\|T\| = \|T'\|$. Thus the map $\mathbf{A}(C) \to \mathbf{D}' : T \to T'$ is an isometric isomorphism. If $R = C_1 + Q_1$ is a second West decomposition, then there is a unitary operator U for which $C = UC_1U^{-1}$; thus $\mathbf{A}(C)$ and $\mathbf{A}(C_1)$ are isometrically isomorphic.

PROPOSITION 8. $\mathbf{D} = \mathbf{A}(C) \oplus \operatorname{Rad} \mathbf{D}$.

Proof (i). If $T \in \mathbf{A}(C) \cap \operatorname{Rad} \mathbf{D}$, then T is both normal and quasinilpotent and hence zero.

(ii) If $T \in \mathbf{D}$ there exists a sequence of polynomials $p_n(C, Q)$ converging to T. Thus

$$p_n(C, Q)' = p_n(C', 0) \to T' \qquad (n \to \infty).$$

By Proposition 7, as $n \to \infty$, $p_n(C, 0) \to S \in \mathbf{A}(C)$ and $S' = T'$. Thus, as $n \to \infty$, $p_n(C, Q) - p_n(C, 0) \to S - T \in \operatorname{Rad} \mathbf{D}$ and hence $\mathbf{D} = \mathbf{A}(C) + \operatorname{Rad} \mathbf{D}$, completing the proof.

These results fail to hold for decompositions of Riesz operators which are not West decompositions. If

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

then C is compact normal, and Q is quasinilpotent but they have no proper invariant subspace in common. Thus the identity representation of the algebra generated by C and Q is irreducible and hence the algebra is semi-simple.

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