ON THE LOWER RANGE OF PERRON'S MODULAR FUNCTION

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The modular function M was introduced by Perron in (**6**). $M(\xi)$ (for irrational ξ) is defined by the property that the inequality

$$|\xi - p/q| < (1 + d)/M(\xi)q^2$$

is satisfied by an infinity of relatively prime pairs (p, q) for positive d, but by at most a finite number of such pairs for negative d. We will write

$$\xi = [x_1, x_2, \ldots] = \frac{1}{x_1+}\frac{1}{x_2+\ldots}$$

for the continued fraction expansion of $\xi \in (0, 1)$ and for any finite collection $y_1, \ldots, y_k$ of positive integers we will write

$$[y_1, \ldots, y_k] = \frac{1}{y_1+}\frac{1}{y_2+}\ldots\frac{1}{+y_k}.$$

It is known (see **6**) that

$$M(\xi) = \limsup_k M_k(\xi),$$

where

$$M_k(\xi) = x_k + [x_{k-1}, x_{k-2}, \ldots, x_1] + [x_{k+1}, x_{k+2}, \ldots].$$

The range of M is known as the Lagrange spectrum.

If ξ^* is a doubly infinite sequence of positive integers,

$$\xi^* = (\ldots, x_{-1}, x_0, x_1, \ldots),$$

we set

$$M^*(\xi^*) = \sup_k M_k^*(\xi^*),$$

where

$$M_k^*(\xi^*) = x_k + [x_{k-1}, x_{k-2}, \ldots] + [x_{k+1}, x_{k+2}, \ldots].$$

The range of M^* is known as the Markov spectrum. It is known (see **3**; **4**; **5**) that the Markov spectrum is closed and contains the Lagrange spectrum. It is clear that if either $\sup_{k>0} x_k$ or $\sup_k x_k$ is infinite, $M(\xi)$ or $M^*(\xi^*)$ is infinite so that it suffices to consider

$$\mathscr{E}_K = \{\xi \mid x_i \leqq K\} \quad \text{and} \quad \mathscr{E}_K^* = \{\xi^* \mid x_i \leqq K\}.$$

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The authors (**2**) have recently shown that the Lagrange spectrum includes all numbers above $7/2 + (7/4)\sqrt{2}$. Perron (**6**) has shown that the values taken by $M(\xi)$ less than $2\sqrt{3}$ are given by $\xi \in \mathscr{E}_2$, and that those less than 3 are isolated values whose inverses form a countable set. He also showed that 3 is their only limit point and is in the Lagrange spectrum. Our aim in this paper is to study the range of $M(\xi)$ for $\xi \in \mathscr{E}_2$. It will be shown that the values of $M(\xi)$ in the neighbourhood of 3 form a nowhere dense set, and indeed a set of Hausdorff dimension less than 1, and that there exist intervals $[a_i, b_i]$, $b_i \to 2\sqrt{3}$ in which no values of $M(\xi)$ occur.

Let $A \oplus B = \{a + b: a \in A, b \in B\}$. The set $\mathscr{E}_2 \oplus \mathscr{E}_2 + \{2\}$ is closed since $\mathscr{E}_2$ is and, by the argument above, it contains $M(\xi_2)$ and $M^*(\xi_2^*)$, and hence all of the Lagrange and Markov spectra up to $2\sqrt{3}$.

2. The sum set $\mathscr{E}_2 \oplus \mathscr{E}_2$. We outline the construction of $\mathscr{E}_2$ corresponding to the construction of the classical Cantor set by leaving out middle thirds. We let l be the greatest value in $\mathscr{E}_2$ and s the least. Clearly, $l = 1/(1 + s)$ and $s = 1/(2 + l)$, hence $l = \sqrt{3} - 1$, $s = (\sqrt{3} - 1)/2$. No points of $\mathscr{E}_2$ may lie between $u = 1/(2 + s) = (3 - \sqrt{3})/3$ and $t = 1/(1 + l) = \sqrt{3}/3$, thus (u, t) is the first interval left out. Let P_n/Q_n be the nth convergent of $\xi \in \mathscr{E}_2$ and for $x \in (0, 1)$ set

$$Z_n(\xi, x) = \frac{xP_n + P_{n-1}}{xQ_n + Q_{n-1}}$$

$$= \frac{1}{x_1+}\,\frac{1}{x_2+}\cdots\frac{1}{+x_n + x}.$$

An interval containing portions of $\mathscr{E}_2$ for n odd is given by $[Z_n(\xi, s), Z_n(\xi, l)]$ and for n even by $[Z_n(\xi, l), Z_n(\xi, s)]$. We assume henceforth that n is even, as the computation for the odd case is similar. We claim that

$$Z_n(\xi, l) + Z_n(\xi, u) > 2Z_n(\xi, t), \qquad Z_n(\xi, s) + Z_n(\xi, t) < 2Z_n(\xi, u).$$

The first follows from the fact that $(Q_nP_{n-1} - P_nQ_{n-1}) = (-1)^n$ and that

$$Z_n(\xi, l) - Z_n(\xi, t) = (t - l)/(lQ_n + Q_{n-1})(tQ_n + Q_{n-1}),$$

$$Z_n(\xi, t) - Z_n(\xi, u) = (u - t)/(uQ_n + Q_{n-1})(tQ_n + Q_{n-1}),$$

and the observations that $t - u = l - t$ and $l > u$. The second inequality is established by more complicated but still trivial arithmetic. Then in $([Z_n(\xi, s), Z_n(\xi, l)] \cap \mathscr{E}_2) \times ([Z_n(\xi, s), Z_n(\xi, l)] \cap \mathscr{E}_2)$ there exist no points between the lines

$$x + y = 2Z_n(\xi, t) \quad \text{and} \quad x + y = Z_n(\xi, l) + Z_n(\xi, u),$$

and none between the lines

$$x + y = Z_n(\xi, s) + Z_n(\xi, t) \quad \text{and} \quad x + y = 2Z_n(\xi, u).$$

THEOREM 2.1. *The intervals*

$$\begin{aligned}(c_{2k}, d_{2k}) &= (2Z_{2k}(\xi, t), Z_{2k}(\xi, l) + Z_{2k}(\xi, u)),\\ (c_{2k}', d_{2k}') &= (Z_{2k}(\xi, s) + Z_{2k}(\xi, t), 2Z_{2k}(\xi, u)),\\ (c_{2k+1}, d_{2k+1}) &= (Z_{2k+1}(\xi, l) + Z_{2k+1}(\xi, u), 2Z_{2k+1}(\xi, t)),\\ (c_{2k+1}', d_{2k+1}') &= (2Z_{2k+1}(\xi, u), Z_{2k+1}(\xi, s) + Z_{2k+1}(\xi, t))\end{aligned}$$

contain no points of $\mathscr{E}_2 \oplus \mathscr{E}_2$. *The intervals* $(a_n, b_n) = (c_{n+2}, d_{n+2})$ *and* $(a_n', b_n') = (c_{n+2}', d_{n+2}')$ *contain no values in the range of* M. *There are intervals of this type as near as we please to* $2\sqrt{3}$.

Proof. Only the last assertion remains to be proved. Taking $\xi = l$ in the above construction and noting that $Z_n(l, x) \to l$ uniformly in x we have:

$$b_{2k} = 2 + Z_{2k}(l, s) + Z_{2k}(l, u) \to 2 + 2l = 2\sqrt{3}.$$

Similarly, $b_{2k+1}, b_{2k}', b_{2k+1}'$ all converge to $2\sqrt{3}$.

3. The range of M **in** $[3, 2 + 2\sqrt{3}/3]$. If $\xi \in \mathscr{E}_2$ is such that

$$(x_{i-1}, x_i, x_{i+1}) = (1, 2, 1)$$

occurs infinitely often, then

$$M(\xi) \geqq 2 + 2\left(\frac{1}{1 + l}\right) = 2 + 2\sqrt{3}/3.$$

Thus, the part of the Lagrange spectrum up to $2 + 2\sqrt{3}/3$ is contained in $M(\mathscr{E}_2')$, where

$$\mathscr{E}_2' = \{\xi |\ \xi \in \mathscr{E}_2 \text{ and } (x_{i-1}, x_i, x_{i+1}) \neq (1, 2, 1) \text{ for all } i\}.$$

This in turn is obviously contained in $\{2\} \oplus \{\mathscr{E}_2' \oplus \mathscr{E}_2'\}$.

THEOREM 3.1. *The Hausdorff-Besicovich dimension of* $\mathscr{E}_2' \oplus \mathscr{E}_2'$ *is less than* 1.

By the above remark, the following corollary is immediate.

COROLLARY 3.1. *The dimension of the part of the range of* M *in* $[3, 2 + 2\sqrt{3}/3]$ *is less than* 1.

We will actually prove that the dimension of $\mathscr{E}_2' \times \mathscr{E}_2'$ is less than 1 and the result will follow from the fact that $\mathscr{E}_2' \oplus \mathscr{E}_2'$ is the projection of $\mathscr{E}_2' \times \mathscr{E}_2'$ through 135°, since projection does not increase dimension. We will show that there is an α and a β, $0 < \alpha, \beta < 1$, such that any rectangle R of the form $[Z_n(\xi, s), Z_n(\xi, l)] \times [Z_m(\eta, s), Z_m(\eta, l)]$ with $\xi, \eta \in \mathscr{E}_2'$ can be "broken up" into subrectangles R_i of the same form with

(1) $\max_i(\text{perimeter } R_i) \leqq \beta$ perimeter R,
(2) $(\text{perimeter } R)^\alpha \geqq \sum_i(\text{perimeter } R_i)^\alpha$,
(3) $R \cap (\mathscr{E}_2' \times \mathscr{E}_2') \subset \bigcup_i R_i$.

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This will prove the existence of coverings of $\mathscr{E}_2' \times \mathscr{E}_2'$ of arbitrarily small perimeter (and hence diameter) for which

$$c \geqq \sum(\text{diameter } R_i)^\alpha,$$

and thus that $\mathscr{E}_2' \times \mathscr{E}_2'$ has dimension $0 \leqq \alpha < 1$.

We take n, m, ξ, and η fixed and deal separately with various cases. We take n and m even since the other cases are similar and we write $I(\alpha, \beta) = [Z_n(\xi, \alpha), Z_n(\xi, \beta)]$, $J(\alpha, \beta) = [Z_m(\eta, \alpha), Z_m(\eta, \beta)]$, $\xi = [x_1, x_2, \ldots]$, $\eta = [y_1, y_2, \ldots]$, $r = [x_n, x_{n-1}, \ldots, x_1]$, and $r' = [y_m, y_{m-1}, \ldots, y_1]$. $|K|$ will stand for the length of an interval K. Because of the obvious symmetry of the two cases, we deal only with the case $|I(s, l)| \geqq |J(s, l)|$.

The subrectangles in each case will be of the form $I(\alpha, \beta) \times J(\gamma, \delta)$, where α, β, γ, and δ are chosen from among the numbers s, l, u, t, $a = 1/(2 + t)$, $b = 1/(2 + u)$, $v = 1/(1 + t)$, $w = 1/(1 + u)$, $z = 1/(1 + b)$, $k = 1/(1 + w)$, $q = 1/(1 + v)$. Note first that these have the desired form, that is, each can be written $[Z_{n'}(\xi', x), Z_{n'}(\xi', l)] \times [Z_{m'}(\eta', s), Z_{m'}(\eta', l)]$. Furthermore, we have:

$$\frac{|I(s, u)|}{|I(s, l)|} = \frac{u - s}{l - s} \frac{l + r}{u + r} \leqq \frac{u - s}{l - s} \frac{l}{u} < 1,$$

and a similar computation shows that $|I(t, l)|/|I(s, l)|$ is also bounded away from 1 so that the first requirement will be met as long as the pair $(\alpha, \beta) = (s, l)$ does not occur.

When $|I(s, l)| \geqq 6|J(s, l)|$, we can simply take the rectangles $I(s, u) \times J(s, l)$ and $I(t, l) \times J(s, l)$. This is so since

$$\frac{|I(u, t)|}{|J(s, l)|} \geqq \frac{6|I(u, t)|}{|J(s, l)|} = \frac{6|t - u|}{|l - s|} \frac{(l + r)(s + r)}{(t + r)(u + r)} \geqq \frac{6t - u}{l - s} > 2,$$

so that the total perimeter decreases while the ratios of the subperimeters to the original perimeter are bounded away from 0.

This allows us to assume henceforth that the ratios of the sides are bounded away from 0 and ∞.

Next, if $x_n = 2$, $x_{n-1} = 1$ or $y_n = 2$, $y_{n-1} = 1$, we can use the single subrectangle $I(s, u) \times J(s, l)$ or $I(s, l) \times J(s, u)$ since $I(t, l)$ or $J(t, l)$ then contains no points of $\mathscr{E}_2'$. Straightforward calculations using the fact that $|I(s, l)|/|J(s, l)|$ and $|J(s, l)|/|I(s, l)|$ are bounded prove that the ratio of the new perimeter to the old is bounded away from 1, and hence that the requirements are met in this case.

We need consider only the following three cases:

Case 1: $x_n = 2$, $x_{n-1} = 2$, $y_m = 2$, $y_{m-1} = 2$;

Case 2: $x_n = 1$, $y_m = 1$;

Case 3: $x_n = 2$, $x_{n-1} = 2$, $y_m = 1$ or $x_n = 1$, $y_m = 2$, $y_{m-1} = 2$.

Set

$$R(\alpha, \beta, r) = \frac{|I(\alpha, \beta)|}{|I(s, l)|} = \frac{\beta - \alpha}{l - s} \frac{(l + r)(s + r)}{(\beta + r)(\alpha + r)}$$

and

$$R'(\alpha, \beta, r') = \frac{|J(\alpha, \beta)|}{|J(s, l)|} = \frac{\beta - \alpha}{l - s} \frac{(l + r')(s + r')}{(\beta + r')(\alpha + r')}.$$

We will need the following lemma, whose proof is elementary.

LEMMA 3.1. *$(l + r)(s + r)/(\alpha + r)(\beta + r)$ is a monotone decreasing function of r in $[s, l]$ if (α, β) is (s, a), (s, u) or (b, u). It is a monotone increasing function of r in $[s, l]$ if (α, β) is (t, k), (t, v), (q, v) or (w, z).*

In Case 1, the rectangles (see Figure 1), $I(s, u) \times J(s, u)$, $I(s, u) \times J(t, v)$, $I(t, v) \times J(s, u)$, $I(t, v) \times J(t, v)$, $I(w, z) \times J(s, a)$, $I(w, z) \times J(b, u)$, $I(w, z) \times J(t, k)$, $I(w, z) \times J(q, v)$, $I(w, z) \times J(w, z)$, $I(s, a) \times J(w, z)$, $I(b, u) \times J(w, z)$, $I(t, k) \times J(w, z)$, and $I(q, v) \times J(w, z)$ satisfy the requirements. The intervals (a, b), (u, t), (k, q), and (v, w) contain no points of $\mathscr{E}_2'$ so that $Z(l, x)$ or $Z(l, x)$ for x in the intervals is not in $\mathscr{E}_2'$. The interval from z to $l' = [121121212\ldots]$ is also outside $\mathscr{E}_2'$ and everything in (l', l) starts with $(1, 2, 1)$, and hence may not be excluded. Thus the new rectangles cover that part of $\mathscr{E}_2' \times \mathscr{E}_2'$ in the old rectangle. To show that conditions (1), (2), and (3) following Corollary 3.1 hold, it will be sufficient to show that

$$A = 2R(s, u, r) + 2R(t, v, r) + R(s, a, r) + R(b, u, r) + R(t, u, r) + R(q, v, n) + 5R(w, z, r),$$

and the similar expression with R and r replaced by R' and r' are bounded away from 1. Since $x_n = x_{n-1} = 2$, we must have $b < r < u$ for large enough n. Thus,

$$1 > 2R(s, u, b) + 2R(t, v, u) + R(s, a, b) + R(b, u, b) + R(t, k, u) + R(q, v, u) + 5R(w, z, u).$$

Computation shows that this latter sum is less than 0.95. The computation for the other expression is the same.

In Case 2, the rectangles (see Figure 2) $I(b, u) \times J(b, u)$, $I(b, u) \times J(t, k)$, $I(b, u) \times J(q, v)$, $I(b, u) \times J(w, z)$, $I(t, k) \times J(b, u)$, $I(t, k) \times J(w, z)$, $I(t, v) \times J(t, v)$, $I(q, u) \times J(b, u)$, $I(q, v) \times J(w, z)$, $I(w, z) \times J(b, u)$, $I(w, z) \times J(t, k)$, $I(w, z) \times J(q, v)$, and $I(w, z) \times J(w, z)$ satisfy the conditions. The inclusion follows from the fact that if $x_n = 1, x_{n+1} = 2$, then $x_{n+2} = 1$ ensures that $I(s, a)$ contains none of $\mathscr{E}_2'$. As above, it is sufficient to show that $4R(b, u, r) + 2R(t, k, r) + 2R(q, v, r) + 4R(b, u, r) + R(t, v, r)$ and the similar expression with R and r replaced by R' and r' are bounded away from 1. Since $a_n = 1$, for sufficiently large n, we must have $t < r < z$ so that Lemma 3.1 ensures that if

$$B = 4R(b, u, t) + 2R(t, k, z) + 2R(q, v, z) + 2R(w, z, z) + R(t, v, z) < 1,$$

the result holds. Another long but straightforward computation shows $B < 0.5$.

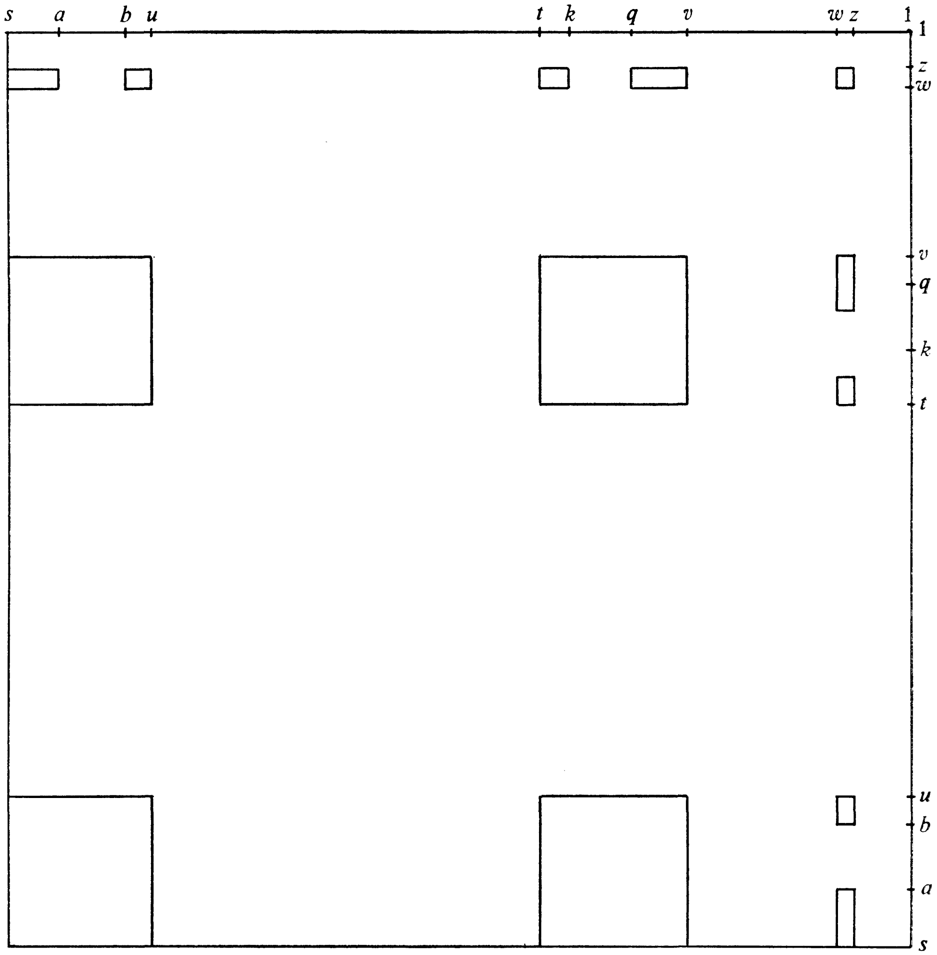

FIGURE 1

In the remaining case, the two possibilities are symmetric, thus we take $x_n = 2$, $x_{n-1} = 2$, $y_m = 1$. The rectangles (see Figure 3) $I(b, u) \times J(s, a)$, $I(b, u) \times J(b, u)$, $I(b, u) \times J(t, k)$, $I(b, u) \times J(q, v)$, $I(b, u) \times J(z, w)$,

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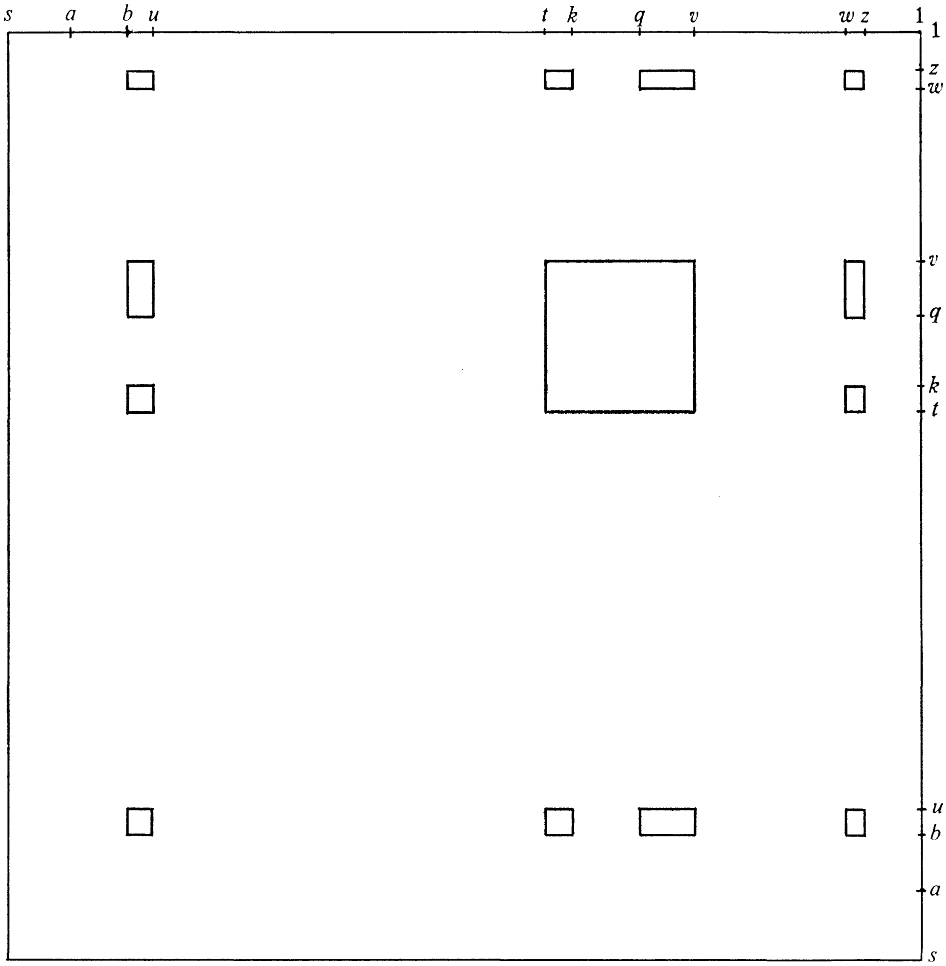

FIGURE 2

$I(t, v) \times J(s, u)$, $I(t, v) \times J(t, v)$, $I(t, k) \times J(z, w)$, $I(q, v) \times J(z, w)$, $I(z, w) \times J(s, a)$, $I(z, w) \times J(b, u)$, $I(z, w) \times J(t, k)$, $I(z, w) \times J(q, v)$, and $I(z, w) \times J(z, w)$ satisfy the requirements. By an argument similar to that above it will be sufficient to show that

$$C = 5R(b, u, t) + 5R(w, z, z) + R(t, k, z) + R(q, v, z) + 2R(t, v, z) < 1$$

and

$$D = 4R(w, z, u) + 2R(q, v, u) + 2R(t, k, u) + 2R(s, a, b) + R(s, u, b) + 2R(b, u, b) + R(t, v, u) < 1.$$

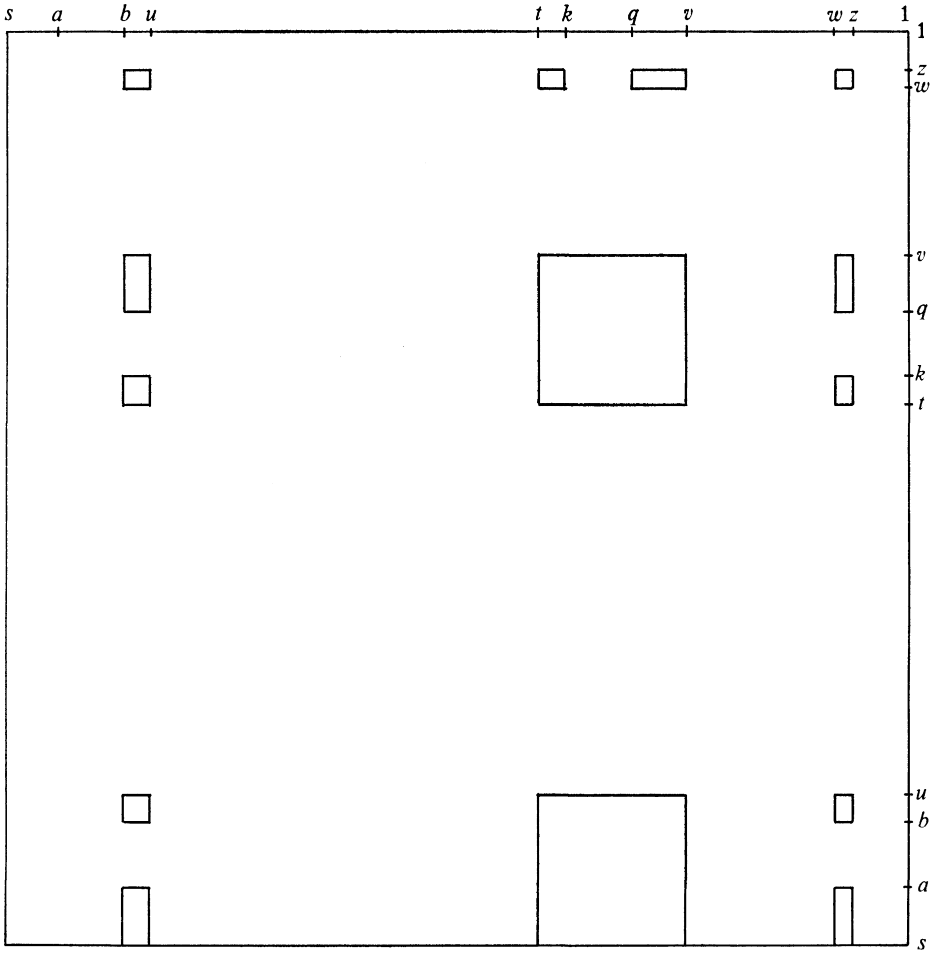

FIGURE 3

Two more computations show that $C < 0.6$ and $D < 0.75$. This completes the proof of Theorem 3.1.

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