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CONJUGACY CLASSES OF SUBGROUPS IN p-GROUPS

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Abstract

The set $C(G)$ of conjugacy classes of subgroups of a group G has a natural partial order. We study p-groups G for which $C(G)$ has antichains of prescribed lengths.

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1. The results

In recent years, there has been a considerable interest in the set $C(G)$ of all conjugacy classes $[S]$ of subgroups S of a group G (see [2], [3] and the references mentioned there). This set $C(G)$ has a natural partial order defined as follows. A class $[S_1]$ is smaller than the class $[S_2]$ if and only if at least one element in $[S_1]$ is contained in an element of $[S_2]$ or, equivalently, if some conjugate of S_1 is contained in S_2. Even for relatively large groups, the poset $C(G)$ is quite small, a feature that is not shared by the lattice of all subgroups of a group G.

In this paper we study some order-theoretic properties of the poset $C(G)$ and investigate its influence on the group G. More precisely, we are interested in the Dilworth number of $C(G)$, that is, the maximum possible cardinality of an antichain in $C(G)$.

DEFINITION. Let G be a group. The *Möbius-width* $w_c(G)$ is the maximum number t of subgroups $S_1, \dots, S_t$ of G with the property that no S_i is conjugate to any subgroup of S_j for every $j \neq i$ (if there is no such t,

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then we set $w_c(G) = \infty$. Moreover, if G is the trivial group, then we define $w_c(G) = 0$).

An important result of Landau [5] states that for every given positive integer n, there exist only finitely many finite groups with precisely n conjugacy classes of elements. A similar result for the width was shown in [1]: for every $n > 1$ there exist only finitely many finite p-groups whose lattice of *all* subgroups has Dilworth number n.

For the Möbius width, an analogous result is not true in general, as for example, all dihedral groups G of 2-power order satisfy $w_c(G) = 3$. For $n > 3$, however, we have the following finiteness theorem:

THEOREM A. *For every $n > 3$ there exist only finitely many primes p and finitely many p-groups G satisfying $w_c(G) = n$.*

Clearly, every antichain of normal subgroups (with containment as the inclusion relation) forms an antichain in $C(G)$, and hence a noncyclic p-group G must satisfy $w_c(G) \geq p + 1$. An an illustration of Theorem A, we determine p-groups of small Möbius width.

THEOREM B. *Let G be a finite p-group with $w_c(G) = p + 1$. Then one of the following occurs:*

(a) *$p = 2$ and G is a dihedral, (generalized) quaternion or a quasidihedral group,*

(b) *$p > 2$ and $G \cong C_p \times C_p$ or G is nonabelian of order p^3 and exponent p^2.*

The next possible value for the Dilworth number of the subgroup lattice of a finite p-group is $2p$ (see [1]). For the Möbius width, the next following value is smaller.

THEOREM C. *Let G be a finite p-group of Möbius width $p + 2$. Then one of the following occurs:*

(a) *$p = 2$ and $G \cong C_2 \times C_4$ or $G \cong \langle a, b \mid a^8 = b^2 = 1, a^b = a^5 \rangle$;*

(b) *$p \geq 3$ and G is nonabelian of order p^3 and exponent p;*

(c) *$p \geq 3$ and $G \cong \langle a, b, c \mid a^p = b^p = c^{p^2} = [b, c] = 1, b^a = bc^{sp}, c^a = cb \rangle$ where $s = 1$ or s is a quadratic nonresidue* $\bmod p$. *If $p = 3$, then we have to add the group $G \cong \langle a, b, x \mid a^9 = b^3 = [a, b] = 1, [a, x] = b, [b, x] = a^3, x^3 = a^3 \rangle$.*

We say that a conjugacy class in $C(G)$ is of type H for some group H, if all of its members are isomorphic to H. The length of a conjugacy class is the number of its members, the cyclic group of order n will be denoted by C_n. All further unexplained notation can be found in [4]; moreover, all groups considered in this paper are finite.

2. Finitely many p-groups

This section is devoted to a proof of Theorem A. But first, we introduce some notation to facilitate the exposition.

DEFINITION. Let G be a group. A collection $S_1, \ldots, S_t$ of subgroups of G is called an *antichain* of G with respect to conjugacy (for short, a *c-antichain*) if for all indices i, j with $i \neq j$, S_i is not conjugate to any subgroup of S_j.

Thus, the Möbius width $w_c(G)$ of a group $G \neq 1$ is nothing else but the maximum taken over all cardinalities t of c-antichains in G.

The following elementary result will be used without further mention.

LEMMA 1. *Let N be a normal subgroup of the group G. Then $w_c(G/N) \leq w_c(G)$.*

In the course of our investigations, we shall frequently consider c-antichains in G that are contained in some normal subgroup N of G.

DEFINITION. Let N be a normal subgroup of the group G. Then we define $w_c^G(N)$ to be the maximum over the lengths of all c-antichains in G, consisting of subgroups contained in N.

Clearly, we have $w_c^G(N) \leq w_c(G)$ and $w_c^G(G) = w_c(G)$.

For the proof of Theorem A, we first investigate certain abelian normal subgroups of G and their connections to $w_c(G)$.

LEMMA 2. *Let G be a p-group and assume that G possesses a normal subgroup N of exponent p and order p^a , say. Then $a \leq w_c(G)$.*

PROOF. Let $1 = N_0 \leq N_1 \leq \cdots \leq N_a = N$ be part of a chief series of G. For $1 \leq i \leq a$, choose $x_i \in N_i \backslash N_{i-1}$. Then all G-conjugates of x_i belong to $N_i \setminus N_{i-1}$. Moreover, all x_i are of order p and hence $\langle x_1 \rangle, \ldots, \langle x_a \rangle$ forms a c-antichain in G . □

The next preparatory result provides some information on the exponent of abelian normal subgroups of G .

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LEMMA 3. *Let N be an noncyclic abelian normal p-subgroup of a group G. If* $\exp(N) = p^a$, *then* $a/2 \le w_c^G(N)$.

PROOF. As N is noncyclic, there exists a direct summand $A = \langle x\rangle \oplus \langle y\rangle$ of N such that $o(x) = p^a$ and $o(y) = p^b$ with $1 \le b \le a$.

Case 1. $b \ge a/2$. For $0 \le i \le a/2$, set $S_i = \langle x^{p^i}, y^{p^{b-i}}\rangle$. Then $S_i \cong C_{p^{a-i}} \times C_{p^i}$, and hence the S_i are pairwise nonisomorphic. As all S_i are of the same order, they clearly form a c-antichain and the result follows here.

Case 2. $b < a/2$. For $0 \le i \le a-b$, consider the cyclic subgroups T_i of N, defined by $T_i = \langle x^{p^i} y\rangle$. Then $|T_i| = p^{a-i}$ for all i. If $T_i^g \le T_j$ for some $g \in G$ and some $i \ne j$, then we must have $T_i^g \le T_j^p$, because T_j is cyclic. As N^p is normal in G, this implies $T_i \le (T_j^p)^{g^{-1}} \le N^p \le \Phi(N)$. But this contradicts the fact that $\langle y\rangle$ is a direct summand of N. Hence, $T_0, \dots, T_{a-b}$ forms a c-antichain in G and we have $w_c^G(N) \ge a-b > a/2$ as claimed. □

Proof of Theorem A. First, note that $n = w_c(G) \ge w(G/G') \ge p+1$, so $p \le n-1$ and there are only finitely many primes p. If all abelian normal subgroups of G are cyclic, then by [4, p. 304], either G is a 2-group of maximal class and hence $w_c(G) = 3 < n$, or $G \cong \langle a, b | a^{2^m} = b^2 = 1,\ a^b = a^{1+2^{m-1}}\rangle$. But in the latter case, we have $G/G' \cong C_2 \times C_{2^{m-1}}$ and so [1] implies $n = w_c(G) \ge w(G/G') \ge m+1$.

Now assume that G contains a noncyclic abelian normal subgroup N, say. We may take N maximal with these properties, and so we have $N = C_G(N)$. First, Lemma 2, applied to $\Omega_1(N)$ yields that the rank $r_p(N)$ of N satisfies $r_p(N) \le w_c^G(N) \le n$. Moreover, Lemma 3 yields that $exp(N)$ divides p^{2n}. Thus, the order of N is bounded by some function of n. Moreover, $G/N = G/C_G(N)$ embeds into $\mathrm{Aut}(N)$ and hence there are only finitely many possibilities for the order of G. (Indeed, from [4, page 302], we can deduce an explicit upper bound for the order of G.) The result follows. □

3. p-groups of small order

In this section, we determine the posets $C(G)$ and their Dilworth number for p-groups G of order $\le p^4$. For the proof of Theorems B and C, it is sufficient to derive lower bounds for $w_c(G)$ in a number of cases. To facilitate the exposition, we do not attempt to derive the exact value here, but rather present a somewhat better bound without proof.

First, we recall some easy facts on the width of abelian groups.

LEMMA 4 ([1]). *Let* p *be a prime.*

(a) *We have* $w_c(C_p \times C_p) = p + 1$ *and* $w_c(C_p \times C_{p^2}) = 2p$.

(b) *If* G *is a noncyclic abelian* p*-group not mentioned in* (a), *then we have* $w_c(G) \geq 3p - 1$.

The proof of the following simple result on groups of order p^3 illustrates the basic method that we shall use several times in this section. During the description of $C(G)$, if we state, without further comment, that a conjugacy class of subgroups contains several others, it is tacitly understood that those listed are the *only* ones with this property. This then determines the poset $C(G)$ and $w_c(G)$ can be read off. The reader is encouraged to draw his own pictures of $C(G)$ as an amusing exercise. Throughout the remainder of this section, p *will denote an odd prime.*

LEMMA 5. *Let* G *be a nonabelian group of order* p^3.

(a) *If* $\exp(G) = p^2$, *then* $w_c(G) = p + 1$.

(b) *If* $\exp(G) = p$, *then* $w_c(G) = p + 2$.

PROOF. (a) Here, we start with the maximal subgroups of G. Indeed, G contains precisely p maximal subgroups of type C_{p^2}, each of which contains exactly one subgroup of order p, namely $G^p = Z(G)$ which forms a conjugacy class of length 1. The remaining maximal subgroup $\Omega_1(G)$ contains $Z(G)$ and p further groups of order p that form a single conjugacy class. Thus, $w_c(G) = p + 1$.

(b) Here, G has $p^2 + p + 1$ subgroups of order p, one of them being $Z(G)$. All others are nonnormal and so they fall into $p + 1$ conjugacy classes of length p. The centre $Z(G)$ belongs to every maximal subgroup of G and each of the remaining classes of type C_p is contained in precisely one maximal subgroup of G, their centraliser. Thus, $w_c(G) = p + 2$. □

Now consider groups of order p^4 where p is odd. From the notation of [4, page 346 f.], we indicate the groups of order p^4 by their numbers, so $G_1, \dots, G_5$ are abelian and for obvious reasons, we do not insist on an ordering of the isomorphism types here. Moreover, $G_6 = \langle a, b | a^{p^3} = b^p = 1,\ a^b = a^{1+p^2} \rangle$ and G_9 corresponds to the case $s = 1$ while G_{10} is the group where s is a nonsquare $\bmod p$. For $p = 3$, there is an extra group of order 3^4 (see [4, p. 349]) which we will denote by G_{ex}. Thus, $G_{ex} = \langle a, b, x | a^9 = b^3 = [a, b] = 1,\ [a, x] = b,\ [b, x] = a^3,\ x^3 = a^3 \rangle$.

Five of the nonabelian groups are very easy to deal with.

LEMMA 6. *Let* $p \geq 3$ *and let* G *be one of the following groups:* $G_6, G_7, G_8, G_{11}, G_{14}$. *Then* $w_c(G) \geq 2p$.

More explicitly, we have: $w_c(G_6) = 2p$, $w_c(G_7) \geq 2p+1$, $w_c(G_8) \geq p^2+p+1$, $w_c(G_{11}) \geq p^2+p+1$ *and* $w_c(G_{14}) \geq p^2+p+1$.

PROOF. The groups G_6 and G_7 map onto $C_p \times C_{p^2}$ and the remaining ones map onto $C_p \times C_p \times C_p$. The result follows from Lemma 4. The proof of the remaining statements is omitted. □

Now we consider the groups of smallest Möbius width.

LEMMA 7. *Let* $p \geq 3$. *Then* $w_c(G_9) = w_c(G_{10}) = p+2$.

PROOF. First, note that G is a split extension of the normal subgroup $N = \langle y, z\rangle \cong C_p \times C_{p^2}$ by the cyclic group $\langle x\rangle$ of order p. Moreover, G is of maximal class and hence $\Omega_1(N) = \langle y, z^p\rangle$ is the unique normal subgroup of order p^2 in G.

We first deal with the case $p \geq 5$. Then G is regular. As $\exp(G) = p^2$, we see that $\Omega_1(G) = \langle x, y, z^p\rangle$ is nonabelian of order p^3 and exponent p. Now there are precisely p^2 cyclic subgroups of order p^2 in G. By the above, none of them is normal in G, and hence each has precisely p conjugates. Thus, there are p classes of type C_{p^2}. Also, the normaliser of a cyclic subgroup of order p^2 in G is a group of order p^3 and exponent p^2 (indeed, one of them is abelian, namely N, and all others are nonabelian).

Now there are three "types" of elements of order p, namely the central ones z^p, the abelian ones contained in $\Omega_1 \backslash \langle z^p\rangle$ and the ones "outside". An inspection of these shows that there are three classes of type C_p with representatives $\langle z^p\rangle, \langle y\rangle$ and $\langle x\rangle$. In fact, among the p maximal subgroups of exponent p^2, the $p-1$ nonabelian ones contain the class with representative $\langle y\rangle$ and N contains $\langle z^p\rangle$.

Next, consider subgroups of type $C_p \times C_p$. Each of the above normal subgroups of exponent p^2 contains precisely one characteristic subgroup of order p^2, their Ω_1. As this is normal in G, it must coincide with $\Omega_1(N)$.

Now the remaining maximal subgroup is $\Omega_1(G)$, which is of exponent p. This clearly contains $\Omega_1(N)$ which is the only one which is normal in G. As it contains precisely p further such subgroups, they are all conjugate in G. Thus, we have determined $C(G)$ and finally, we see that $w_c(G) = p+2$.

The case $p = 3$ is similar. □

The extra group G_{ex} has a "similar" Moebius-poset and we omit the proof of the following result.

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LEMMA 8. *Let* $p = 3$. *Then* $w_c(G_{ex}) = 5 = p + 2$.

The remaining groups can be dealt with by using similar methods and we only present the result.

LEMMA 9. (a) *We have* $w_c(G_{12}) \geq 2p + 1$ *if* $p = 3$, *and* $w_c(G_{12}) \geq 2p + 2$ *if* $p \geq 5$.

(b) *We have* $w_c(G_{13}) \geq 2p + 2$ *if* $p = 3$, *and* $w_c(G_{13}) \geq 2p + 3$ *if* $p \geq 5$.

(c) *We have* $w_c(G_{15}) \geq 2p + 2$ *if* $p = 3$, *and* $w_c(G_{15}) \geq 2p + 3$ *if* $p \geq 5$.

Part b) of Theorem B follows from the results of this section, because all noncyclic groups of order at least p^4 have Möbius width $\geq p + 2$.

4. Larger groups ?

The proof of Theorem C is by showing that all noncyclic p-groups of Möbius width $\leq p + 2$ are of order $\leq p^4$. Now in Section 3, we have determined all such groups of this order, and so, by way of contradiction, we may assume that there exists a noncyclic p-group H of order $\geq p^5$ satisfying $w_c(H) \leq p + 2$. Clearly then, there exists a noncyclic factor group $\widetilde{G}$ of H with the following properties: $|\widetilde{G}| = p^5$ and $w_c(\widetilde{G}) \leq p + 2$. Obviously, $\widetilde{G}$ is nonabelian, so let M be a minimal normal subgroup of $\widetilde{G}$ contained in $\widetilde{G}'$. Thus, $M \leq \widetilde{G}' \cap Z(\widetilde{G})$, and hence $G = \widetilde{G}/M$ has a nontrivial Schur multiplicator. By Section 3, we have $G \cong G_9, G_{10}$ or G_{ex}. We shall keep this notation for the rest of this section.

LEMMA 10. *Let* $G = G_9$ *or* G_{10} *if* $p \geq 5$, *or* $G = G_9$ *if* $p = 3$ *or* $G = G_{ex}$. *Then* $w_c(\widetilde{G}) \geq 2p$.

PROOF. We first show that $Z(\widetilde{G})$ is elementary abelian of rank two. For this, we use some information on the quotient G of $\widetilde{G}$ that we know about. In all cases, $G = \widetilde{G}/M$ contains precisely $p - 1$ maximal subgroups $E_1/M, \ldots, E_{p-1}/M$ which are nonabelian of exponent p^2 and of order p^3. For $1 \leq i \leq p - 1$, we have $M \leq Z(E_i)$. As E_i/M has trivial Schur multiplicator, we must have $M \cap E_i' = 1$. As E_i/M is of class two, this implies $[E_i, E_i'] \leq M \cap E_i' = 1$, and hence E_i is of class two. As $M \leq Z(E_i)$, this shows that $M \times E_i' \leq Z(E_i)$. Let $Z/M = Z(\widetilde{G}/M)$, so Z/M is of order p. Then $ME_i'/M \leq Z(\widetilde{G}/M) = Z/M$, and hence we have $M \times E_i' = Z$. As $M \times E_i' \leq Z(E_i)$ for $i = 1, 2$, and $E_1E_2 = \widetilde{G}$, we get $Z \leq Z(\widetilde{G})$ and our claim is proved.

We now show that $w_c(\widetilde{G}) \geq 2p$. Indeed, Z contains p normal subgroups $C_1, \dots, C_p$ of order p, distinct from M. Thus, we have $C_jM/M = Z(\widetilde{G}/M)$ for all j. Now let $D_1/M, \dots D_p/M$ be representatives of the p classes of cyclic subgroups of order p^2 in $\widetilde{G}/M$. As these are pairwise incomparable with $Z(\widetilde{G}/M)$, the subgroups $C_1, \dots, C_p, D_1, \dots D_p$ of $\widetilde{G}$ form a c-antichain in $\widetilde{G}$, and we therefore have $w_c(\widetilde{G}) \geq 2p$. □

We now deal with the exceptional case when $p = 3$.

Lemma 11. *Let $G = G_{10}$ and assume that $p = 3$ and $s = -1$. Then $w_c(\widetilde{G}) \geq 6$.*

Proof. First, $\widetilde{G}/M$ is of maximal class, and hence it has precisely one normal subgroup of order 3. Moreover, by Lemma 7, $G = \widetilde{G}/M$ has a c-antichain $S_1/M, \dots, S_5/M$ of cyclic subgroups of order 3, precisely one of them is normal in G, say S_1/M. Let W/M be the maximal subgroup of G isomorphic to $C_3 \times C_9$. We show that one of the following conditions hold:

(α) W contains two distinct characteristic subgroups or order 9;

or

(β) W contains a subgroup U of order 9 with $M \cap U = 1$.

In both cases, it follows that $w_c(\widetilde{G}) \geq 6$. Indeed, if ($\alpha$) holds, then we can choose a characteristic subgroup C of order 9 of W with $C \neq S_1$. Then C is normal in $\widetilde{G}$ and $S_1, \dots, S_5, C$ is a c-antichain in $\widetilde{G}$. In the situation (β), no conjugate of U contains M. As all conjugates of $S_1, \dots, S_5$ contain M, we have that $S_1, \dots, S_5, U$ is a c-antichain and in both cases, it follows that $w_c(\widetilde{G}) \geq 6$.

We now prove the above claim. Indeed, if W contains an abelian subgroup A of rank ≥ 3, then we clearly can choose U as a suitable subgroup of A and we have (β). If W is abelian, there are two more cases: if $W \cong C_3 \times C_{27}$, we have ($\alpha$) and if $W \cong C_9 \times C_9$, we have (β). So let W be nonabelian. As $W/M \cong C_3 \times C_9$, the list of all groups of order 3^4 gives two more possibilities for W. If $W \cong \langle a, b \mid a^{27} = b^3 = 1,\ a^b = a^{10} \rangle$, we have $\Omega_1(W) \neq W^3$ and (α) holds. Finally, in $W \cong \langle a, b \mid a^9 = b^9 = 1,\ a^b = a^4 \rangle$, we must have $M = W' =< a^3 >$ and $U :=< b >$ has trivial intersection with M. The result follows. □

By the remarks on the beginning of this section, there are no noncyclic p-groups of order $\geq p^5$ and Möbius width $\leq p + 2$, and so parts (b) and (c) of Theorem C are proved.

5. The case $p = 2$

In this final section, we consider groups of even order. Here, the situation is quite different as the 2-groups G of maximal class satisfy $w_c(G) = 3$. However, it turns out that there are only finitely many additional examples.

PROPOSITION 12. *Let G be a 2-group satisfying $w_c(G) \leq 4$. Then one of the following holds*:
(i) *G is cyclic*;
(ii) $G \cong C_2 \times C_4$;
(iii) $G/G' \cong C_2 \times C_2$ *and G is of maximal class*;
(iv) $G \cong \langle a, b \mid a^8 = b^2 = 1, \ a^b = a^5 \rangle$.

PROOF. Let G be a counterexample of least possible order. Lemma 4 implies that G is nonabelian. If $G/G' \cong C_2 \times C_2$, then [4, p. 339 f.] shows that we have (iii). By Lemma 4 again, we have $G/G' \cong C_2 \times C_4$. Let M be a minimal normal subgroup of G contained in G'. Then $M \leq G' \cap Z(G)$ and hence G is a covering group of $Q := G/M$. As $w_c(Q) \leq w_c(G)$, induction applies. Clearly, Q cannot be of type (i). If Q is of type (iii), then G is of type (iii). As the group mentioned in (iv) has a trivial Schur multiplicator, this case cannot occur here, so let Q be of type (ii). An inspection of all groups of order 16 shows that we are left with three possibilities. First, G may well be of type (iv). The next possibility would be

$$G \cong \langle a_1, a_2, x \mid a_1^4 = a_2^2 = x^2 = [a_1, a_2] = [a_2, x] = 1, \ a_1^x = a_1 a_2 \rangle.$$

Here, $\Omega_1(G) \cong C_2 \times C_2 \times C_2$ contains seven Klein groups each of which has at most two conjugates. As G contains cyclic subgroups of order 4, we have $w_c(G) \geq 5$ here. The last possibility is $G \cong \langle a, b \mid a^4 = b^4 = 1, \ a^b = a^3 \rangle$. Here, $\Omega_1(G) = \langle a^2, b^2 \rangle$ is of order 4, and hence G contains six cyclic subgroups of order 4. At least one of them is normal and the remaining ones have at most two conjugates. These classes, together with $\Omega_1(G)$, form a c-antichain of length 5 and our result follows. □

References

[1] R. Brandl, 'The Dilworth number of subgroup lattices', *Arch. Math.* **50** (1988), 502–510.
[2] T. Hawkes and I.M. Isaacs, 'On the poset of p-subgroups of a p-solvable group', *J. London Math. Soc.* (2) **38** (1988), 77–86.

[3] T. Hawkes, I.M. Isaacs and M. Özaydin, 'On the Möbius function of a finite group', *Rocky Mountain J. Math.* **19** (1989), 1003–1034.
[4] B. Huppert, *Endliche Gruppen* I (Springer-Verlag, Berlin, 1967).
[5] E. Landau, 'Über die Klassenzahl der binären quadratischen Formen von negativer Discriminante', *Math. Ann.* **56** (1903), 671–676.

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