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SQUARE OF BROWNIAN MOTION

To Professor Kiyoshi Ito on the occasion of his 60th birthday

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1. Introduction.

Let X_t be a stochastic process and Y_t be its square process. The present note is concerned with the solution of the equation $X_t^2 = Y_t$ assuming Y_t is given. In [4], F. A. Grünbaum proved that certain statistics of Y_t are enough to determine those of X_t when it is a centered, nonvanishing, Gaussian process with continuous correlation function. In connection with this result, we are interested in sample function-wise inference, though it is far from generalities. A glance of the equation $X_t^2 = Y_t$ shows that the difficulty is related how to pick up a sign of $\pm\sqrt{Y_t}$. Thus if we know that X_t has nice sample process such as the zero crossings are finite, no tangencies, in any finite time interval, then observations of these statistics will make it sure to find out sample functions of X_t from those of Y_t (see [2]). The purpose of this note is to consider the above problem from this point of view.

In section 2, we shall construct a new standard random walk as an explicit functional of the square process S_n^2 of the given standard random walk S_n availing of sample functions properties, reflection at $x = 0$, of $\sqrt{S_n^2}$. In section 3, we shall consider the same problem to the previous section for Brownian motion. In this case, we make use of the local time at $x = 0$ of reflecting Brownian motion on $[0, \infty)$ to get a new Brownian motion from an "inverse map" of the map $B_t \to B_t^2$.

As to the terminology about Markov processes we refer to [1] and [6].

2. Random walk.

Before to state theorems, we prepare two lemmas.

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LEMMA 1. *Let* $[X_n; n \geqq 0]$ *be a Markov chain on all nonnegative integers* $\{0, 1, 2, \cdots\}$ *with the transition probability* $p_{00} = p_{01} = \frac{1}{2}$ *and* $p_{i,i\pm 1} = \frac{1}{2}$ $(i \geqq 1)$. *Then* $u(X_n - X_{n-1})$ $(n \geqq 1)$ *are independent with common distribution* $P_0\{u(X_n - X_{n-1}) = \pm 1\} = \frac{1}{2}$. *Where* $u(x) = x$ *or* $u(x) = -1$ *according as* $x = \pm 1$ *or* $x = 0$.

Proof. 1°. Since

$$(1) \qquad P_i\{u(X_1 - X_0) = a\} = \tfrac{1}{2} \qquad \text{for } a = \pm 1 \text{ and } i \geqq 0\,,$$

we see

$$(2) \qquad \begin{aligned} P_0\{u(X_n - X_{n-1}) = a\} &= E_0[P_{X_{n-1}}\{u(X_1 - X_0) = a\}] \\ &= \sum_{k \geqq 0} p_{0k}^{(n-1)} P_k\{u(X_1 - X_0) = a\} \\ &= \tfrac{1}{2} \sum_{k \geqq 0} p_{0k}^{(n-1)} = \tfrac{1}{2}\,.^{1)} \end{aligned}$$

That is, each $u(X_n - X_{n-1})$ has the same distribution with the fair coin-tossing game.

Next we shall prove that the system $[u(X_n - X_{n-1}); n \geqq 1]$ is independent with respect to P_0.

2°. First we compute the probability

$$p_n = P_1\{u(X_1 - X_0) = a_1, \cdots, u(X_n - X_{n-1}) = a_n\}\,, \qquad (a_j = \pm 1)\,.$$

Let $\tau = \mathrm{Min}\{k \geqq 1 \mid X_k = 0\}$ be the hitting time to $x = 0$. Since $P_k\{\tau < \infty\} = 1$, we have

$$(3) \qquad p_n = \sum_{\ell=1}^{\infty} P_1\{u(X_1 - X_0) = a_1, \cdots, u(X_n - X_{n-1}) = a_n, \ \tau = \ell\}\,.$$

For given a_j's, consider the following two cases;

Case 1. There exists a k such that

$$(4) \qquad \begin{aligned} &1 \leqq k \leqq n, \ 1 + a_1 + \cdots + a_j \geqq 1\,, \\ &\qquad (j = 1, \cdots, k-1) \text{ and } 1 + a_1 + \cdots + a_k = 0\,, \end{aligned}$$

and

Case 2. $\qquad 1 + a_1 + \cdots + a_j \geqq 1, \ (j = 1, 2, \cdots, n)\,.$

In the case 1, using the strong Markov property, we have

1) $p_{ij}^{(n)}$ denotes the transition probability in n steps.

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$$\begin{aligned} p_n &= P_1\{u(X_1 - X_0) = a_1, \cdots, u(X_n - X_{n-1}) = a_n,\ \tau = k\} \\ &= \frac{1}{2} P_{1+a_1}\{u(X_1 - X_0) = a_2, \cdots, u(X_{n-1} - X_{n-2}) = a_n,\ \tau = k-1\} \\ &= \cdots \\ &= \frac{1}{2^k} P_{1+a_1+\cdots+a_k}\{u(X_1 - X_0) = a_{k+1}, \cdots, u(X_{n-k} - X_{n-k-1}) = a_n\}, \end{aligned}$$

so that (4) implies

$$p_n = \frac{1}{2^k} P_0\{u(X_1 - X_0) = a_{k+1}, \cdots, u(X_{n-k} - X_{n-k-1}) = a_n\}, \quad \text{with } X_{-1} = 0. \tag{5}$$

In the case 2, by the same computations as above, we get

$$\begin{aligned} p_n &= \sum_{\ell=n+1}^{\infty} P_1\{u(X_1 - X_0) = a_1, \cdots, u(X_n - X_{n-1}) = a_n,\ \tau = \ell\} \\ &= \frac{1}{2^n} \sum_{\ell=n+1}^{\infty} P_{1+a_1+\cdots+a_n}\{\tau = \ell - n\} \\ &= \frac{1}{2^n}. \end{aligned} \tag{6}$$

3°. Now we shall prove that the probability

$$q_n = P_0\{u(X_1 - X_0) = a_1, \cdots, u(X_n - X_{n-1}) = a_n\} \tag{7}$$

is equal to $1/2^n$ by the induction for n together with (5) and (6). Since $q_1 = 1$ by (1), assume that $q_k = 1/2^k$ for any $k \leqq n-1$ and $\{a_1, \cdots, a_k\}$ with $a_j = \pm 1$. Then, it holds that

$$\begin{aligned} q_n &= E_0[P_{X_1}\{u(X_1 - X_0) = a_2, \cdots, u(X_{n-1} - X_{n-2}) = a_n\}; u(X_1 - X_0) = a_1]^{2)} \\ &= P_0\{X_1 = 0, u(X_1 - X_0) = a_1\} P_0\{u(X_1 - X_0) = a_2, \cdots, u(X_{n-1} - X_{n-2}) = a_n\} \\ &\quad + P_0\{X_1 = 1, u(X_1 - X_0) = a_1\} P_1\{u(X_1 - X_0) = a_2, \cdots, u(X_{n-1} - X_{n-2}) \\ &\qquad = a_n\}. \end{aligned}$$

Therefore, by the assumption of the induction together with (1), (5) and (6), we get

$$\begin{aligned} q_n &= \frac{1}{2^{n-1}} P_0\{X_1 = 0,\ u(X_1 - X_0) = a_1\} + \frac{1}{2^{n-1}} P_0\{X_1 = 1,\ u(X_1 - X_0) = a_1\} \\ &= \frac{1}{2^{n-1}} P_0\{u(X_1 - X_0) = a_1\} = \frac{1}{2^n}. \end{aligned}$$

2) $E_x[f; A] = \int_A f(w) P_x(dw)$.

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That is

$$q_n = \prod_{k=1}^{n} P_0\{u(X_k - X_{k-1}) = a_k\} .$$

This proves the lemma.

Consider a Markov chain on $\{0, 1, 2, \cdots\}$ determined by

$$(8) \qquad p_{01} = 1 \quad \text{and} \quad p_{j,j\pm 1} = \tfrac{1}{2} \ (j \geqq 1)$$

Let

$$(9) \qquad \begin{aligned} A_n &= \sum_{k=1}^{n} \chi_N(X_k) , \quad (n \geqq 1) , \quad A_0 = 0 ,^{3)} \\ A_n^{-1} &= \mathrm{Min}\,\{k \geqq 1 \mid A_k \geqq n\} , \quad A_0^{-1} = 0 \end{aligned}$$

be the additive functional of X_n and its inverse respectively.

Then we have the following lemma.

LEMMA 2. *The time changed process* $\hat{X}_n = X(A_n^{-1})$ *is the Markov chain on* $\{0, 1, 2, \cdots\}$ *with the transition probabilitiy*

$$(10) \qquad \hat{p}_{01} = 1, \ \hat{p}_{10} = 0, \ \hat{p}_{11} = \hat{p}_{12} = \tfrac{1}{2} \quad \textit{and} \quad \hat{p}_{i,i\pm 1} = \tfrac{1}{2} \ (i \geqq 2) .$$

Proof. Since it is known that the time changed process $\hat{X}_n$ is a Markov chain, it remains only to check the relations of (10). For this, suppose $X_0 = 1$, then

$$A_1^{-1} = 1 \quad \text{iff} \ X_1 = 2 \quad \text{and} \quad A_1^{-1} = 2 \quad \text{iff} \ X_1 = 0 .$$

By definition of A_n, this implies

$$\hat{X}_1 = X(A_1^{-1}) = X_1 \ \text{or} \ X_2 \ \text{according as} \ X_1 = 2 \ \text{or} \ X_1 = 0 ,$$

so that we have

$$\begin{aligned} P_1\{\hat{X}_1 = 1\} &= P_1\{X_1 = 0, \ X_2 = 1\} = \tfrac{1}{2} , \\ P_1\{\hat{X}_1 = 2\} &= P_1\{X_1 = 2\} = \tfrac{1}{2} . \end{aligned}$$

The rests of (10) are obvious.

THEOREM 1. *Let* $[S_n\,; n \geqq 0]$ *be the standard random walk on all integers* $\boldsymbol{Z}$. *Then we can construct a new standard random walk from the square process* $[S_n^2\,; n \geqq 0]$.

3) χ_N denotes the indicator function of the set $N = \{1, 2, 3, \cdots\}$

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Proof. Let X_n be the root process of S_n^2: $X_n = \sqrt{S_n^2} = |S_n|$. Then $[X_n; n \geqq 0]$ is a Markov chain with the transition matrix (8). So, by lemma 2, the time changed process $\hat{X}_n$ of X_n by the additive functional A_n of (9) is a Markov chain on $\{0, 1, 2, 3, \cdots\}$ with the transition matrix (10). Because of $\hat{p}_{01} = 1$, $\hat{p}_{10} = 0$, we can apply lemma 1 to $[\hat{X}_n - 1; n \geqq 0]$ to get the new standard random walk

$$\hat{S}_n = \sum_1^n u(\hat{X}_k - \hat{X}_{k-1}) .$$

This proves the theorem.

3. Brownian motion.

THEOREM 2. *Let $[B_t; t \geqq 0]$ be the standard Brownian motion on real line $\boldsymbol{R}$. Then we can construct a new standard Brownian motion from the square process $[B_t^2; t \geqq 0]$.*

In fact, it is known that the root X_t of B_t^2 is the reflecting Brownian motion on $[0, \infty)$ and the local time T_t of X_t at $x = 0$ exists:

$$T_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \operatorname{meas} \{s \mid s < t,\ X_s < \varepsilon\} . \tag{11}$$

So, if we set

$$\hat{B}_t = X_t - T_t \tag{12}$$

then it will be seen that $[\hat{B}_t; P_a]$ $(a \geq 0)$ is a new standard Brownian motion starting from a.

To prove Theorem 2, suppose we are given a Brownian motion $\hat{B}_t$. Then it is known that, considering (12) as a stochastic equation—the Skorohod's equation—with unknown X_t and T_t, it has the unique non-anticipating solution X_t, the reflecting barrier Brownian motion and T_t, the local time of X_t (see [5], [7]). Therefore, Theorem 2 is easily derived by virtue of the above results.

Finally, we shall give a direct proof of Theorem 2 applying a theorem of Doob ([3], Theorem 11.9). The proof will be given in several steps as follows.

1°. As the first step, bring the well known formula ([5])

$$P_0\{X_t \varepsilon dx, T_t \varepsilon dy\} = \left(\frac{2}{\pi t^3}\right)^{1/2} \exp\{-(x+y)^2/2t\}(x+y)dxdy, x, y > 0 , \tag{13}$$

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to obtain

$$(14)\qquad P_0\{\hat{B}_t \varepsilon da\} = \left(\frac{1}{2\pi t}\right)^{1/2} \exp\{-a^2/2t\}da\ , \qquad a \in R\ ,$$

$$(15)\qquad P_0\{T_t \varepsilon dy\} = \left(\frac{2}{\pi t}\right)^{1/2} \exp\{-y^2/2t\}dy\ , \qquad y > 0\ ,$$

$$(16)\qquad \begin{cases} E_a(X_t) = \left(\dfrac{1}{2\pi t}\right)^{1/2}\left[2t \exp(-a^2/2t) + 2a\displaystyle\int_0^a \exp(-u^2/2t)du\right]\ , \\ E_a(X_t^2) = a^2 + t\ . \end{cases}$$

Moreover, we need the followings;

$$(17)\qquad \begin{cases} E_a(e^{-p\sigma}) = e^{-\sqrt{2p}a} \qquad a > 0,\ \sigma = \text{Min}\{t \mid X_t = 0\}\ , \\ P_a\{\sigma \varepsilon dt\} = \left(\dfrac{a^2}{2\pi t^3}\right)^{1/2} \exp\{-a^2/2t\}dt\ . \end{cases}$$

2°. With these formulas, we shall show that

$$(18)\qquad E_a(\hat{B}_t) = a\ , \qquad a > 0\ ,$$

$$(19)\qquad E_a(\hat{B}_t^2) = a^2 + t\ , \qquad a > 0\ ,$$

(20) $[\hat{B}_t, F_t, P_a]$ is a martingale with $E_a\{(\hat{B}_t - \hat{B}_s)^2 \mid F_s\} = t - s\ , \qquad t > s\ ,$

where $F_t = \sigma\{X_s \mid s \leqq t\}$ denotes the σ-algebra generated by $X_s, s \leqq t$.
Proof of (18). Since

$$T_t(w) = \begin{cases} 0 & , \quad t \leq \sigma(w)^{4)} \\ T_{t-\sigma(w)}(\theta_\sigma w)\ , & \quad t \geq \sigma(w) \end{cases}$$

and

$$P_a\{\sigma \varepsilon ds, T_{t-s} \varepsilon dy\} = P_a\{\sigma \varepsilon ds\} P_0\{T_{t-s} \varepsilon dy\}\ ,$$

we see that

$$E_a(\hat{B}_t) = E_a(X_t - T_t) = E_a(X_t) - E_a(T_t\,;\, t > \sigma)$$
$$= E_a(X_t) - \int_0^t E_0(T_{t-s}) P_0(\sigma \varepsilon ds)\ ,$$
$$\int_0^\infty e^{-pt} E_a(\hat{B}_t)dt = \int_0^\infty e^{-pt} E_a(X_t)dt - \int_0^\infty e^{-pt} E_0(T_t)dt \int_0^\infty e^{-pt} P_0\{\sigma \varepsilon ds\}\ .$$

So, by the formulas in step 1°, the right hand side becomes

4) θ_t denotes the shift operator on the basic space $W = \{w\}$.

$$\frac{a}{p}\left(1+\frac{1}{\sqrt{2p}a}\right)e^{-\sqrt{2p}a}+\frac{a}{p}(1-e^{-\sqrt{2p}a})-\frac{1}{\sqrt{2p^3}}e^{-\sqrt{2p}a}$$
$$=a/p\,.$$

Consequently, we get $E_a(\hat{B}_t)=a$.

Proof of (19). Observe that

$$\int_0^\infty e^{-pt}E_a(X_tT_t)dt=\int_0^\infty e^{-pt}E_a(X_tT_t\,;\,t>\sigma)dt$$
$$=\int_0^\infty e^{-pt}dtE_a(X_{t-\sigma}(\theta_\sigma w)T_{t-\sigma}(\theta_\sigma w)\,;\,t>\sigma)dt$$
$$=\int_0^\infty e^{-pt}dt\left[\iint_{x,y>0}xy\int_0^t P_0\{X_{t-s}\varepsilon dx,\,T_{t-s}\varepsilon dy\}P_a\{\sigma\varepsilon ds\}\right]$$
$$=\int_0^\infty e^{-pt}dt\left[\iint_{x,y>0}xy(x+y)dxdy\int_0^t\frac{a}{\sqrt{2\pi s^3}}e^{-a^2/2s^2}\frac{2}{\sqrt{2\pi(t-s)^3}}\times e^{-(x+y)^2/2(t-s)}ds\right]$$
$$=2e^{-\sqrt{2p}a}\iint_{x,y>0}xye^{-\sqrt{2p}(x+y)}dxdy$$
$$=e^{-\sqrt{2p}a}/2p^2\,,$$

and

$$\int_0^\infty e^{-pt}E_a(T_t^2)dt=e^{-\sqrt{2p}a}/p^2$$

imply

$$\int_0^\infty e^{-pt}E_a(\hat{B}_t^2)dt=\int_0^\infty e^{-pt}[E_a(X_t^2)-2E_a(X_tT_t)+E_a(T_t^2)]dt$$
$$=a^2/p+1/p-2e^{-\sqrt{2p}a}/2p^2+e^{-\sqrt{2p}a}/p^2$$
$$=a^2/p+1/p$$

so that $E_a(\hat{B}_t^2)=a^2+t$.

Proof of (20). Noting (18), we get

$$E_a[\hat{B}_{t+s}\,|\,\boldsymbol{F}_s]=E_a[X_{t+s}-T_{t+s}\,|\,\boldsymbol{F}_s]$$
$$=E_a[X_t(\theta_s w)-T_s(w)-T_t(\theta_s w)\,|\,\boldsymbol{F}_s]$$
$$=E_{X_s}[X_t-T_t]-T_s=E_{X_s}(\hat{B}_t)-T_s$$
$$=X_s-T_s=\hat{B}_s\,.$$

Thus we know that $\hat{B}_t$ is a martingale. Next, to show the latter part of (20), it is enough to use (18) and (19) at before the last step in the following computations;

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$$
\begin{aligned}
E_a[(\hat{B}_{t+s} - \hat{B}_s)^2 | \boldsymbol{F}_s] \\
&= E_a[((X_{t+s} - T_{t+s}) - (X_s - T_s))^2 | \boldsymbol{F}_s] \\
&= E_a[(X_t(\theta_s w) - T_t(\theta_s w) - X_s)^2 | \boldsymbol{F}_s] \\
&= E_{X_s}(X_t - T_t)^2 - 2X_s E_{X_s}(X_t - T_t) + X_s^2 \\
&= E_{X_s}(\hat{B}_t^2) - 2X_s E_{X_s}(\hat{B}_t) + X_s^2 \\
&= X_s^2 + t - 2X_s^2 + X_s^2 \\
&= t\,.
\end{aligned}
$$

Therefore, $[\hat{B}_t : P_a]$ $(a \geq 0)$ is a Brownian motion starting from a by a Doob's theorem [3]. Since it is easily seen $[-\hat{B}_t : P_a]$ $(a < 0)$ is a Brownian motion with $-\hat{B}_0 = a$, we conclude the proof of the theorem.

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