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INDUCTION OF CHARACTERS AND FINITE *p*-GROUPS

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Abstract. Let G be a finite p-group, where p is an odd prime number, H a subgroup of G and $\theta \in \mathrm{Irr}(H)$ an irreducible character of H. Assume also that $|G : H| = p^2$. Then the character θ^G of G induced by θ is either a multiple of an irreducible character of G, or has at least $\frac{p+1}{2}$ distinct irreducible constituents.

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1. Introduction. Let G be a finite group. Denote by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. Throughout this work, we use the notation of [**2**]. In addition, we are going to denote by $\mathrm{Lin}(G) = \{\lambda \in \mathrm{Irr}(G) \mid \lambda(1) = 1\}$ the set of linear characters.

Let Γ be a character of G. Then Γ can be expressed as a nontrivial integral linear combination of distinct irreducible characters of G. Denote by $\eta(\Gamma)$ the number of distinct irreducible constituents of Γ.

Let G be a finite p-group, where p is a prime number, H be a subgroup of G and $\theta \in \mathrm{Irr}(H)$. Denote by θ^G the character of G induced by θ. If H is a normal subgroup, then either $\eta(\theta^G) = 1$, i.e. θ^G is a multiple of an irreducible, or $\eta(\theta^G) \geq p$, i.e. θ^G is an integral linear combination of at least p distinct irreducible characters of G (see Lemma 2.2). In Theorem 4.15, it is shown that given any prime $p > 2$ and any integer $l \geq 2$, there exist a p-group G, a subgroup H of G with $|G : H| = p^l$ and $\theta \in \mathrm{Irr}(H)$ such that $\eta(\theta^G) = \frac{p+1}{2}$. Therefore Lemma 2.2 does not remain true without the hypothesis that H is normal in G. But given any prime $p > 2$ and any integer $n > 0$, do there exist a p-group G, a subgroup H of G and $\theta \in \mathrm{Irr}(H)$ with $\eta(\theta^G) = n$? If we also required, in addition, that $|G : H| = p^2$ and $1 < n < \frac{p+1}{2}$, then the answer is no. More specifically:

THEOREM A. *Let G be a finite p-group, where p is an odd prime number, H be a subgroup of G and $\theta \in \mathrm{Irr}(H)$. Assume also that $|G : H| = p^2$. Then either $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq \frac{p+1}{2}$.*

For a fixed prime $p > 3$, Theorem A implies that there exists a "gap" among the possible values that $\eta(\theta^G)$ can take for any finite p-group G, any subgroup H of G with $|G : H| = p^2$, and any character $\theta \in \mathrm{Irr}(H)$. But, do there exist a p-group G, a subgroup H of G and $\theta \in \mathrm{Irr}(H)$ with $1 < \eta(\theta^G) < \frac{p+1}{2}$ and $|G : H| > p^2$? The answer is yes. In Theorem 4.23, given any prime p such that 3 divides $p - 1$, we provide a p-group G, a subgroup H of G with $|G : H| = p^3$ and a character $\lambda \in \mathrm{Lin}(H)$ such that $\eta(\lambda^G) = \frac{p+2}{3}$. Does it mean then that, for a fixed prime $p > 5$, there are no "gaps" among the possible values that $\eta(\theta^G)$ can take for any finite p-group G, any subgroup H of G with $|G : H| = p^3$, and any character $\theta \in \mathrm{Irr}(H)$? We do not know the answer of that question.

2. Preliminaries.

LEMMA 2.1. *Let G be a finite group, N be a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$. Let G_θ be the stabilizer of θ in G. Then $\eta(\theta^G) = \eta(\theta^{G_\theta})$.*

Proof. Observe that all the irreducible constituents of θ^{G_θ} lie above θ. Thus by Clifford theory it follows that $\eta(\theta^G) = \eta(\theta^{G_\theta})$. □

LEMMA 2.2. *Let G be a finite p-group, H be a normal subgroup of G and $\theta \in \mathrm{Irr}(H)$. Then either $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq p$.*

Proof. In [**1**, Lemma 4.1], it is proved that, if in addition to the previous hypothesis, θ is G-invariant, then $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq p$. Thus by induction on $|G : H|$ and Lemma 2.1, the result follows. □

Let G be a group, H be a subgroup of G and $\theta \in \mathrm{Irr}(H)$. Denote by $\mathrm{Irr}(G \mid \theta) = \{\chi \in \mathrm{Irr}(G) \mid [\chi_H, \theta] \neq 0\}$ the set of irreducible characters of G lying above θ.

LEMMA 2.3. *Let G be a finite p-group, H be a subgroup of G and $\theta \in \mathrm{Irr}(H)$. Let Z_1 be a subgroup of the center $\mathbf{Z}(G)$ of G such that $|HZ_1 : H| = p$. Then θ extends to HZ_1 and*

$$\eta(\theta^G) = \sum_{\nu \in \mathrm{Irr}(HZ_1|\theta)} \eta(\nu^G).$$

In particular, if $\nu \in \mathrm{Irr}(HZ_1 \mid \theta)$ we have that

$$\eta(\theta^G) \geq \eta(\nu^G) + (p-1). \tag{2.4}$$

Proof. Observe that θ extends to HZ_1 since $Z_1 \leq \mathbf{Z}(G)$ and $|HZ_1 : H| = p$. Thus there are exactly p characters in $\mathrm{Irr}(HZ_1 \mid \theta)$. Let $\alpha \in \mathrm{Lin}(H \cap Z_1)$ be the unique character such that $\theta_{H\cap Z_1} = \theta(1)\alpha$. Since $(\theta^{HZ_1})_{Z_1} = (\theta_{H\cap Z_1})^{Z_1}$, we have that $(\theta^{HZ_1})_{Z_1} = \theta(1)\sum_{\nu\in \mathrm{Lin}(Z_1|\alpha)} \nu$. Therefore

$$\text{for any } \nu, \mu \in \mathrm{Irr}(HZ_1 \mid \theta), \text{ if } \nu \neq \mu \text{ then } \nu_{Z_1} \neq \mu_{Z_1}. \tag{2.5}$$

Observe that for any $\chi \in \mathrm{Irr}(G)$ and any $\beta \in \mathrm{Lin}(Z_1)$, if $[\chi_{Z_1}, \beta] \neq 0$ then $\chi_{Z_1} = \chi(1)\beta$. By (2.5), it follows that if $\chi, \psi \in \mathrm{Irr}(G)$, $\nu, \mu \in \mathrm{Irr}(HZ_1 \mid \theta)$, $\nu \neq \mu$, $[\chi_{Z_1}, \nu] \neq 0$ and $[\psi_{Z_1}, \mu] \neq 0$, then $\chi \neq \psi$. Thus the irreducible constituents of θ^G lying over distinct extensions of θ in HZ_1 are distinct characters. It follows that

$$\eta(\theta^G) = \sum_{\nu \in \mathrm{Irr}(HZ_1|\theta)} \eta(\nu^G).$$

Since $\eta(\nu^G) \geq 1$ for any $\nu \in \mathrm{Irr}(HZ_1)$, (2.4) follows. □

3. Proof of Theorem A. Let G and $\theta \in \mathrm{Irr}(H)$ be a minimal counterexample of the statement of Theorem A with respect to the order $|G|$ of G. That is we are assuming that

$$|G : H| = p^2,\ 1 < \eta(\theta^G) < \frac{p+1}{2} \tag{3.1}$$

and *for any finite p-group G_1, any subgroup H_1 of G_1, and any $\theta_1 \in \mathrm{Irr}(H_1)$, if*

$$|G_1 : H_1| = p^2 \text{ and } |G_1| < |G| \text{ then either } \eta({\theta_1}^{G_1}) = 1 \text{ or } \eta({\theta_1}^{G_1}) \geq \tfrac{p+1}{2}. \tag{3.2}$$

Set $\overline{L} = L/\mathrm{core}_G(\mathrm{Ker}(\theta))$ for any subgroup L of G such that $L \geq \mathrm{core}_G(\mathrm{Ker}(\theta))$. Observe that $H \geq \mathrm{core}_G(\mathrm{Ker}(\theta))$ and $|\overline{G} : \overline{H}| = |G : H|$. Observe also that we can regard θ as a character of $H/\mathrm{core}_G(\mathrm{Ker}(\theta))$ and $\eta(\theta^{\overline{G}}) = \eta(\theta^G)$.

By working with the group $G/\mathrm{core}_G(\mathrm{Ker}(\theta))$ and (3.2), we may assume that

$$\mathrm{core}_G(\mathrm{Ker}(\theta)) = 1.$$

Thus $\overline{L} = L$ for all subgroups L of G.

Denote by Z the center $\mathbf{Z}(G)$ of G.

Claim 3.3. *$Z < H$. Let $\nu \in \mathrm{Lin}(Z)$ be the unique character of Z lying below θ. Then $\nu \in \mathrm{Lin}(Z)$ is a faithful character of Z and Z is a cyclic group.*

Proof. Suppose Z is not contained in H. Let $Z_1 \leq Z$ be such that $|HZ_1 : H| = p$. Lemma 2.3 implies that $\eta(\theta^G) \geq p$, a contradiction with (3.1). Thus $Z \leq H$. Since $Z = H$ implies that H is normal, by Lemma 2.2 we must have that $Z < H$.

Since $\mathrm{Ker}(\theta) \cap Z$ is normal in G and $\mathrm{core}_G(\mathrm{Ker}(\theta)) = 1$, it follows that θ_Z is a faithful character of Z. Therefore $\nu \in \mathrm{Lin}(Z)$ is faithful and Z is cyclic. □

Claim 3.4. $\mathrm{core}_G(H) = Z$.

Proof. Assume that there exists a normal subgroup N of G such that $N \leq H$ and N/Z is a chief factor of G. Fix $\beta \in \mathrm{Irr}(N)$ such that $[\theta_N, \beta] \neq 0$. Since $\nu \in \mathrm{Lin}(Z)$ is a faithful character, we can check that $\mathbf{C}_G(N)$ is a normal subgroup of G of index p. Also the stabilizer G_β of β in G is $\mathbf{C}_G(N)$.

If $H \cap \mathbf{C}_G(N) < H$, by Clifford theory we have that there exists some $\alpha \in \mathrm{Irr}(H \cap \mathbf{C}_G(N))$ such that $\alpha^H = \theta$. Thus $\eta(\theta^G) = \eta(\alpha^G)$. Since $|\mathbf{C}_G(N)| < |G|$ and $|\mathbf{C}_G(N) : H \cap \mathbf{C}_G(N)| = p^2$, by (3.2) we have that $\eta(\alpha^{\mathbf{C}_G(N)}) = 1$ or $\eta(\alpha^{\mathbf{C}_G(N)}) \geq \frac{p+1}{2}$. By Lemma 2.1 we have then that $\eta(\alpha^G) = 1$ or $\eta(\alpha^G) \geq \frac{p+1}{2}$ and therefore $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq \frac{p+1}{2}$, a contradiction with (3.1). We may assume then that $H < \mathbf{C}_G(N)$.

Since $|\mathbf{C}_G(N) : H| = p$, H is normal in $\mathbf{C}_G(N)$ and thus by Lemma 2.2 we have that either $\eta(\theta^{\mathbf{C}_G(N)}) = 1$ or $\eta(\theta^{\mathbf{C}_G(N)}) = p$. By Lemma 2.1 and the previous statement, we have that $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq p$, a contradiction with (3.1). Thus such N cannot exist and so $\mathrm{core}_G(H) = Z$. □

Let Y/Z be a chief factor of G. By the previous claim, it follows that $HY > H$. Since Y/Z has order p, we have that $|HY : H| = p$. Since $|G : H| = p^2$, it follows that $|G : HY| = p$ and thus HY is a normal subgroup of G.

Set $C = \mathbf{C}_G(Y)$.

Claim 3.5. *$|G : C| = p$. Also, given any $\mu \in \mathrm{Lin}(Y)$ which is an extension of the faithful character $\nu \in \mathrm{Lin}(Z)$, we have that the stabilizer G_μ of μ in G is C.*

Proof. Since $\nu \in \mathrm{Lin}(Z)$ is a faithful character of the center Z of G and Y/Z is a chief factor of the p-group G, it follows that the index of the centralizer C of Y in G is p. □

Claim 3.6. *HY/Z is an elementary abelian p-group. Also, we may assume that $\mathbf{Z}(HY) \geq Y$ and thus $C = HY$.*

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Proof. Since $|HY : H| = p$, we have that $(HY)' = \langle [h, k] \mid h, k \in HY \rangle \leq H$. Observe that $(HY)'$ is normal in G since HY is normal in G and $(HY)'$ is a characteristic subgroup of HY. Since $\mathrm{core}_G(H) = Z$, it follows then that $(HY)' \leq Z$. Also, since Y/Z is of order p and $Z < H$, $(HY)^p = \langle k^p \mid k \in HY \rangle$ is a characteristic subgroup of the normal subgroup HY of G and it is contained in H. It follows then that $(HY)^p \leq Z$ and thus HY/Z is an elementary abelian p-group.

Observe that the center $\mathbf{Z}(HY)$ of HY contains Z. If $\mathbf{Z}(HY) = Z$, then there is a unique character in $\mathrm{Irr}(H)$ lying above ν since HY/Z is an elementary abelian p-group and $\nu \in \mathrm{Lin}(Z)$ is a faithful character, and so $\eta(\theta^G) = 1$ or $\eta(\theta^G) = p$, that is a contradiction with (3.1) and therefore it must follow that $\mathbf{Z}(HY) > Z$. By replacing Y for a normal subgroup of G contained in $\mathbf{Z}(HY)$ if necessary, we may assume then that $Y \leq \mathbf{Z}(HY)$ and thus $\mathbf{C}_G(Y) = HY$. □

CLAIM 3.7. *The character* $\theta \in \mathrm{Irr}(H)$ *extends to* $HY = C$. *Thus* θ^C *is the sum of the* p *distinct extensions of* θ.

Proof. Since $|HY : H| = p$, we have that either $\theta^{HY} \in \mathrm{Irr}(HY)$ or θ^{HY} is the sum of the p distinct extensions of θ.

Suppose that $\theta^C \in \mathrm{Irr}(C)$. Let $\mu \in \mathrm{Lin}(Y)$ be the unique character of Y such that $[(\theta^{HY})_Y, \mu] \neq 0$. Since $G_\mu = C$, then $\theta^G \in \mathrm{Irr}(G)$. Thus θ^{HY} is the sum of the p distinct extensions of θ. □

Let $\rho_1, \ldots, \rho_p \in \mathrm{Irr}(HY)$ be the p distinct extensions of θ. Since $|G : HY| = p$, by Lemma 2.2 we must have that

$$\rho_i^G \in \mathrm{Irr}(G). \tag{3.8}$$

Since $\mathbf{Z}(C) \geq Y$, there is a unique character $\mu_i \in \mathrm{Lin}(Y)$ lying below ρ_i.

CLAIM 3.9. $\mathbf{Z}(C) = Y$.

Proof. Clearly $Y \leq \mathbf{Z}(C)$. Assume that $Y < \mathbf{Z}(C)$. Let $X \leq \mathbf{Z}(C)$ such that X/Y is a chief factor of G and $Y < X \leq HY = C$. Observe that such X exists since HY is normal in G, and X is abelian since $X \leq \mathbf{Z}(C)$. We are going to conclude that $\nu \in \mathrm{Lin}(Z)$ is not a faithful character, which is a contradiction with Claim 3.3.

STEP 3.10. *The subgroup* $[X, G]$ *generates* $Y = [X, G]Z$ *modulo* Z.

Proof. Since Y and X are normal subgroups of G with $Y \triangleleft X$ and $|X/Y| = p$, the chief factor X/Y of the p-group G is centralized by G. So $[X, G] \leq Y$. Suppose that $[X, G]Z < Y$. Since $|Y/Z| = p$, we must have $[X, G] \leq Z = \mathbf{Z}(G)$. So commutation in G induces a bilinear map

$$d : (xZ, g\mathbf{C}_G(X)) \mapsto [x, g]$$

of $X/Z \times G/\mathbf{C}_G(X)$ into the cyclic group Z. This map d is non-singular on the right since $[X, g] = 1$ if and only if $g \in \mathbf{C}_G(X)$. It is non-singular on the left since $[x, G] = 1$ if and only if $x \in Z$. Because $|X : Z| = p^2$ and d is a non-singular bilinear form of $X/Z \times G/\mathbf{C}_G(X)$ into the cyclic group Z, we have $|G : \mathbf{C}_G(X)| = p^2$. Since $\lambda \in \mathrm{Lin}(X \mid \nu)$ extends the faithful character $\nu \in \mathrm{Irr}(Z)$, this implies that $\mathbf{C}_G(X) = G_\lambda$. Thus $|G : G_\lambda| = p^2$. Since $X \leq \mathbf{Z}(C)$, C fixes λ. But then $|G : C| = p$, $C \leq G_\lambda$ and $|G : G_\lambda| = p^2$. This contradiction proves the claim. □

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Given any character $\rho \in \mathrm{Irr}(C)$, since $X \leq \mathbf{Z}(C)$, we have that $\frac{1}{\rho(1)}\rho_X \in \mathrm{Lin}(X)$ is the unique character lying below ρ.

STEP 3.11. *There exist some* $\lambda \in \mathrm{Lin}(X)$*, some* $g \in G\backslash C$ *and* $i \in \{2, \ldots, p-1\}$ *such that* $[(\theta^C)_X, \lambda] \neq 0$*,* $[(\theta^C)_X, \lambda^g] \neq 0$ *and* $[(\theta^C)_X, \lambda^{g^i}] \neq 0$*.*

Proof. Since $1 < \eta(\theta^G) < \frac{p+1}{2}$ and $\rho_1^G, \ldots, \rho_p^G$ are the irreducible constituents of θ^G, there exist at least 3 distinct $j, k, l \in \{1, 2, \ldots, p\}$ such that $\rho_j^G = \rho_k^G = \rho_l^G$. Since X is normal in G, by Clifford Theory it follows that $\frac{1}{\rho_j(1)}(\rho_j)_X$, $\frac{1}{\rho_k(1)}(\rho_k)_X$ and $\frac{1}{\rho_l(1)}(\rho_l)_X$ are G-conjugates. Set $\lambda = \frac{1}{\rho_j}(\rho_j)_X$. Then there exists some $g \in G\backslash C$ such that $\lambda^g = \frac{1}{\rho_k(1)}(\rho_k)_X$. Since $X \leq \mathbf{Z}(C)$ and $|G : C| = p$, there exists some $i \in \{2, \ldots, p-1\}$ such that $(\lambda)^{g^i} = \frac{1}{\rho_l(1)}(\rho_l)_X$. □

Fix $g \in G\backslash C$ as in 3.11. Since X/Y is cyclic of order p, $H \cap X > Z$, and $H \cap Y = Z$ we may choose

$$x \in H \text{ such that } X = \langle x, Y \rangle . \tag{3.12}$$

Since $X \leq \mathbf{Z}(C)$, we have $[X, C] = 1$. Suppose that $[x, g^{-1}] \in Z$. Then x centralizes both g^{-1} and C modulo Z. Hence $xZ \in \mathbf{Z}(G/Z)$, which is false by Step 3.10. Hence $[x, g^{-1}] \in Y\backslash Z$ and so

$$Y = Z\langle y \rangle \text{ is generated over } Z \text{ by } y = [x, g^{-1}]. \tag{3.13}$$

Since $[Y, G] \leq Z$ we have that $z = [y, g^{-1}] \in Z$. If $z = 1$, then $G = C\langle g \rangle$ centralizes $Y = Z\langle y \rangle$, since C centralizes $Y < X$ because $X \leq \mathbf{Z}(C)$, and G centralizes Z. This is impossible because $Z = \mathbf{Z}(G) < Y$. Thus

$$z = [y, g^{-1}] \text{ is a non-trivial element of } Z. \tag{3.14}$$

By (3.13) we have $y = [x, g^{-1}] = x^{-1}x^{g^{-1}}$. By (3.14) we have $z = [y, g^{-1}] = y^{-1}y^{g^{-1}}$. Finally $z^{g^{-1}} = z$ since $z \in Z$. Since $X = Z\langle x, y \rangle$ is abelian since $X \leq \mathbf{Z}(C)$, it follows that

$$z^{g^{-j}} = z, \; y^{g^{-j}} = yz^j \text{ and } x^{g^{-j}} = xy^j z^{\binom{j}{2}}, \tag{3.15}$$

for any integer $j = 0, 1, \ldots, p-1$. Because $g^{-p} \in C$ centralizes X since $X \leq \mathbf{Z}(C)$, we have

$$z^p = 1 \text{ and } y^p z^{\binom{p}{2}} = 1.$$

Since $p > 2$ is odd by hypothesis, p divides $\binom{p}{2} = \frac{p(p-1)}{2}$ and $z^{\binom{p}{2}} = 1$. Therefore $y^p = z^p = 1$. It follows that y^i, z^i and $z^{\binom{i}{2}}$ depend only on the residue of i modulo p, for any integer $i \geq 0$. such that $X = Y\langle x \rangle$ and $x \in C$. Thus by (3.14) we have that

$$z^{\binom{j}{2}} \neq 1 \text{ for any integer } 0 < j < p. \tag{3.16}$$

Let $\lambda \in \mathrm{Lin}(X)$ and $i \in \{2, \ldots, p-1\}$ be as in Step 3.11. Set $\varpi = \frac{1}{\theta(1)}\theta_{X\cap H}$. We can check that $\varpi \in \mathrm{Lin}(X \cap H)$. Since $(\theta^C)_X = (\theta_{H\cap X})^X$, we have that λ, λ^g and λ^{g^i} are

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extensions of ϖ. Since $x \in (H \cap X)$, by the previous statement we have that

$$\lambda(x) = \lambda^{g}(x) = \lambda^{g^i}(x). \tag{3.17}$$

By (3.15) we have that

$$\lambda^{g}(x) = \lambda(x^{g^{-1}}) = \lambda(xy) = \lambda(x)\lambda(y).$$

Thus by (3.17), we get

$$\lambda(y) = 1. \tag{3.18}$$

Therefore

$$\begin{aligned}\lambda^{g^i}(x) &= \lambda(x^{g^{-i}})\\ &= \lambda\big(xy^i z^{\binom{i}{2}}\big) \quad \text{by (3.15)}\\ &= \lambda(x)\lambda(y^i)\lambda\big(z^{\binom{i}{2}}\big)\\ &= \lambda(x)\lambda\big(z^{\binom{i}{2}}\big),\end{aligned}$$

where the last line follows from (3.18). By (3.17), we have that $\lambda(z^{\binom{i}{2}}) = 1$. But $\lambda_Z = \nu \in \mathrm{Lin}(Z)$ is a faithful character and $z^{\binom{i}{2}} \neq 1$ by (3.16). This is a contradiction and the claim is proved. □

Since $\mathbf{Z}(HY) = Y$, we have that $\mathbf{Z}(H) = Z$. Thus HY is a class 2 group with HY/Z elementary abelian. Therefore $\theta \in \mathrm{Irr}(H)$ is the only character in H lying above $\nu \in \mathrm{Lin}(Z)$. Hence an irreducible character of G lies over θ if and only if it lies over ν. Since $\mathrm{Irr}(G \mid \nu)$ has either 1 element or at least p by Lemma 2.2, it follows that $\eta(\nu^G) = 1$ or $\eta(\nu^G) \geq p$, and therefore either $\eta(\theta^G) = 1$ or $\eta(\theta^G) \geq p$. But $1 < \eta(\theta^G) < \frac{p+1}{2}$, and that is our final contradiction and thus the statement of Theorem A holds.

4. Examples. In this section, we will prove that the group G, the subgroup H and the character $\lambda \in \mathrm{Lin}(H)$ that satisfy Hypothesis 4.1 have the properties that $|G : H| = p^2$ and $\eta(\lambda^G) = \frac{p+1}{2}$. And then, given any integer $n \geq 2$, we construct a group G with a subgroup H and a character $\lambda \in \mathrm{Lin}(H)$ such that $|G : H| = p^n$ and $\eta(\lambda^G) = \frac{p+1}{2}$.

HYPOTHESIS 4.1. *Fix an odd prime p. Let G be the semidirect product of a cyclic group C of order p and an elementary abelian group A of order* p^3. *Assume* $C = \langle c \rangle$ *and*

$$A = \langle a \rangle \times \langle [a, c] \rangle \times \langle [a, c, c] \rangle, \tag{4.2}$$

for some a in A. Observe that the subgroup $\{e\} \times \{e\} \times \langle [a, c, c] \rangle$ *is the center of the group G. Set* $Z = \{e\} \times \{e\} \times \langle [a, c, c] \rangle$.

Fix ω a primitive complex p-th root of unity. Let $\alpha \in \mathrm{Lin}(\langle a \rangle)$, $\beta \in \mathrm{Lin}(\langle [a, c] \rangle)$ *and* $\gamma \in \mathrm{Lin}(\langle [a, c, c] \rangle)$ *be the unique linear characters such that* $\alpha(a) = \beta([a, c]) = \gamma([a, c, c]) = \omega$.

Set

$$H = \langle a \rangle \times \{e\} \times \langle [a, c, c] \rangle \text{ and } \lambda = 1_{\langle a \rangle} \times 1_{\{e\}} \times \gamma \in \mathrm{Lin}(H). \tag{4.3}$$

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Observe that H is a subgroup of A of index p. Thus $|G:H| = p^2$. Observe also that λ extends to A and there are exactly p distinct extensions of λ to A, namely

$$\mathrm{Irr}(A \mid \lambda) = \{1_{\langle a\rangle} \times \beta^r \times \gamma \mid r = 0, 1, \dots, p-1\}. \tag{4.4}$$

Set $\Lambda_r = 1_{\langle a\rangle} \times \beta^r \times \gamma$.

LEMMA 4.5. *Assume Hypothesis* 4.1. *Given any integer i with $0 < i$, we have that*

$$(\Lambda_r)^{c^i} = \alpha^{ri+\frac{i(i-1)}{2}} \times \beta^{r+i} \times \gamma.$$

Proof. Observe that $(\Lambda_r)^c = \alpha^r \times \beta^r\beta \times \gamma = \alpha^r \times \beta^{r+1} \times \gamma$ since $a^c = a[a, c]$ and $[a, c]^c = [a, c][a, c, c]$. Assume by induction that

$$(\Lambda_r)^{c^n} = \alpha^{rn+\frac{n(n-1)}{2}} \times \beta^{r+n} \times \gamma. \tag{4.6}$$

Then

$$\begin{aligned}(\Lambda_r)^{c^{n+1}} &= \left((\Lambda_r)^{c^n}\right)^c \\ &= \left(\alpha^{rn+\frac{n(n-1)}{2}} \times \beta^{r+n} \times \gamma\right)^c \text{ by (4.6)} \\ &= \alpha^{rn+\frac{n(n-1)}{2}+r+n} \times \beta^{r+n+1} \times \gamma,\end{aligned}$$

where the last line follows since $a^c = a[a, c]$ and $[a, c]^c = [a, c][a, c, c]$. We can check that $rn + \frac{n(n-1)}{2} + r + n = r(n+1) + \frac{(n+1)(n)}{2}$. Thus

$$(\Lambda_r)^{c^{n+1}} = \alpha^{r(n+1)+\frac{(n+1)n}{2}} \times \beta^{r+(n+1)} \times \gamma,$$

and the result follows by induction. □

LEMMA 4.7. *Assume Hypothesis* 4.1. *Let r be an integer such that $0 < r < p$. Then $(\Lambda_r)^{c^j}$ is an extension of λ if and only if either $j \equiv 0 \bmod p$ or $j \equiv (1-2r) \bmod p$. If $i \equiv (1-2r) \bmod p$ then $(\Lambda_r)^{c^i} = \Lambda_{1-r}$.*

Proof. By Lemma 4.5, we have that $(\Lambda_r)^{c^i}$ is an extension of λ if and only if $\alpha^{ir+\frac{i(i-1)}{2}} = 1_{\langle a\rangle}$. Since α is a faithful linear character of a cyclic group of order p, $\alpha^{ir+\frac{i(i-1)}{2}} = 1_{\langle a\rangle}$ if and only if $(ir + \frac{i(i-1)}{2}) \equiv 0 \bmod p$. Observe that $(ir + \frac{i(i-1)}{2}) \equiv 0 \bmod p$ if and only if either $i \equiv 0 \bmod p$ or $(r + \frac{i-1}{2}) \equiv 0 \bmod p$. Therefore $(\Lambda_r)^{c^i}$ is an extension of λ if and only if either $i \equiv 0 \bmod p$ or $i \equiv (1-2r) \bmod p$.

If $i \equiv (1-2r) \bmod p$, then $(\Lambda_r)^{c^i} = \Lambda_{1-r}$ by Lemma 4.5. □

LEMMA 4.8. *Assume Hypothesis* 4.1. *Then $1 < \eta(\lambda^G) \leq \frac{p+1}{2}$.*

Proof. By the previous lemma, it follows that the stabilizer of Λ_r is a proper subgroup of G. Since $|G:A| = p$ and $\Lambda_r \in \mathrm{Lin}(A)$, we have that

$$(\Lambda_r)^G \in \mathrm{Irr}(G) \text{ for any integer } r. \tag{4.9}$$

Since $p > 2$, it follows that there exist two distinct integers k, l such that $0 < k, l < p$ and $k \neq (1-2l) \bmod p$. Thus by Lemma 4.7 we have that Λ_k and Λ_l are not

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G-conjugates. It follows that $(\Lambda_k)^G \neq (\Lambda_l)^G$. Since $(\Lambda_k)^G \neq (\Lambda_l)^G$, $(\Lambda_k)^G, (\Lambda_l)^G \in \mathrm{Irr}(G)$ and both Λ_k and Λ_l lie above λ, we have that $\eta(\lambda^G) \geq 2$.

Observe that $r \equiv (1-r) \bmod p$ if and only if $2r \equiv 1 \bmod p$. Thus given any r such that $0 < r < p$ and $2r \not\equiv 1 \bmod p$, by Lemma 4.7 we have that $\Lambda_r, \Lambda_{1-r} \in \mathrm{Irr}(A)$ are two distinct G-conjugate extensions of λ. Thus $\eta(\lambda^G) \leq \frac{p+1}{2}$. □

PROPOSITION 4.10. *Assume Hypothesis* 4.1. *Then* $|G:H| = p^2$ *and* $\eta(\lambda^G) = \frac{p+1}{2}$.

Proof. By Lemma 4.8, we have that $1 < \eta(\lambda^G) \leq \frac{p+1}{2}$. Thus by Theorem A, it follows that $\eta(\lambda^G) = \frac{p+1}{2}$. □

Denote by 1_H the principal character of H.

LEMMA 4.11. *Let p be a prime number, G be a p-group and H be a subgroup of G with* $|G:H| = p^n$. *Then* $\eta((1_H)^G) \geq n(p-1)+1$.

Proof. We are going to use a double induction, first on $|G|$ and then on n, where $|G:H| = p^n$. Using induction on the order of G, without lost of generality we may assume that $\mathrm{core}_G(H) = 1$.

Let Z_1 be a subgroup of the center $\mathbf{Z}(G)$ of G with $|Z_1| = p$. Observe that $H \cap Z_1 = 1$ since $\mathrm{core}_G(H) = 1$. Thus $|HZ_1 : H| = p$. By Lemma 2.3, we have that

$$\eta((1_H)^G) \geq \eta((1_{HZ_1})^G) + (p-1). \tag{4.12}$$

Since $|G:HZ_1| = p^{n-1}$, by induction on n we have that

$$\eta((1_{HZ_1})^G) \geq (n-1)(p-1)+1.$$

The result follows by (4.12) and the previous statement. □

LEMMA 4.13. *Let G_0 be a p-group and Γ be a character of G_0. Assume that* $[\Gamma, 1_{G_0}] = 0$. *Let* $N = G_0 \times G_0 \times \cdots \times G_0$ *be the direct product of p-copies of G_0. Set*

$$\Delta = \Gamma \times 1_{G_0} \times \cdots \times 1_{G_0}.$$

Let $C = \langle c \rangle$ be a cyclic group of order p. Observe that C acts on N by

$$c : (n_0, n_1, \ldots, n_{p-1}) \mapsto (n_{p-1}, n_0, \ldots, n_{p-2}) \tag{4.14}$$

for any $(n_0, n_1, \ldots, n_{p-1}) \in N$.

Let G be the direct product of N and C, i.e, G is the wreath product of G_0 and C. Then $\eta(\Delta^G) = \eta(\Gamma)$.

Proof. Let $\delta \in \mathrm{Irr}(N)$ be a constituent of Δ. Observe that δ is of the form $\gamma \times 1_{G_0} \times \cdots \times 1_{G_0}$, for some $\gamma \in \mathrm{Irr}(G_0)$ such that $[\gamma, \Gamma] \neq 0$. Observe that $\gamma \neq 1_{G_0}$ since $[\Gamma, 1_{G_0}] = 0$. By (4.14), we have that δ is G-invariant if and only if $\gamma = 1_{G_0}$. Thus $\delta^G \in \mathrm{Irr}(G)$ for any constituent $\delta \in \mathrm{Irr}(N)$ of Δ. Observe that the G-orbit of $\delta \in \mathrm{Irr}(N)$ is

$$\{\gamma \times 1_{G_0} \times \cdots \times 1_{G_0}, 1_{G_0} \times \gamma \times \cdots \times 1_{G_0}, \cdots, 1_{G_0} \times \ldots \times 1_{G_0} \times \gamma\}.$$

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Thus if $\delta, \epsilon \in \mathrm{Irr}(N)$ are two distinct constituents of Δ, then $\delta^G \neq \epsilon^G$. It follows that $\eta(\Delta^G) = \eta(\Gamma)$. □

THEOREM 4.15. *Let p be an odd prime number and $n \geq 2$ be an integer. There exist a p-group G, a subgroup H of G and $\lambda \in \mathrm{Lin}(H)$, such that $|G : H| = p^n$ and $\eta(\lambda^G) = \frac{p+1}{2}$.*

Proof. If $n = 2$, then the result follows by Lemma 4.10. By induction on n, we may assume that the result holds for any integer n such that $n - 1 \geq 2$.

Fix a p-group G_0, a subgroup $H_0 \leq G_0$ and $\lambda_0 \in \mathrm{Lin}(H_0)$ such that:

$$|G_0 : H_0| = p^{n-1} \text{ and } \eta\left(\lambda_0^{G_0}\right) = \frac{p+1}{2}. \tag{4.16}$$

Let N and G be as in Lemma 4.13. Let

$$H = H_0 \times G_0 \times \ldots \times G_0.$$

Then H is a subgroup of N and $|G : H| = |G : N||N : H_0| = p|G_0 : H_0| = p^n$.

Set $\lambda = \lambda_0 \times 1_{G_0} \times \ldots \times 1_{G_0}$. Observe that $\lambda \in \mathrm{Lin}(H)$ since $\lambda_0 \in \mathrm{Lin}(H_0)$. We can check that $\eta(\lambda^N) = \eta(\lambda_0^{G_0})$. Thus by (4.16) we have that $\eta(\lambda^N) = \frac{p+1}{2}$.

By Lemma 4.11, we have that $\lambda_0 \neq 1_{H_0}$. Thus $[\lambda_0^{G_0}, 1_{G_0}] = 0$. By Lemma 4.13 we have then that $\eta(\lambda^N) = \eta(\lambda^G)$ and the result is proved. □

LEMMA 4.17. *Let p be a prime number such that $p - 1$ is divisible by 3. Fix $r \in \{1, \ldots, p-1\}$. Then the set $\{r(1 - i^3) \bmod p \mid i = 0, \ldots, p-1\}$ has $\frac{p+2}{3}$ elements. Also, given any $e \in \{r(1 - i^3) \bmod p \mid i = 1, \ldots, p-1\}$, there are exactly 3 distinct solutions in $\{1, \ldots, p-1\}$ of the equation $e \equiv r(1 - x^3) \bmod p$*

Proof. Let u be a generator of the units of the field F of p elements. Then $U = \langle u^{\frac{p-1}{3}} \rangle$ is a subgroup of order 3 and any element in U is a solution of $x^3 \equiv 1 \bmod p$. Thus given any integer $n \neq r$, if the equation $x^3 \equiv r - n \bmod p$ has a solution, then it has exactly 3 distinct solutions in F. Therefore the set $\{r(1 - i^3) \bmod p \mid i = 1, \ldots, p-1\}$ has $\frac{p-1}{3}$ distinct elements. Since $0^3 = 0$, the set $\{(r(1 - i^3) \bmod p \mid i = 0, \ldots, p-1\}$ has $\frac{p-1}{3} + 1 = \frac{p+2}{3}$ elements. □

HYPOTHESIS 4.18. *Let $p > 5$ be a prime number such that $p - 1$ is divisible by 3. Let F be a field of p elements and $F[x]$ be the truncated polynomial algebra generated over F by some x satisfying only $x^4 = 0$. So $F[x]$ is a vector space of dimension 4 over F with 1, x, x^2 and x^3 as a basis. Let m be an isomorphism of the additive group $F[x]^+$ of $F[x]$ onto a multiplicative group M. Then M is an elementary abelian multiplicative group of order p^4 with $m(1)$, $m(x)$, $m(x^2)$, $m(x^3)$ as generators. Let U be the subgroup of the unit group $F[x]^\times$ generated by $1 + x$ and $1 + x^2$. The general element of U is*

$$(1+x)^i(1+x^2)^j = 1 + ix + \left(\binom{i}{2} + j\right)x^2 + \left(\binom{i}{3} + ij\right)x^3 \tag{4.19}$$

for arbitrary integers i, j, since $x^4 = 0$. Because $p > 3$, it follows that U is elementary abelian of order p^2, and that (4.19) holds for any $i, j \in F$. The group U acts naturally on the group M, so that

$$m(y)^u = m(yu) \tag{4.20}$$

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for all $y \in F[x]$ *and* $u \in U$. *Let* G *be the semidirect product of* M *and* U. *Then* G *is a multiplicative group with order* p^6.

Let H *be the subgroup*

$$H = \langle m(1), m(x), m(x^3)\rangle = \{m(a_0 + a_1x + a_3x^3) \mid a_0, a_1, a_3 \in F\}. \tag{4.21}$$

Fix a primitive p*-th root of unity* ω. *Fix an integer* $r > 0$ *such that* $3r \equiv -1 \bmod p$. *Thus* $r \equiv \frac{-1}{3} \bmod p$ *and* $r \not\equiv 0 \bmod p$. *Let* $\lambda \in \mathrm{Lin}(H)$ *be the character given by*

$$\lambda(m(a_0 + a_1x + a_3x^3)) = \omega^{ra_0 + ra_1 + a_3}. \tag{4.22}$$

THEOREM 4.23. *Assume Hypothesis* 4.18. *Then*

$$\lambda^G = \chi_0 + 3\sum_{i=1}^{\frac{p-1}{3}} \chi_i \tag{4.24}$$

where $\chi_i \in \mathrm{Irr}(G)$ *and* $\chi_i \neq \chi_j$ *if* $i \neq j$ *for* $i, j = 0, 1, \ldots, \frac{p-1}{3}$. *Thus* $\eta(\lambda) = \frac{p+2}{3}$.

Proof. The center $\mathbf{Z}(G)$ of G is the subgroup $\langle m(x^3)\rangle$ of order p. Let γ be the faithful linear character of $\mathbf{Z}(G)$ sending $m(x^3)$ to ω. Then $\mathrm{Lin}(M \mid \gamma)$ consists of the p^3 linear characters μ_{f_0,f_1,f_2}, for $f_0, f_1, f_2 \in F$ given by

$$\mu_{f_0,f_1,f_2}(m(a_0 + a_1x + a_2x^2 + a_3x^3)) = \omega^{f_0a_0 + f_1a_1 + f_2a_2 + a_3} \tag{4.25}$$

for all $a_0, a_1, a_2, a_3 \in F$. If $e, i, j \in F$, then (4.19) and (4.20) imply that the conjugate character $\mu_{e,0,0}^{(1+x)^{-i}(1+x^2)^{-j}}$ to $\mu_{e,0,0}$ sends

$$\begin{aligned}
m(1) &\mapsto \mu_{e,0,0}\left(m\left(1 + ix + \left(\binom{i}{2} + j\right)x^2 + \left(\binom{i}{3} + ij\right)x^3\right)\right) = \omega^{e + \binom{i}{3} + ij},\\
m(x) &\mapsto \mu_{e,0,0}\left(m\left(x + ix^2 + \left(\binom{i}{2} + j\right)x^3\right)\right) = \omega^{\binom{i}{2} + j},\\
m(x^2) &\mapsto \mu_{e,0,0}(m(x^2 + ix^3)) = \omega^i,\\
m(x^3) &\mapsto \mu_{e,0,0}(m(x^3)) = \omega.
\end{aligned}$$

It follows that

$$\mu_{e,0,0}^{(1+x)^{-i}(1+x^2)^{-j}} = \mu_{e + \binom{i}{3} + ij, \binom{i}{2} + j, i} \tag{4.26}$$

for any $e, i, j \in F$. If we fix e, then the above equation implies that distinct pairs $(i, j) \in F \times F$ yield distinct conjugates $\mu_{e,0,0}^{(1+x)^{-i}(1+x^2)^{-j}} \in \mathrm{Lin}(M \mid \gamma)$. Hence the G-orbit L_e of $\mu_{e,0,0}$ has exactly p^2 members. Furthermore the above equation implies that the only member of that orbit with the form $\mu_{f,0,0}$ is $\mu_{e,0,0}$. We conclude that the orbits

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L_e, for $e \in F$, are p distinct G-orbits in $\mathrm{Lin}(M \mid \gamma)$, each with size p^2. Since the normal subgroup M of index p^2 is exactly the stabilizer of $\mu_{e,0,0} \in \mathrm{Lin}(M)$ in G, the induced characters

$$\chi_e = \mu_{e,0,0}^G \text{ are precisely the distinct members of } \mathrm{Irr}(G \mid \gamma). \tag{4.27}$$

Then

$$\lambda^M = \sum_{f \in F} \mu_{r,r,f} \text{ and } \lambda^G = \sum_{f \in F} \mu_{r,r,f}^G. \tag{4.28}$$

CLAIM 4.29. *Let* $i \in \{1, \dots, p-1\}$, $e = r(1 - i^3)$ *and* $j = r - \binom{i}{2}$. *Then*

$$\mu_{e,0,0}^{(1+x)^{-i}(1+x^2)^{-j}} = \mu_{r,r,i}. \tag{4.30}$$

Proof. For a fixed i, we have

$$\begin{aligned} e + \binom{i}{3} + ij &= e + \binom{i}{3} + i\left(r - \binom{i}{2}\right) \\ &= e + \frac{i(i-1)(i-2)}{6} + i\left(r - \frac{i(i-1)}{2}\right) \\ &= i^3\left(\frac{1}{6} - \frac{1}{2}\right) + i^2\left(\frac{1}{2} - \frac{1}{2}\right) + i\left(r + \frac{1}{3}\right) + e \\ &\equiv \frac{-i^3}{3} + e \bmod p, \quad \text{since } r \equiv \tfrac{-1}{3} \bmod p \\ &\equiv \frac{-i^3}{3} + r(1 - i^3) \bmod p, \quad \text{since } e = r(1 - i^3) \\ &\equiv r - i^3\left(r + \frac{1}{3}\right) \equiv r \bmod p, \end{aligned}$$

where the last line follows since $r \equiv \frac{-1}{3} \bmod p$. Thus $(e + \binom{i}{3} + ij, \binom{i}{2} + j, i) = (r, r, i)$ in $F \times F \times F$ and so by (4.26) we get (4.30). □

By the previous claim and (4.28), we have that

$$\lambda^G = \sum_{i=0}^{p-1} \mu_{r(1-i^3),0,0}^G.$$

By Lemma 4.17, we have then

$$\lambda^G = \mu_{r,0,0}^G + 3 \sum_{e \in \{r(1-i^3) \mid i=1,\dots,p-1\}} \mu_{e,0,0}^G. \tag{4.31}$$

By (4.27) we have that $\mu_{e,0,0}^G \in \mathrm{Irr}(G)$ and $\mu_{e,0,0}^G \neq \mu_{f,0,0}^G$ if $e \not\equiv f \bmod p$. Thus by Lemma 4.17 and (4.31), we conclude that $\eta(\lambda^G) = \frac{p+2}{3}$ and the proof is complete. □

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