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THE SPEED OF A BIASED WALK ON A GALTON–WATSON TREE WITHOUT LEAVES IS MONOTONIC FOR LOW VALUES OF BIAS

HE SONG,* *Huaiyin Normal University*
LONGMIN WANG,** *Nankai University*
KAINAN XIANG,*** *Xiangtan University*

Abstract

We show that for $\lambda \in [0, m_1/(1+\sqrt{1-1/m_1})]$, the biased random walk's speed on a Galton–Watson tree without leaves is strictly decreasing, where $m_1 \geq 2$. Our result extends the monotonic interval of the speed on a Galton–Watson tree.

Keywords: Galton–Watson tree; biased random walk; speed; monotonicity

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1. Introduction

In this paper, we investigate the properties of biased random walks denoted as RW_λ on Galton–Watson trees $\mathbb{T}$. We focus our attention on the following inquiry: Is the speed of RW_λ monotonic non-increasing as a function of its bias λ when the Galton–Watson tree has no leaves?

Let $\mathbb{T}$ denote a Galton–Watson tree with the root e, ν represent the offspring distribution with $m = \mathbb{E}(\nu) > 1$. Denote the associated probability space as $(\Omega, \mathbb{P})$. Note that $\mathbb{T}$ is supercritical and the extinction probability $q = \mathbb{P}[\mathbb{T} \text{ is finite}] < 1$. Let $\nu(x)$ denote the number of children of a vertex $x \in \mathbb{T}$. For any $x \in \mathbb{T} \setminus \{e\}$, let x_* be the parent of x, which refers to the neighbor of x that lies on a geodesic path from x to e. We write xi, $1 \leq i \leq \nu(x)$, as the children of x.

For any $\lambda \geq 0$, a λ-biased random walk RW_λ, $(X_n)_{n=0}^\infty$, on the Galton–Watson tree $\mathbb{T}$ is defined as follows. The transition probability from x to an adjacent vertex y is

$$p(x, y) = \begin{cases} \dfrac{1}{\nu(x)} & \text{if } x = e, \\ \dfrac{\lambda}{\lambda + \nu(x)} & \text{if } y = x_*, \ x \neq e, \\ \dfrac{1}{\lambda + \nu(x)} & \text{otherwise.} \end{cases}$$

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* Postal address: School of Mathematics and Statistics, Huaiyin Normal University Huaiyin 223300, P. R. China. Email: songhe@hytc.edu.cn
** Postal address: School of Statistics and Data Science, Nankai University Tianjin 300071, P. R. China. Email: wanglm@nankai.edu.cn
*** Postal address: School of Mathematics and Computational Science, Xiangtan University Xiangtan City 210000, Hunan Province, P. R. China. Email: kainan.xiang@xtu.edu.cn

$(X_n)_{n=0}^{\infty}$ is a reversible Markov chain for $\lambda > 0$. Let $\mathbf{P}_x$ be the quenched probability of RW_λ starting at x and $\mathbb{P}_x$ the annealed probability obtained by the semi-direct product $\mathbb{P}_x = \mathbb{P} \times \mathbf{P}_x$. Denote the respectively associated expectations by $\mathbf{E}_x$ and $\mathbb{E}_x$. A motivation for the introduction of RW_λ on trees is its ability to obtain almost uniform samples from the set of self-avoiding walks of a given length on a lattice [6]. For further insights into the motivations for biased random walks on graphs, see the existing surveys [3, 12].

On a general tree, [8] showed that there exists a critical parameter λ_c for RW_λ, where λ_c is equal to the exponential of the Hausdorff dimension of the tree boundary. It was found that RW_λ exhibits transient behavior for $\lambda < \lambda_c$ and recurrent behavior for $\lambda > \lambda_c$. It was also proved that, for almost every Galton–Watson tree conditioned on non-extinction, RW_λ is transient for $0 \le \lambda < m$. In [9], it was shown that, conditional on non-extinction, RW_m is null recurrent, while RW_λ is positive recurrent for $\lambda > m$.

Let $|x|$ be the graph distance between x and e for any vertex $x \in \mathbb{T}$. Note that $|x|$ is also the generation of x. Fix $X_0 = e$. The speed ℓ_λ of RW_λ is defined as the almost sure limit, if it exists, of the ratio $|X_n|/n$ as $n \to \infty$. In this paper, the dependence of ℓ_λ with respect to the environment will often be omitted. A transient RW_λ may exhibit zero speed when an excessive amount of time is allocated to the leaves. In [11], it was proved that, conditional on non-extinction, ℓ_λ exists almost surely and ℓ_λ is deterministic and positive if and only if $\lambda \in (\mathbb{E}(\nu q^{\nu-1}), m)$. From [10], $\ell_1 = \mathbb{E}((\nu - 1)/(\nu + 1))$. An expression for ℓ_λ, shown in (1), was specified in [2], though an artificial parent to e was added there. For related results, see [7].

The following problem was raised in [11] (see also [12]), and was called the Lyons–Pemantle–Peres monotonicity problem in [4].

Problem 1. *([11].)* Assume $\mathbb{P}(\nu = 0) = 0$, namely that the Galton–Watson tree $\mathbb{T}$ has no leaf, meaning that the extinction probability $q = 0$. Is the speed ℓ_λ of RW_λ on $\mathbb{T}$ monotonic non-increasing in $\lambda \in [0, m)$?

It was conjectured in [11, 12] that Problem 1 should have a positive answer. If we consider general trees, monotonicity does not hold. Moreover, we should notice that speed might not exist on general trees. For instance, on a binary tree with pipes (a binary tree to every vertex of which is joined a unary tree), which is a multi-type Galton–Watson tree, the speed is $(2 - \lambda)(\lambda - 1)/(\lambda^2 + 3\lambda - 2)$ for $1 \le \lambda \le 2$ [12]. For any $0 < \lambda_1 < \lambda_2$, by the repeated filtering method we can produce a tree such that the speed of RW_{λ_1} is less than that of RW_{λ_2} [12]. Notice that these examples are not Galton–Watson trees and show the complexity of Problem 1. Therefore, the monotonicity of ℓ_λ would represent a highly significant and fundamental characteristic of Galton–Watson trees.

The Lyons–Pemantle–Peres monotonicity problem for Galton–Watson trees without leaves was proven to have a positive solution for $\lambda \le m_1/1160$ in [4], where $m_1 = \min\{k \ge 1 : P[\nu = k] > 0\}$ is the minimal degree of the Galton–Watson tree. This result was improved in [1] to $\lambda \le \frac{1}{2}$ by a completely different approach. In [5], the Einstein relation was obtained for RW_λ on Galton–Watson trees, which implies that Problem 1 holds in a neighborhood of m. These slow advances indicate that Problem 1 is rather difficult. For more information on RW_λ on $\mathbb{T}$, see [3, 12] and references therein. For the monotonicity of the speed of biased random walks on groups, see [14, 15].

The main result of our study is presented as follows.

Theorem 1. *The speed ℓ_λ of RW_λ on a Galton–Watson tree $\mathbb{T}$ without leaves is strictly decreasing in $\lambda \in [0, m_1/(1 + \sqrt{1 - 1/m_1})]$ when $m_1 \ge 2$.*

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2. Proof of Theorem 1

Inspired by [1], based on some new observations, we prove Theorem 1. Let $\mathbb{T}_*$ be the tree obtained from $\mathbb{T}$ by adding an artificial parent e_* to the root e. For any vertex $x \in \mathbb{T}_*$, let $\tau_x = \min\{n \geq 0, X_n = x\}$, where $\min \emptyset = \infty$ and $(X_n)_{n=0}^{\infty}$ is a λ-biased random walk on $\mathbb{T}_*$. For $x \neq e_*$, let $\beta(x) := \beta(x, \lambda) = \mathbf{P}_x(\tau_{x_*} = \infty)$ be the quenched probability of never reaching the parent x_* of x when starting from x. Since $\mathbb{T}$ has no leaf and $\lambda < m$, we have $\beta(x) > 0$ due to transience. Let $(\beta_i)_{i\geq 0}$ be generic independent and identically distributed random variables distributed under $\mathbb{P}$ as $\beta(e)$, and independent of ν.

The following expression for ℓ_λ was given in [2]:

$$\ell_\lambda = \frac{\mathbb{E}\left((\nu-\lambda)\beta_0/\left(\lambda-1+\sum_{i=0}^{\nu}\beta_i\right)\right)}{\mathbb{E}\left((\nu+\lambda)\beta_0/\left(\lambda-1+\sum_{i=0}^{\nu}\beta_i\right)\right)}, \quad \lambda < m. \tag{1}$$

Notice that (1) holds trivially when $\lambda = 0$. In this context, it is important to note that RW_λ on $\mathbb{T}_*$ and RW_λ on $\mathbb{T}$ have a slight difference, but due to $\lambda < m$ and transience these two biased random walks have the same speed when starting at e. Indeed, we have the following result.

Lemma 1. *For $\lambda < m$, RW_λ on $\mathbb{T}_*$ and RW_λ on $\mathbb{T}$ have the same speed when starting at e.*

Proof. Since the random walk on RW_λ $(X_n)_{n=0}^{\infty}$ on $\mathbb{T}_*$ is transience, the edge ee_* (ee_* denotes the edge that connects the vertices e and e_*) can be visited only a finite number of times. We can define $K = \sup\{n, X_n \notin \{e, e_*\}\}$, where K is finite. And RW_λ $(X_n)_{n=K+1}^{\infty}$ is a biased random walk on $\mathbb{T}$. This implies the lemma.

For the reader's convenience, we provide a more detailed proof as follows.

For RW_λ $(X_n)_{n=0}^{\infty}$ on $\mathbb{T}_*$ with $X_0 = e$, define the following regenerative times [16]:

- $\tau_0 = 0$, $\sigma_0 = \inf\{n \geq \tau_0 : X_n \notin \{e, e_*\}\}$;
- $\tau_1 = \inf\{n \geq \sigma_0 : X_n = e\}$, $\sigma_1 = \inf\{n \geq \tau_1 : X_n \notin \{e, e_*\}\}$ when $\tau_1 < \infty$;
- for any $i \geq 1$, $\tau_{i+1} = \inf\{n \geq \sigma_i : X_n = e\}$, $\sigma_{i+1} = \inf\{n \geq \tau_{i+1} : X_n \notin \{e, e_*\}\}$ when $\tau_{i+1} < \infty$.

Since RW_λ $(X_n)_{n=0}^{\infty}$ is transient, there exists a unique $\hat{i}$ such that $\tau_{\hat{i}} < \infty$ and $\tau_{\hat{i}+1} = \infty$. Define a random walk $(Y_n)_{n=0}^{\infty}$ as follows:

$$(Y_n)_{n=0}^{\infty} = \big(X_{\tau_0}, \underbrace{X_{\sigma_0}, \ldots, X_{\tau_1}}, \underbrace{X_{\sigma_1}, \ldots, X_{\tau_2}}, \ldots, \underbrace{X_{\sigma_{\hat{i}-1}}, \ldots, X_{\tau_{\hat{i}}}}, X_{\sigma_{\hat{i}}}, X_{\sigma_{\hat{i}}+1}, \ldots\big).$$

It is evident that the sequence $(Y_n)_{n=0}^{\infty}$ is just an RW_λ on $\mathbb{T}$ starting at e. It is known that, almost surely, both $\lim_{n\to\infty}(|Y_n|/n)$ and $\lim_{n\to\infty}(|X_n|/n)$ exist and are deterministic. By construction, there exists a random function $s(\cdot)$ on non-negative integers such that, almost surely, $Y_n = X_{s(n)}$, $n \geq 1$, and $\lim_{k\to\infty}(s(k)/k) = 1$. Therefore, almost surely,

$$\lim_{n\to\infty}\frac{|Y_n|}{n} = \lim_{n\to\infty}\frac{|X_{s(n)}|}{n} = \lim_{n\to\infty}\frac{|X_{s(n)}|}{s(n)} = \lim_{n\to\infty}\frac{|X_n|}{n}.$$

This implies the lemma. □

For any $n \geq 1$, let $\beta_n(x) := \beta_{n,\lambda}(x)$ denote the probability of hitting level n before x_* when $|x| \leq n$. Recall that, for a given vertex x, xi is its ith child and $\nu(x)$ is the number of its children.

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Then $\beta_n(x) = 1$ if $|x| = n$ and, for $|x| < n$,

$$\beta_n(x) = \frac{\sum_{i=1}^{\nu(x)} \beta_n(xi)}{\lambda + \sum_{i=1}^{\nu(x)} \beta_n(xi)}.$$

Fix $\lambda \in [0, m)$, and consider the derivative of $\beta_n(x)$ at λ; we have

$$\begin{aligned}
&\lim_{\lambda_1 \to \lambda} \frac{\beta_{n,\lambda_1}(x) - \beta_{n,\lambda}(x)}{\lambda_1 - \lambda} \\
&= \left(\frac{\sum_{i=1}^{\nu(x)} \beta_{n,\lambda_1}(xi)}{\lambda_1 + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda_1}(xi)} - \frac{\sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)}{\lambda + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)} \right) / (\lambda_1 - \lambda) \\
&= \lim_{\lambda \to \lambda_1} \frac{\left(\lambda + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)\right) \sum_{i=1}^{\nu(x)} \beta_{n,\lambda_1}(xi) - \left(\lambda_1 + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda_1}(xi)\right) \sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)}{\left(\lambda_1 + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda_1}(xi)\right)\left(\lambda + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)\right)} / (\lambda_1 - \lambda) \\
&= \lim_{\lambda \to \lambda_1} \frac{\lambda \sum_{i=1}^{\nu(x)} \beta_{n,\lambda_1}(xi) - \lambda_1 \sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)}{\left(\lambda_1 + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda_1}(xi)\right)\left(\lambda + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)\right)} / (\lambda_1 - \lambda) = -\frac{\sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)}{\left(\lambda + \sum_{i=1}^{\nu(x)} \beta_{n,\lambda}(xi)\right)^2}.
\end{aligned}$$

We can deduce that each $\beta_n(x)$ has a continuous derivative in λ when $|x| \le n$. Also, $\beta_n(e) \downarrow \beta(e)$ as $n \uparrow \infty$, almost surely. To continue, we need Lemmas 2 and 3. For any natural number d, let $\mathbb{T}_{d+1}$ denote the $(d+1)$-regular tree.

Let $\mathbf{P}_x$ represent the probability measure of RW_λ starting at x on $\mathbb{T}_{d+1}$ with a fixed root e, and τ_{x_*} denote the hitting time of x_*.

Lemma 2. *For any $\lambda \ge 0$ and any vertex $x \in \mathbb{T}_{d+1} \setminus \{e\}$ with parent x_*,*

$$\mathbf{P}_x(\tau_{x_*} = \infty) = 1 - \frac{\lambda \wedge d}{d}.$$

Proof. We can project the biased random walk on $\mathbb{Z}$ as the tree is regular. □

Let us analyze the meaning of $\beta(e)$ within the framework of electric networks. Consider any weighted graph (say, an electric network) $G = (V(G), E(G))$ with a non-negative edge weight function c (here the weights are called conductances). Suppose $a \in V(G)$ and $Z \subseteq V(G)$. Write $\mathbf{P}_a^{G,c}(a \to Z) = \mathbf{P}_a^{G,c}(\tau_Z < \tau_a^+)$, where $\tau_Z = \inf\{n \ge 0 : X_n \in Z\}$, $\tau_a^+ = \inf\{n \ge 1 : X_n = a\}$, $(X_n)_{n \ge 0}$ is the random walk associated with electric network G, and $\mathbf{P}_a^{G,c}$ is the law of $(X_n)_{n \ge 0}$ starting at a. Let $\pi(x) = \sum_{y \in V(G): y \sim x} c(\{x, y\})$ for all $x \in V(G)$, where $y \sim x$ means y is adjacent to x. Then $\pi(\cdot)$ is a stationary measure of $(X_n)_{n \ge 0}$. Call

$$\mathcal{C}_G(a \leftrightarrow Z) := \mathcal{C}_{G,c}(a \leftrightarrow Z) = \pi(a) \mathbf{P}_a^{G,c}(a \to Z)$$

the effective conductance between a and Z. Note that $\mathbb{P}_a^{G,c}(a \to \infty) = \mathbb{P}_a\{X_i \ne a \text{ for } i \ge 1\}$. Then call

$$\mathcal{C}_G(a \leftrightarrow \infty) := \mathcal{C}_{G,c}(a \leftrightarrow \infty) = \pi(a) \mathbf{P}_a^{G,c}(a \to \infty)$$

the effective conductance from a to ∞ in G.

To emphasize the concentration on $\mathbb{T}_*$, denote $\beta(e) = \beta(e, \lambda)$ by $\beta_{\mathbb{T}_*}(e, \lambda)$. When $\lambda > 0$, on $\mathbb{T}_*$ endow any edge $\{x, y\}$ where $x, y \ne e_*$ with a weight $\lambda^{-|x| \wedge |y| - 1}$, and the edge $\{e_*, e\}$ with weight 1; denote this weight function by c_0. Then, for $\lambda > 0$, the RW_λ on $\mathbb{T}_*$ is the random walk associated with the weighted graph (electric network) $\mathbb{T}_*$;

$$\beta_{\mathbb{T}_*}(e, \lambda) = \mathbf{P}_{e_*}^{\mathbb{T}_*, c_0}(e_* \to \infty) = \mathcal{C}_{\mathbb{T}_*, c_0}(e_* \leftrightarrow \infty).$$

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Lemma 3. *Assume the Galton–Watson tree $\mathbb{T}$ has no leaf. Then, almost surely,*

$$1-\frac{\lambda\wedge m_1}{m_1}\le\beta_{\mathbb{T}_*}(e,\lambda)\le 1-\frac{\lambda}{m_2},\quad \lambda\in[0,m),$$

where $m_2=\sup\{k\ge 1 : P[\nu=k]>0\}$.

Proof. When $m_2=\infty$, $\beta(e)\le 1-\lambda/m_2$ holds trivially. Clearly, the lemma holds true when $\lambda=0$, so we assume $m_2<\infty$ (namely ν takes finitely many values) and $0<\lambda<m$. In a natural way, we can embed an m_1-ary tree $\mathbb{H}^1$ into $\mathbb{T}$ and also embed $\mathbb{T}$ into an m_2-ary tree $\mathbb{H}^2$ such that the roots of $\mathbb{H}^1$ and $\mathbb{H}^2$ are the root e of $\mathbb{T}$. Similarly to $\mathbb{T}_*$, let each $\mathbb{H}^i_*$ be obtained from $\mathbb{H}^i$ by adding the artificial parent e_* of e to $\mathbb{H}^i$.

Like the electric network $(\mathbb{T}_*, c_0)$, we endow each $\mathbb{H}^i_*$ with a weight function c_i, and view c_0 and c_1 as functions on the set of edges of $\mathbb{H}^2_*$ by letting $c_0(\{x,y\})=0$ ($c_1(\{x,y\})=0$) when $\{x,y\}$ is not an edge of $\mathbb{T}_*$ ($\mathbb{H}^1_*$). Then $c_1(\{x,y\})\le c_0(\{x,y\})\le c_2(\{x,y\})$ for any edge $\{x,y\}$ of $\mathbb{H}^2_*$.

Notice that

$$\beta_{\mathbb{T}_*}(e,\lambda)=\mathbf{P}^{\mathbb{T}_*,c_0}_{e_*}(e_*\to\infty)=\mathbf{P}^{\mathbb{H}^2_*,c_0}_{e_*}(e_*\to\infty)=\mathcal{C}_{\mathbb{H}^2_*,c_0}(e_*\leftrightarrow\infty),$$
$$\beta_{\mathbb{H}^i_*}(e,\lambda)=\mathbf{P}^{\mathbb{H}^i_*,c_i}_{e_*}(e_*\to\infty)=\mathbf{P}^{\mathbb{H}^2_*,c_i}_{e_*}(e_*\to\infty)=\mathcal{C}_{\mathbb{H}^2_*,c_i}(e_*\leftrightarrow\infty),\quad i=1,2.$$

Recall Rayleigh's monotonicity principle from [13, Section 2.4]: Let G be an infinite connected graph with two non-negative edge weight functions c and c' such that $c\le c'$ everywhere. Then, for any vertex a of G, $\mathcal{C}_{G,c}(a\leftrightarrow\infty)\le\mathcal{C}_{G,c'}(a\leftrightarrow\infty)$.

Therefore, we have $\mathcal{C}_{\mathbb{H}^2_*,c_1}(e_*\leftrightarrow\infty)\le\mathcal{C}_{\mathbb{H}^2_*,c_0}(e_*\leftrightarrow\infty)\le\mathcal{C}_{\mathbb{H}^2_*,c_2}(e_*\leftrightarrow\infty)$. In other words, $\beta_{\mathbb{H}^1_*}(e,\lambda)\le\beta_{\mathbb{T}_*}(e,\lambda)\le\beta_{\mathbb{H}^2_*}(e,\lambda)$. Hence, by Lemma 2, we obtain that

$$1-\frac{\lambda\wedge m_1}{m_1}\le\beta_{\mathbb{T}_*}(e,\lambda)\le 1-\frac{\lambda}{m_2}.$$

The lemma holds. □

Put

$$A_n(x)=\frac{\lambda}{\left(\lambda+\sum_{i=1}^{\nu(x)}\beta_n(xi)\right)^2},\qquad B_n(x)=\frac{\sum_{i=1}^{\nu(x)}\beta_n(xi)}{\left(\lambda+\sum_{i=1}^{\nu(x)}\beta_n(xi)\right)^2}.$$

Now we are in a position to prove the following lemma on the derivative of $\beta(e,\lambda)$.

Lemma 4. *For a Galton–Watson tree $\mathbb{T}$ without leaves, $\beta(e)=\beta(e,\lambda)$ almost surely has a continuous derivative $\beta'(e)=\beta'(e,\lambda)$ in $\lambda\in[0,m_1)$, with*

$$0<-\beta'(e,\lambda)\le\frac{\beta(e,\lambda)}{m_1-\lambda},\quad \lambda\in[0,m_1).$$

Proof. Differentiating (1) for $\lambda<m$ yields $-\beta_n'(x,\lambda)=A_n(x)\sum_{i=1}^{\nu(x)}-\beta_n'(xi,\lambda)+B_n(x)$, where $\beta_n'(x,\lambda)$ is the derivative in λ. Then

$$-\beta_n'(e,\lambda)=\sum_{k=0}^{n-1}\sum_{|x|=k}B_n(x)\prod_{i=0}^{k-1}A_n(x_i),\quad \lambda<m, \tag{2}$$

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where x_i is the ancestor at generation i of x. And, for any $k \in [0, n-1]$,

$$\beta_n(e, \lambda) = \sum_{|x|=k} \beta_n(x, \lambda) \prod_{i=0}^{k-1} \frac{1}{\lambda + \sum_{j=1}^{\nu(x_i)} \beta_n(x_i j, \lambda)}, \qquad \lambda < m.$$

Here, $x_i j$ is the jth child of the ancestor x_i.

Clearly, $\beta_n(x, \lambda)$ is non-increasing in n. By Lemma 3, $\lambda + \sum_{i=1}^{\nu(x)} \beta_n(xi, \lambda) \geq m_1$, $\lambda < m$. Hence, for $\lambda < m$,

$$A_n(x) \leq \frac{1}{m_1} \frac{\lambda}{\lambda + \sum_{i=1}^{\nu(x)} \beta_n(xi, \lambda)}, \qquad B_n(x) \leq \frac{1}{m_1} \beta_n(x, \lambda).$$

For $\lambda < m$,

$$\sum_{|x|=k} B_n(x) \prod_{i=0}^{k-1} A_n(x_i) \leq \frac{1}{m_1^{k+1}} \sum_{|x|=k} \beta_n(x, \lambda) \prod_{i=0}^{k-1} \frac{\lambda}{\lambda + \sum_{j=1}^{\nu(x_i)} \beta_n(x_i j, \lambda)} = \frac{\lambda^k}{m_1^k} \frac{1}{m_1} \beta_n(e, \lambda). \tag{3}$$

By (2) and (3), almost surely,

$$0 \leq -\beta_n'(e, \lambda) \leq \frac{\beta_n(e, \lambda)}{m_1 - \lambda} \leq \frac{1}{m_1 - \lambda}, \qquad \lambda < m_1.$$

From this, given any small enough $\varepsilon > 0$, we see that, almost surely, as a sequence of functions on $[0, m_1 - \varepsilon]$ $\{(\beta_n(e, \lambda) : \lambda \in [0, m_1 - \varepsilon])\}_{n \geq 1}$ is equi-continuous. Combining this with $\beta_n(e, \lambda) \downarrow \beta(e, \lambda)$ as $n \uparrow \infty$ for all $\lambda \in [0, m)$ almost surely, by the Ascoli–Arzelà theorem, $\{\beta_n(e, \lambda) : \lambda \in [0, m_1 - \varepsilon]\}_{n \geq 1}$ converges uniformly to $(\beta(e, \lambda) : \lambda \in [0, m_1 - \varepsilon])$ almost surely.

Note that, for any vertex $x \in \mathbb{T}$, $(\{(\beta_n(x, \lambda) : 0 \leq \lambda < m)\}_{n \geq 1}, (\beta(x, \lambda) : 0 \leq \lambda < m))$ has the same distribution as $(\{(\beta_n(e, \lambda) : 0 \leq \lambda < m)\}_{n \geq 1}, (\beta(e, \lambda) : 0 \leq \lambda < m))$. We obtain that, almost surely, for any vertex $x \in \mathbb{T}$, $\{\beta_n(x, \lambda) : \lambda \in [0, m_1 - \varepsilon]\}_{n \geq 1}$ converges uniformly to $(\beta(x, \lambda) : \lambda \in [0, m_1 - \varepsilon])$. Hence, by the definitions of $A_n(x)$ and $B_n(x)$, we have that, almost surely, for any vertex x, $A_n(x)$ and $B_n(x)$ converge uniformly in $\lambda \in [0, m_1 - \varepsilon]$ to some continuous functions $A(x)$ and $B(x)$ respectively.

Notice (2) and (3). By the dominated convergence theorem, we see that, almost surely, $\big(\beta_n'(x, \lambda) : \lambda \in [0, m_1 - \varepsilon]\big)$ converges uniformly to some continuous function $(F_\lambda : \lambda \in [0, m_1 - \varepsilon])$. By the dominated convergence theorem again, almost surely, $\int_0^\lambda \beta_n'(e, s)\, \mathrm{d}s$ converges to $\int_0^\lambda F_s\, \mathrm{d}s$, which is also $\beta(e, \lambda) - 1$ for all $\lambda \leq m_1 - \varepsilon$.

Since ε is arbitrary, $\beta(e, \lambda)$ is almost surely differentiable in $\lambda \in [0, m_1)$. Further, almost surely,

$$0 \leq -\beta'(e, \lambda) \leq \frac{\beta(e)}{m_1 - \lambda}, \qquad \lambda \in [0, m_1).$$

By checking (2) and the definitions of $A_n(x)$ and $B_n(x)$, when taking limits we indeed have, almost surely,

$$0 < -\beta'(e, \lambda) \leq \frac{\beta(e)}{m_1 - \lambda}, \qquad \lambda \in [0, m_1).$$

□

By symmetry, we have

$$\ell_\lambda = \mathbb{E}\bigg(\frac{\nu - \lambda}{\nu + 1} \frac{\sum_{i=0}^{\nu} \beta_i}{\lambda - 1 + \sum_{i=0}^{\nu} \beta_i}\bigg) \Big/ \mathbb{E}\bigg(\frac{\nu + \lambda}{\nu + 1} \frac{\sum_{i=0}^{\nu} \beta_i}{\lambda - 1 + \sum_{i=0}^{\nu} \beta_i}\bigg).$$

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By Lemma 4, each β_i has a derivative in $\lambda \in [0, m_1)$, and so does ℓ_λ. Write each β_i' and ℓ_λ' for the derivatives in $\lambda \in [0, m_1)$ of β_i and ℓ_λ respectively. Then, by straightforward calculus [1], for $\lambda \in [0, m_1)$, $\ell_\lambda' < 0$ is equivalent to

$$\begin{aligned}\mathbb{E}\bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\Bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{\big(\lambda-1+\sum_{i=0}^{\nu}\beta_i\big)^2}\Bigg)\\ -\mathbb{E}\bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\Bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{\big(\lambda-1+\sum_{i=0}^{\nu}\beta_i\big)^2}\Bigg)\\ <\frac{1}{\lambda}\mathbb{E}\bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg).\end{aligned} \tag{4}$$

Lemma 5. *For a Galton–Watson tree $\mathbb{T}$ without leaves, when $m_1 \geq 2$ (4) is true for $\lambda \in [0, m_1/(1+\sqrt{1-1/m_1})]$.*

Proof. Note that, by Lemma 4,

$$0 < -\beta_i'(\lambda) \leq \frac{\beta_i(\lambda)}{m_1-\lambda}, \quad \lambda < m_1, \qquad i \geq 0.$$

By Lemma 3,

$$\lambda - 1 + \sum_{i=0}^{\nu}\beta_i \geq \lambda - 1 + \bigg(1-\frac{\lambda}{m_1}\bigg)\times(m_1+1) = m_1 - \frac{\lambda}{m_1} > m_1 - 1, \qquad \lambda < m_1. \tag{5}$$

Then, for any $\lambda < 1$, $0 \leq \beta_i(\lambda) + (1-\lambda)\beta_i'(\lambda) < \beta_i(\lambda)$ and

$$\mathbb{E}\bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\Bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{\big(\lambda-1+\sum_{i=0}^{\nu}\beta_i\big)^2}\Bigg) \geq 0.$$

When $\lambda < 1$,

$$\begin{aligned}\mathbb{E}\bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\Bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{\big(\lambda-1+\sum_{i=0}^{\nu}\beta_i\big)^2}\Bigg)\\ <\frac{1}{m_1-\lambda/m_1}\mathbb{E}\bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg).\end{aligned}$$

Since $\lambda < 1$ and $m_1 \geq 2$, $m_1 - \lambda/m_1 > \lambda$ and we obtain that, when $\lambda < 1$,

$$\begin{aligned}\mathbb{E}\bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\Bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{\big(\lambda-1+\sum_{i=0}^{\nu}\beta_i\big)^2}\Bigg)\\ -\mathbb{E}\bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\Bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{\big(\lambda-1+\sum_{i=0}^{\nu}\beta_i\big)^2}\Bigg)\\ <\frac{1}{\lambda}\mathbb{E}\bigg(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg)\mathbb{E}\bigg(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\bigg);\end{aligned}$$

namely, (4) holds.

When $\lambda = 1$, (4) becomes

$$\mathbb{E}\left(\frac{\nu}{\nu+1}\right)\mathbb{E}\left(\frac{1}{\nu+1}\frac{1}{\sum_{i=0}^{\nu}\beta_i}\right) - \mathbb{E}\left(\frac{1}{\nu+1}\right)\mathbb{E}\left(\frac{\nu}{\nu+1}\frac{1}{\sum_{i=0}^{\nu}\beta_i}\right) < \mathbb{E}\left(\frac{\nu}{\nu+1}\right)\mathbb{E}\left(\frac{1}{\nu+1}\right),$$

while, by Lemma 3, $\sum_{i=0}^{\nu}\beta_i(1) \geq m_1 - 1/m_1 > 1$, which implies the above inequality.

When $m_1 > \lambda > 1$,

$$\beta_i + (1-\lambda)\beta_i' \leq \frac{m_1 - 1}{m_1 - \lambda}\beta_i.$$

Combining this with (5), we obtain, for $1 < \lambda < m_1$,

$$\begin{aligned}\mathbb{E}&\left(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right)\mathbb{E}\left(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{(\lambda-1+\sum_{i=0}^{\nu}\beta_i)^2}\right)\\ &\leq \frac{(m_1-1)/(m_1-\lambda)}{m_1-\lambda/m_1}\mathbb{E}\left(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right)\mathbb{E}\left(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right).\end{aligned}$$

When

$$\frac{(m_1-1)/(m_1-\lambda)}{m_1-\lambda/m_1} \leq \frac{1}{\lambda}$$

and $1 < \lambda < m_1$, namely $\lambda \in (1, m_1/(1+\sqrt{1-1/m_1})]$, we have

$$\begin{aligned}\mathbb{E}&\left(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right)\mathbb{E}\left(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{(\lambda-1+\sum_{i=0}^{\nu}\beta_i)^2}\right)\\ &\leq \frac{1}{\lambda}\mathbb{E}\left(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right)\mathbb{E}\left(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right).\end{aligned} \quad (6)$$

Note when $1 < \lambda < m_1$, $\sum_{i=0}^{\nu}(\beta_i + (1-\lambda)\beta_i') > \sum_{i=0}^{\nu}\beta_i > 0$ and

$$\mathbb{E}\left(\frac{1}{\nu+1}\frac{\sum_{i=0}^{\nu}\beta_i}{\lambda-1+\sum_{i=0}^{\nu}\beta_i}\right)\mathbb{E}\left(\frac{\nu}{\nu+1}\frac{\sum_{i=0}^{\nu}(\beta_i+(1-\lambda)\beta_i')}{(\lambda-1+\sum_{i=0}^{\nu}\beta_i)^2}\right) > 0.$$

Therefore, combining with (6), we see that for $\lambda \in (1, m_1/(1+\sqrt{1-1/m_1})]$, (4) holds.

We have thus finished proving Theorem 1. □

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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