BULL. AUSTRAL. MATH. SOC. MOS 60E05
VOL. 6 (1972), 1-9.

An inequality for characteristic functions

C.R. Heathcote and J.W. Pitman

The paper is concerned with an extension of the inequality $1 - u(2^n t) \le 4^n[1-u(t)]$ for $u(t)$ the real part of a characteristic function. The main result is that the inequality in fact holds for all positive integer n and not only powers of 2. Certain consequences are deduced and a brief discussion is given of the circumstances under which equality holds.

Suppose the random variable X has characteristic function $Ee^{itX} = \psi(t)$ with real and imaginary parts respectively

$$u(t) = E\cos tX \ , \quad v(t) = E\sin tX \ .$$

For all real t and $n = 1, 2, \ldots$, the inequalities

$$(1) \qquad 1 - u(2^n t) \le 4^n[1-u(t)] \ ,$$

(see, for example, Lukacs [4], page 69, *et seq.* for applications), and

$$(2) \qquad 2u^2(t) \le 1 + u(2t) \ ,$$

(Feller [2], page 527) are well known. Proofs use either properties of trigonometric functions or standard inequalities. (1) in particular is used frequently.

In this note we prove a generalization of (2) and show that an inequality like (1) holds for all positive integers n and not only powers

Received 30 July 1971. The proof given for the lemma was supplied by Professor E.J.G. Pitman. This paper was prepared while the second author held a CSIRO Postgraduate Studentship.

of 2 . We establish a number of related inequalities involving $u(t)$, $v(t)$ and $|\psi(t)|$, and conclude by stating a result concerning conditions for equality in (2) (the proof of this and related propositions will be given elsewhere).

Our argument is based essentially on the family of random variables

$$Z(t) = \cos tX + i\sin tX \ , \quad -\infty < t < \infty \ ,$$

whose expectation is the characteristic function $\psi(t)$. We observe that properties of $Z(t)$ other than its expected value may also be useful. For example if we use the identity $\cos^2\theta = (1+\cos 2\theta)/2$ we find that the variance of the real part of $Z(t)$ is

$$\operatorname{var}(\cos tX) = \{1+u(2t)-2u^2(t)\}/2 \ .$$

Since a variance cannot be negative we have an almost trivial proof of (2). We can recast (2) as

$$\begin{aligned} 1 - u(2t) &\le 2[1-u^2(t)] \\ &= 2[1-u(t)][1+u(t)] \le 4[1-u(t)] \ , \end{aligned}$$

and this gives (1) by iteration. Similarly

$$\operatorname{var}(\sin tX) = \{1-u(2t)-2v^2(t)\}/2 \ ,$$

and hence

$$2v^2(t) \le 1 - u(2t) \ . \tag{3}$$

More generally, for any function g which is convex on the interval $[-1, 1]$, Jensen's inequality gives

$$g[u(t)] = g(E\cos tX) \le Eg(\cos tX)$$

$$g[v(t)] = g(E\sin tX) \le Eg(\sin tX)$$

from which a variety of inequalities involving $u(t)$ and $v(t)$ can be obtained by taking functions g for which $Eg(\cos tX)$ or $Eg(\sin tX)$ can be expressed in terms of $\psi(t)$. Thus (2) and (3) arise when $g(x) = x^2$.

We use the above reasoning to extablish our main result which is the following:

THEOREM. *If* $u(t)$ *is the real part of a characteristic function then*

https://doi.org/10.1017/S0004972700044191 Published online by Cambridge University Press

for all real t *and* $n = 1, 2, \ldots$,

$$(4) \qquad 1 - u(nt) \leq n\left[1-u^n(t)\right]$$

$$(5) \qquad \leq n^2[1-u(t)] .$$

REMARK. The restriction of (4) and (5) to an integral n cannot generally be relaxed. For example, if $u(t) = \cos t$ and $t = 2\pi$, the right hand sides of both (4) and (5) are identically zero. However, the left hand side will be positive for any non-integral n , thus reversing the inequality.

To prove the theorem we will require a trigonometric inequality which is (4) for $u(t) = \cos t$.

LEMMA. *For all real* x *and* $n = 1, 2, \ldots$,

$$(6) \qquad 1 - \cos nx \leq n\left(1-\cos^n x\right) ,$$

or equivalently

$$(7) \qquad \cos^n x \leq (n-1+\cos nx)/n .$$

Proof of the Lemma. For all real x and $n = 0, 1, \ldots$, the inequality

$$(8) \qquad \left|\frac{\sin nx}{\sin x}\right| \leq n$$

is easily proved by induction. We use (8) to show that

$$(9) \qquad f(x) = n\cos^n x - \cos nx \leq n - 1$$

for all x and $n = 1, 2, \ldots$, and this gives (6) . We have that

$$f'(x) = - n^2\cos^{n-1}x\sin x + n\sin nx ,$$

so that at a maximum of $f(x)$ either

$$(10) \qquad \sin x = 0$$

or

$$(11) \qquad \sin x \neq 0 \text{ and } n\cos^{n-1}x = \frac{\sin nx}{\sin x} .$$

It is easy to check that (10) implies (9). When (11) is true,

https://doi.org/10.1017/S0004972700044191 Published online by Cambridge University Press

$$\begin{aligned} f(x) &= \frac{\cos x \sin nx}{\sin x} - \cos nx \\ &= \frac{\cos x \sin nx - \sin x \cos nx}{\sin x} \\ &= \frac{\sin(n-1)x}{\sin x} \le n - 1 \quad \text{by (8).} \end{aligned}$$

Proof of the Theorem. Suppose first that n is an even integer. The function x^n is then convex on $[-1, 1]$ and from Jensen's inequality

$$\begin{aligned} u^n(t) &= (E\cos tX)^n \le E\cos^n tX \\ &\le E[(n-1+\cos ntX)/n] \quad \text{by (7)} \\ &= [n-1+u(nt)]/n , \end{aligned}$$

that is, $1 - u(nt) \le n[1-u^n(t)]$, and we have proved (4) for even n .

If n is odd, x^n is no longer a convex function on $[-1, 1]$. However, (4) can be established using what is essentially the method used to prove Jensen's inequality itself.

The theorem is vacuously true if $n = 1$, so let us take n odd and $n \ge 3$. Further, if $n(1-u^n(t)) \ge 2$ the inequality (4) is trivial and we therefore assume

$$n - 2 < nu^n(t) .$$

Then certainly

$$(12) \qquad 1 \le nu^{n-1}(t) \quad \text{and} \quad 0 < u(t) .$$

But, if $1 \le na^{n-1}$ and $0 < a$, it is easily seen that the graph of the function x^n on $[-1, 1]$ lies on or above the line tangent to the curve x^n at (a, a^n) , that is

$$x^n \ge a^n + (x-a)na^{n-1} , \quad -1 \le x \le 1 .$$

Because of (12) we can set $x = \cos tX$, $a = u(t)$ to obtain

$$\cos^n tX > u^n(t) + [\cos tX-u(t)]nu^{n-1}(t) .$$

Taking expectations yields

$$E\cos^n tX \geq u^n(t) .$$

But this is precisely what is required to complete the proof of (4) by an application of (7), as in the even case.

Finally, we deduce (5) from (4) by using the fact that for $|x| \leq 1$,

(13) $$1 - x^n \leq n(1-x) .$$

We note that for $n = 2$, (4) reduces to (2), and that (1) is a special case of (5) for powers of 2 .

COROLLARY 1. *If* $\psi(t)$ *is a characteristic function, then for all real* t *and* $n = 1, 2, \ldots$

(14) $$1 - |\psi(nt)| \leq n\left(1-|\psi(t)|^n\right) \leq n^2\left(1-|\psi(t)|\right) .$$

Proof. Because of (13) we have only to establish the first inequality in (14). Let us fix a particular t , and set

$$\psi(t) = |\psi(t)|e^{i\lambda(t)} .$$

Define now, for real s

$$\theta(s) = \psi(s)e^{i\lambda(t)s/t} = |\psi(s)|e^{i[\lambda(s)-\lambda(t)s/t]} .$$

θ is a characteristic function, so we have by a rearrangement of (4):

$$n[Re\theta(t)]^n \leq n - 1 + Re\theta(nt) ,$$

which is, from the definition of θ :

$$n|\psi(t)|^n \leq n - 1 + |\psi(nt)|\cos[\lambda(nt)-n\lambda(t)] .$$

Certainly then

$$n|\psi(t)|^n \leq n - 1 + |\psi(nt)| ,$$

and (14) follows.

We can use Corollary 1 to prove an extension of the following result due to Cramér ([*1*], p. 27):

If $\psi(t)$ is a characteristic function such that $|\psi(t)| \leq A < 1$ for $B \leq t \leq 2B$, then for $0 < t < B$

https://doi.org/10.1017/S0004972700044191 Published online by Cambridge University Press

$$|\psi(t)| \le 1 - \left(1-A^2\right)\frac{t^2}{8B^2} .$$

The corollary below states inequalities of a similar character for characteristic functions which satisfy weaker conditions, but even with the same conditions as above we get the sharper result:

$$|\psi(t)| < \left(\frac{B+tA}{B+t}\right)^{\frac{t}{B+t}} < 1 - (1-A)\frac{t^2}{4B^2} < 1 - \left(1-A^2\right)\frac{t^2}{8B^2}$$

for all $0 < t < B$.

COROLLARY 2. *Let* $\psi(t)$ *be a characteristic function with real part* $u(t)$ *, and let* $0 \le A < 1$ *,* $0 < b \le B$ *. Suppose that* $|\psi(t)| \le A$ *when* $B \le t \le B+b$ *. Then for* $0 < t < b$

$$(15) \qquad |\psi(t)| < \left(\frac{B+tA}{B+t}\right)^{\frac{t}{B+t}} < 1 - \frac{(1-A)t^2}{(B+t)^2} < 1 - \frac{(1-A)t^2}{(B+b)^2} .$$

The same result holds if we replace $|\psi(t)|$ *by* $u(t)$ *.*

Proof. Suppose the conditions of the corollary hold, and let $0 < t < b$. Then there exists an integer $n \ge 2$ such that

$$n - 1 < B/t \le n$$

and hence

$$(16) \qquad B \le nt < B+t < B+b .$$

We now apply Corollary 1 to obtain

$$|\psi(t)| \le \left(1 - \frac{1-|\psi(nt)|}{n}\right)^{\frac{1}{n}} < \left(1 - \frac{1-A}{n}\right)^{\frac{1}{n}}$$

$$< \left(1 - \frac{(1-A)t}{B+t}\right)^{\frac{t}{B+t}} = \left(\frac{B+tA}{B+t}\right)^{\frac{t}{B+t}} ,$$

since, from (16), $\frac{1}{n} > \frac{t}{B+t}$. This proves the first inequality in (15). The second inequality follows from the standard result that

$$(1-x)^{\alpha} < 1 - \alpha x \quad \text{for} \quad 0 < x < 1 , \quad 0 < \alpha < 1 ,$$

https://doi.org/10.1017/S0004972700044191 Published online by Cambridge University Press

and the third is obvious. The same proof holds for $u(t)$, using the theorem instead of Corollary 1.

In a similar way we can prove that if $\psi(t)$ is a characteristic function for which $1 - |\psi(t)| \le \alpha$ when $0 \le t \le b$, then for $t > b$,

$$1 - |\psi(t)| < \left[1-(1-\alpha)^{(t+b)/b}\right](t+b)/b < (t+b)^2\alpha/b^2 < 4\alpha t^2/b^2 .$$

Again we can replace $|\psi(t)|$ by $u(t)$.

The methods used in the proof of the theorem can also be applied to obtain inequalities involving $v(t)$. Thus to prove (17) below one first establishes that $v^n(t) \le E\sin^n tX$ whenever $n\left[1-v^n(t)\right] < 2$. The lemma can be applied after writing $\sin^n tX = \cos^n\left(\frac{\pi}{2} - tX\right)$, to give for all real t and $n = 1, 2, \ldots$,

$$\begin{aligned} n^2[1-v(t)] \ge n\left[1-v^n(t)\right] &\ge 1 - u(nt) \quad \text{if} \quad n \equiv 0 \\ &\ge 1 - v(nt) \quad \text{if} \quad n \equiv 1 \\ &\ge 1 + u(nt) \quad \text{if} \quad n \equiv 2 \\ &\ge 1 + v(nt) \quad \text{if} \quad n \equiv 3 \end{aligned} \tag{17}$$

where "$\equiv$" denotes "congruent modulo 4". We note that (17) yields (3) for $n = 2$. Another inequality can be obtained if we observe that

$$v^2(nt) \le 1 - u^2(nt) \le 2[1-u(nt)]$$

and then apply the theorem to get

$$v^2(nt) \le 2n\left[1-u^n(t)\right] \le 2n^2[1-u(t)] .$$

Finally, let us examine conditions under which equality is attained in the inequalities (4) and (5). To avoid triviality we consider only $n \ge 2$. It is apparent from the proof of (5) that this inequality is strict unless $u(t) = 1$. Also, for $n \ge 3$, an inspection of the proofs of Theorem 1 and the lemma reveals that equality obtains in (4) if and only if $u^n(t) = 1$. When $n = 3$ we can in fact improve on (4) by applying the type of argument used in the proof of the theorem for odd integers with the identity $4\cos^3 x = \cos 3x + 3\cos x$ to obtain the inequality

$$u(3t) \ge 4u^3(t) - 3u(t) \quad \text{if} \quad u(t) \ge \frac{1}{2} . \tag{18}$$

https://doi.org/10.1017/S0004972700044191 Published online by Cambridge University Press

(4) for $n = 3$ gives us

$$u(3t) \geq 3u^3(t) - 2 , \tag{19}$$

but we find that (18) is strictly stronger than (19) except when $u(t) = 1$. Also equality holds everywhere in (18) for $u(t) = \cos at$.

However for $n \geq 4$ this approach breaks down, because $\cos^n x$ can no longer be expressed as a linear function of $\cos nx$ and $\cos x$. It seems that the best we can do is use the inequality (7) to obtain (4).

For $n = 2$, (4) is equivalent to (2) and equality is attained when

$$u(2t) = 2u^2(t) - 1 . \tag{20}$$

If (20) holds for $t \neq 0$, we must have $\mathrm{var}(\cos tX) = 0$ so that $\cos tX$ is a degenerate random variable. Thus X is a discrete random variable with support contained in the point set

$$\{nA \pm a\}_{n \in I} \tag{21}$$

where $A = 2\pi/t$ and $a = \left[\cos^{-1} u(t)\right]/t$. Conversely, if X has support contained in a set of the form (21) it is clear that (20) holds for $t = 2\pi/A$. If (20) holds for $t = t_0$, then (20) also holds for all integral multiples of t_0 , because $\cos nx$ can be written as a polynomial in $\cos x$. Furthermore, it can be shown that if S is the set of all points for which (20) holds, then either S is contained in the union of two lattices

$$\{n\alpha\}_{n \in I} \text{ and } \{n\beta\}_{n \in I}$$

for some real α and β , or else S is the whole real line, when $u(t) = \cos at$ for some a , and X has support contained in $\{-a, +a\}$.

We find from this last case that if a characteristic function satisfies the functional equation (20) for all real t , then $u(t) = \cos at$ for some a . It is interesting to note, however, that without the restriction to characteristic functions (that is, non-negative definite $u(t)$ with $u(0) = 1$), (20) has solutions other than the cosine; (see Kuczma [3], p. 103).

References

[1] Harald Cramér, *Random variables and probability distributions*, 3rd ed. (Cambridge University Press, Cambridge, 1970).

[2] William Feller, *An introduction to probability theory and its applications*, Vol. 2, 2nd ed. (John Wiley & Sons, New York, London, Sydney, Toronto, 1971).

[3] Marek Kuczma, *Functional equations in a single variable* (Polish Scientific Publishers, Warsaw, 1968).

[4] Eugene Lukacs, *Characteristic functions*, 2nd ed. (Griffin, London, 1970).

Department of Statistics,
School of General Studies,
Australian National University,
Canberra, ACT.

https://doi.org/10.1017/S0004972700044191 Published online by Cambridge University Press