POSITIVE SOLUTIONS OF INTEGRODIFFERENTIAL AND DIFFERENCE EQUATIONS WITH UNBOUNDED DELAY

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Abstract. We establish a necessary and sufficient condition for the existence of a positive solution of the integrodifferential equation

$$x'(t)+\int_0^\infty x(t-s)\,dn(s)=0,$$

where n is an increasing real-valued function on the interval $[0,\infty)$; that is, if and only if the characteristic equation

$$-\lambda+\int_0^\infty e^{\lambda s}\,dn(s)=0$$

admits a positive root.

Consider the difference equation $x_{n+1}-x_n+\sum_{k=0}^{\infty}c_kx_{n-k}=0$, where $(c_k)_{k\geq 0}$ is a sequence of non-negative numbers. We prove this has positive solution if and only if the characteristic equation $-\lambda+\sum_{k=0}^{\infty}\lambda^{-k}c_k=0$ admits a root in $(0,1)$.

For general results on integrodifferential equations we refer to the book by Burton [**1**] and the survey article by Corduneanu and Lakshmikantham [**2**]. Existence of a positive solution and oscillations in integrodifferential equations or in systems of integrodifferential equations recently have been investigated by Ladas, Philos and Sficas [**5**], Györi and Ladas [**4**], Philos and Sficas [**12**], Philos [**9**], [**10**], [**11**].

Recently, there has been some interest in the existence or the non-existence of positive solutions or the oscillation behavior of some difference equations. See Ladas, Philos and Sficas [**6**], [**7**].

The purpose of this paper is to investigate the positive solutions of integrodifferential equations (Section 1) and difference equations (Section 2) with unbounded delay. We obtain also some results for integrodifferential and difference inequalities.

1. Integrodifferential equations. Consider the integrodifferential equation

$$x'(t)+\int_0^\infty x(t-s)\,dn(s)=0, \tag{E}$$

where n is an increasing real-valued function on the interval $[0,\infty)$. It will be supposed that $n(0)=0$ and that n is not identically zero on $[0,\infty)$.

By a *solution* of (E) we mean a continuous real-valued function x defined on the real line $\mathbb{R}$, which is differentiable on $[0,\infty)$ and satisfies (E) for all $t\geq 0$.

The *characteristic equation* of (E) is

$$-\lambda+\int_0^\infty e^{\lambda s}\,dn(s)=0. \tag{$*$}$$

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THEOREM 1. *Equation* (E) *has a non-negative solution which is eventually positive if and only if the characteristic equation* $(*)$ *admits a positive root.*

A real-valued function h defined on $\mathbb{R}$ is said to be *non-negative* if $h(t) \geq 0$ for every $t \in \mathbb{R}$, and it is called *eventually positive* if there exists a $T \in \mathbb{R}$ such that $h(t) > 0$ for all $t \geq T$.

Proof of Theorem 1. If $\lambda_0 > 0$ is a root of the characteristic equation $(*)$, then $x(t) = e^{-\lambda_0 t}(t \in \mathbb{R})$ is a solution of (E) with $x(t) > 0$ for every $t \in \mathbb{R}$.

Assume, conversely, that (E) has a solution x such that $x(t) \geq 0$ for all $t \in \mathbb{R}$ and $x(t) > 0$ for every $t \geq T$, where T is a real number. Then from (E) it follows that $x'(t) \leq 0$ for $t \geq 0$ and consequently x is decreasing on the interval $[0, \infty)$.

Consider the set Λ of all $\lambda > 0$ for which there exists a $t_\lambda \geq 0$ such that $x'(t) + \lambda x(t) \leq 0$ for all $t \geq t_\lambda$. The set Λ is nonempty. Indeed, by taking into account the hypotheses on n, we can see that, there is a $\tau > 0$ so that

$$\lambda_0 \equiv \int_0^\tau dn(s) > 0.$$

Since x is decreasing on $[0, \infty)$, for every $t \geq \tau$, from (E) we obtain

$$\begin{aligned} 0 = x'(t) + \int_0^\infty x(t-s)\, dn(s) &\geq x'(t) + \int_0^\tau x(t-s)\, dn(s) \\ &\geq x'(t) + \left[\int_0^\tau dn(s)\right] x(t) = x'(t) + \lambda_0 x(t), \end{aligned}$$

which means that $\lambda_0 \in \Lambda$. Thus, Λ is nonempty. Clearly, Λ is a subinterval of $(0, \infty)$ with $\inf \Lambda = 0$. Next, we will show that Λ is bounded from above.

By the hypotheses on n, we can choose $\sigma > \varepsilon > 0$, so that

$$A \equiv \int_\varepsilon^\sigma dn(s) > 0.$$

Then, by taking into account the fact that x is decreasing on $[0, \infty)$, from (E) we find for $t \geq \sigma$

$$\begin{aligned} 0 = x'(t) + \int_0^\infty x(t-s)\, dn(s) &\geq x'(t) + \int_\varepsilon^t x(t-s)\, dn(s) \\ &\geq x'(t) + \left[\int_\varepsilon^t dn(s)\right] x(t-\varepsilon) \geq x'(t) + \left[\int_\varepsilon^\sigma dn(s)\right] x(t-\varepsilon). \end{aligned}$$

That is, $x'(t) + Ax(t-\varepsilon) \leq 0$ for all $t \geq \sigma$. Thus from Ladas, Sficas and Stavroulakis **[8]** it follows that

$$x(t) > Bx(t-\varepsilon) \quad \text{for all large } t, \tag{1}$$

where $B = (A\varepsilon/2)^2$. Since x is decreasing on $[0, \infty)$, we always have $B < 1$. In fact, we have

$$\sup \Lambda \leq \theta = -\varepsilon^{-1} \ln B.$$

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Indeed, if not, θ belongs to Λ and hence there is a $t_\theta \ge 0$ such that

$$x'(t) + \theta x(t) \le 0 \quad \text{for} \quad t \ge t_\theta.$$

So, if we define

$$u_\theta(t) = e^{\theta t} x(t) \qquad (t \ge t_\theta),$$

then we have for all $t \ge t_\theta$

$$u'_\theta(t) = e^{\theta t}[x'(t) + \theta x(t)] \le 0$$

and consequently u_θ is decreasing on $[t_\theta, \infty)$.

Hence, for every $t \ge t_\theta + \varepsilon$, we obtain

$$e^{\theta(t-\varepsilon)} x(t-\varepsilon) \equiv u_\theta(t-\varepsilon) \ge u_\theta(t) \equiv e^{\theta t} x(t).$$

Thus,

$$x(t) \le e^{-\theta\varepsilon} x(t-\varepsilon) = Bx(t-\varepsilon) \quad \text{for} \quad t \ge t_\theta + \varepsilon,$$

which contradicts (1).

Now, we set $\tilde{\lambda} = \sup \Lambda$, $0 < \tilde{\lambda} < \infty$. Moreover we consider an arbitrary number $\mu \in (0, \tilde{\lambda})$. Then $r \equiv \tilde{\lambda} - \mu \in \Lambda$ and hence there exists a $t_r \ge 0$ such that

$$x'(t) + rx(t) \le 0 \quad \text{for all} \quad t \ge t_r.$$

Without loss of generality, we may assume that $t_r \ge T$ and hence $x(t) > 0$ for every $t \ge t_r$. For any t, s with $t \ge t_r$ and $0 \le s \le t - t_r$, we have

$$\frac{x(t-s)}{x(t)} = \exp\left[-\ln \frac{x(t)}{x(t-s)}\right] = \exp\left[-\int_{t-s}^{t} \frac{\dot{x}(\xi)}{x(\xi)} d\xi\right] \ge e^{rs}.$$

That is $x(t-s) \ge e^{rs} x(t)$ for $t \ge t_r$ and $0 \le s \le t - t_r$.

Thus, from (E) it follows that for $t \ge t_r$

$$0 = x'(t) + \int_0^\infty x(t-s)\, dn(s) \ge x'(t) + \int_0^{t-t_r} x(t-s)\, dn(s)$$

$$\ge x'(t) + \left[\int_0^{t-t_r} e^{rs}\, dn(s)\right] x(t).$$

We claim that

$$\int_0^{t-t_r} e^{rs}\, dn(s) \le \tilde{\lambda} \quad \text{for all} \quad t \ge t_r. \tag{2}$$

Otherwise, there exists a $\hat{t}_r > t_r$ such that

$$\hat{\lambda} \equiv \int_0^{\hat{t}_r - t_r} e^{rs}\, dn(s) > \tilde{\lambda}$$

and therefore we have for $t \ge \hat{t}_r$

$$0 \ge x'(t) + \left[\int_o^{t-t_r} e^{rs}\, dn(s)\right] x(t) \ge x'(t) + \hat{\lambda} x(t).$$

Hence $\hat{\lambda} \in \Lambda$ which contradicts to the fact that $\hat{\lambda} > \tilde{\lambda} \equiv \sup \Lambda$. Thus (2) has been established.

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Finally, from (2) it follows that

$$\int_0^\infty e^{rs}\,dn(s) \le \tilde{\lambda} \quad \text{or} \quad \int_0^\infty e^{(\lambda-\mu)s}\,dn(s) \le \tilde{\lambda}.$$

As $\mu \in (0, \tilde{\lambda})$ is arbitrary, we obtain

$$\int_0^\infty e^{\tilde{\lambda}s}\,dn(s) \le \tilde{\lambda}.$$

So, if we define

$$F(\lambda) = -\lambda + \int_0^\infty e^{\lambda s}\,dn(s) \quad \text{for} \quad \lambda \ge 0,$$

then we have $F(\tilde{\lambda}) \le 0$. On the other hand, we have $F(0) = \int_0^\infty dn(s) > 0$. Hence, there is a $\lambda_0 \in (0, \tilde{\lambda}]$ with $F(\lambda_0) = 0$.

Then $\lambda_0 > 0$ is a root of the characteristic equation (*) and the proof of Theorem 1 is complete.

Consider the integrodifferential inequality

$$y'(t) + \int_0^\infty y(t-s)\,dn(s) \le 0. \tag{I}$$

By a *solution* of (I) we mean a continuous real-valued function y defined on $\mathbb{R}$, which is differentiable on $[0, \infty)$ and satisfies (I) for all $t \ge 0$.

The proof of Theorem 1 can be used to establish the following result.

THEOREM 1′. *Inequality* (I) *has a non-negative solution which is eventually positive if and only if* (*) *admits a positive root.*

Now, let us consider the equation

$$N'(t) = N(t)\left[\alpha - \int_0^\infty N(t-s)\,dn(s)\right], \tag{Ê}$$

where α is a positive constant, and $\int_0^\infty dn(s) < \infty$. This equation can arise in a study of the dynamics of a single-species population model; see for example J. M. Cushing [**3**].

By a *solution* of (Ê) we mean a continuous real valued function N defined on $\mathbb{R}$, which is differentiable on $[0, \infty)$ and satisfies (Ê) for every $t \ge 0$.

Equation (Ê) has a unique positive equilibrium N_0 which is given by

$$\alpha = N_0 \int_0^\infty dn(s).$$

[By our assumptions on n, we have $0 < \int_0^\infty dn(s) < \infty$.]

Consider the equation

$$-\lambda + N_0 \int_0^\infty e^{\lambda s}\,dn(s) = 0. \tag{$\hat{*}$}$$

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THEOREM 2. *Assume that* (*) *has no positive roots. Then there is no solution N of* (Ê) *such that*

$$N(t) \geq N_0 \quad \textit{for every} \quad t \in \mathbb{R}, \quad \textit{and} \quad N(t) > N_0 \quad \textit{for all large } t.$$

Proof. Let N be a solution of (Ê) with $N(t) \geq N_0$ for every $t \in \mathbb{R}$, and $N(t) > N_0$ for all large t.

Set

$$y(t) = \ln(N(t)N_0^{-1}) \quad \text{for} \quad t \in \mathbb{R}$$

and observe that y is a non-negative function on $\mathbb{R}$ which is eventually positive. The function y satisfies

$$y'(t) + N_0 \int_0^\infty [e^{y(t-s)} - 1]\, dn(s) = 0 \quad \text{for all} \quad t \geq 0.$$

Since $e^w - 1 \geq w$ for $w \geq 0$, we get

$$y'(t) + N_0 \int_0^\infty y(t-s)\, dn(s) \leq 0 \quad \text{for} \quad t \geq 0.$$

Thus, y is a solution of the inequality

$$y'(t) + \int_0^\infty (t-s) d\tilde{n}(s) \leq 0,$$

where $\tilde{n} = N_0 n$.

An application of Theorem 1′ completes the proof of Theorem 2.

2. Difference equations. Consider the difference equation

$$x_{n+1} - x_n + \sum_{k=0}^{\infty} c_k x_{n-k} = 0, \tag{E}$$

where $(c_k)_{k \geq 0}$ is a sequence of non-negative numbers which is not identically zero.

By a *solution* of (E) we mean a sequence $(x_n)_{n \in \mathbb{Z}}$ ($\mathbb{Z}$ is the set of all integers) which satisfies (E) for all $n \geq 0$.

The *characteristic equation* of (E) is

$$\lambda - 1 + \sum_{k=0}^{\infty} \lambda^{-k} c_k = 0. \tag{*}$$

THEOREM 1. *Equation* (E) *has a non-negative solution which is eventually positive if and only if the characteristic equation* (*) *admits a root in* (0, 1).

A sequence $(h_n)_{n \in \mathbb{Z}}$ is said to be *non-negative* if $h_n \geq 0$ for every $n \in \mathbb{Z}$, and it is called *eventually positive* if there exists a $m \in \mathbb{Z}$ such that $h_n > 0$ for all $n \geq m$.

Proof of Theorem 1. If $\lambda_0 \in (0, 1)$ is a root of the characteristic equation (*), then $x_n = \lambda^n (n \in \mathbb{Z})$ is a solution of (E) with $x_n > 0$ for every $n \in \mathbb{Z}$.

Assume, conversely, that (E) has a solution $(x_n)_{n \in \mathbb{Z}}$ which is non-negative and eventually positive. Let $m \geq 0$ be an integer such that $x_n > 0$ for $n \geq m$. Then from (E) we

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obtain for every $n \geq m$,

$$0 = x_{n+1} - x_n + \sum_{k=0}^{\infty} c_k x_{n-k} = x_{n+1} - x_n + \sum_{k=0}^{n} c_k x_{n-k}$$
$$+ \sum_{k=n+1}^{\infty} c_k x_{n-k} \geq x_{n+1} - x_n + \sum_{k=0}^{n} c_k x_{n-k}.$$

That is

$$x_{n+1} - x_n + \sum_{k=0}^{n} c_k x_{n-k} \leq 0 \quad \text{for every} \quad n \geq m.$$

Thus from Theorem 1 in Ladas, Philos and Sficas [7] it follows that there is a $\bar{\lambda} \in (0, 1)$ such that

$$\bar{\lambda} - 1 + \sum_{k=0}^{\infty} \bar{\lambda}^{-k} c_k \leq 0.$$

Set

$$F(\lambda) = \lambda - 1 + \sum_{k=0}^{\infty} \lambda^{-k} c_k \quad \text{for} \quad \lambda \in [\bar{\lambda}, 1].$$

We have $F(\bar{\lambda}) \leq 0$. On the other hand, we observe that

$$F(1) = \sum_{k=0}^{\infty} c_k > 0.$$

Thus, there exists a $\hat{\lambda} \in [\bar{\lambda}, 1)$ with $F(\hat{\lambda}) = 0$. Then $\hat{\lambda} \in (0, 1)$ is a root of the characteristic equation and the proof of Theorem 1 is complete.

Consider the difference inequality

$$y_{n+1} - y_n + \sum_{k=0}^{\infty} c_k y_{n-k} \leq 0. \tag{I}$$

By a *solution* of (I) we mean a sequence $(y_n)_{n \in \mathbb{Z}}$ that satisfies (I) for all $n \geq 0$.
The proof of Theorem 1 can be used to establish the following result.

THEOREM 1′. *Inequality* (I) *has a non-negative solution which is eventually positive if and only if* (∗) *admits a root in* (0, 1).

Now, let us consider the difference equation

$$N_{n+1} - N_n = N_n\left(a - \sum_{k=0}^{\infty} c_k N_{n-k}\right), \tag{Ê}$$

where a is a positive constant and $\sum_{k=0}^{\infty} c_k < \infty$.

By a *solution* of (Ê) we mean a sequence $(N_n)_{n\in\mathbb{Z}}$ which satisfies $(\hat{E})$ for all $n \ge 0$. Equation (Ê) has a unique positive *equilibrium* N^* given by

$$a = N^* \sum_{k=0}^{\infty} c_k .$$

[By our assumptions, we have $0 < \sum_{k=0}^{\infty} c_k < \infty$.]

Consider the equation

$$\lambda - 1 + N^* \sum_{k=0}^{\infty} \lambda^{-k} c_k = 0. \qquad (\text{¥})$$

THEOREM 2. *Assume that* (¥) *has no roots in* $(0, 1)$. *Then there is no solution* $(N_n)_{n\in\mathbb{Z}}$ *of* (Ê) *such that*

$$N_n \ge N^* \quad \text{for every} \quad n \in \mathbb{Z}, \quad \text{and} \quad N_n > N^* \text{ for all large } n.$$

Proof. Let $(N_n)_{n\in\mathbb{Z}}$ be a solution of (Ê) with $N_n \ge N^*$ for every $n \in \mathbb{Z}$, and $N_n > N^*$ for all large n. Set

$$y_n = \frac{N_n}{n^*} - 1 \quad \text{for} \quad n \in \mathbb{Z}.$$

Then we observe that $(y_n)_{n\in\mathbb{Z}}$ is a non-negative sequence which is eventually positive. This sequence satisfies

$$y_{n+1} - y_n + N^*(1 + y_n) \sum_{k=0}^{\infty} c_k y_{n-k} = 0 \quad \text{for all} \quad n \ge 0.$$

Thus, we get

$$y_{n+1} - y_n + N^* \sum_{k=0}^{\infty} c_k y_{n-k} \le 0 \quad \text{for} \quad n \ge 0.$$

That is, the sequence $(y_n)_{n\in\mathbb{Z}}$ is a solution of the difference inequality

$$y_{n+1} - y_n + \sum_{k=0}^{\infty} \tilde{c}_k y_{n-k} \le 0,$$

where $\tilde{c}_k = N^* c_k$ for $k \ge 0$. An application of Theorem 1′ completes our proof.

Consider now the equation

$$N_{n+1} = N_n\left(c - bN_n - \sum_{k=0}^{\infty} c_k N_{n-k}\right), \qquad (\text{E}^*)$$

where $c > 1$ and $b > 0$ are constants, and $(c_k)_{k\ge 0}$ is a sequence of non-negative real numbers which is not identically zero and such that $\sum_{k=0}^{\infty} c_k < \infty$. By a *solution* of (E*) we mean a sequence $(N_n)_{n\in\mathbb{Z}}$ which satisfies (E*) for all $n \ge 0$.

We call such solutions $(N_n)_{n\in\mathbb{Z}}$ of (E*), which satisfy the condition below, *positive solutions*:

$$N_n \ge 0 \quad \text{for} \quad n < 0 \quad \text{and} \quad N_n > 0 \quad \text{for} \quad n \ge 0.$$

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Equation (E*) has a unique positive *equilibrium* N^* which is given by

$$\left(b+\sum_{k=0}^{\infty} c_k\right) N^* = c-1.$$

THEOREM 3. *Any positive solution of* (E*) *is bounded.*

Proof. We show that every positive solution $(N_n)_{n\in\mathbb{Z}}$ of (E*) is bounded for all $n \geq 0$. To see this observe that

$$\frac{N_{n+1}}{N_n} \leq c - bN_n \Rightarrow \frac{N_{n+1}}{N_n} \leq c - bN_n + 1 - 1 \Rightarrow \frac{N_{n+1}-N_n}{N_n} \leq (c-1) - bN_n, \quad \text{for all} \quad n \geq 0. \tag{i}$$

If $N_n \to +\infty$, as $n \to \infty$, then (i) implies

$$N_{n+1} - N_n < 0 \quad \text{for all large } n.$$

Since $N_n \to \infty$, there is $m \in \mathbb{Z}$ such that, for all $n > m$ we have $bN_n > c-1$. If there exists $n_0 > m$ such that

$$N_{n_0} > N_{n_0+1} > \ldots > N_{n_0+k} > \ldots,$$

then $\overline{\lim}\, N_n < \infty$ and the solution $(N_n)_{n\geq 0}$ will be bounded.

We suppose now the existence of n_0 such that

$$N_{n_0} > \frac{c-1}{b}, \qquad N_{n_0+1} - N_{n_0} \geq 0,$$

so that this implies the absurdity

$$0 \leq \frac{N_{n_0+1} - N_{n_0}}{N_0} \leq (c-1) - bN_{n_0} < 0.$$

Hence, there exists a constant $M > 0$ such that

$$0 < N_n \leq M, \quad \text{for all} \quad n \geq 0.$$

REMARK. When the equation (E*) has a positive solution such that $N_n \geq N^*$ $(n \in \mathbb{Z})$ and $N_n > N^*$ for all large n, then the characteristic equation (*) has a root in $(0, 1)$ and has no root greater or equal to one.

Indeed, we set $y_n = N_n - N^*$ in (E*) and, together with

$$c - bN^* - N^*\left(\sum_{k=0}^{\infty} c_k\right) = 1,$$

we obtain

$$y_{n+1} - y_n = -bN^* y_n - N^* \sum_{k=0}^{\infty} c_k y_{n-k} - by^2 - y_n \sum_{k=0}^{\infty} c_k y_{n-k}. \tag{E$_0$}$$

From (E_0) we have

$$y_{n+1} - y_n + N^* \sum_{k=0}^{\infty} c_k y_{n-k} \leq 0, \qquad \forall n \geq 0. \tag{E$_1$}$$

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According to Theorem 1′ (since $y_n > 0$) the characteristic equation

$$\lambda - 1 + N^* \sum_{k=0}^{\infty} c_k \lambda^{-k} = 0 \qquad (*)$$

has a root in $(0, 1)$.

Furthermore, if $\lambda \geq 1$, then

$$0 \leq \lambda - 1 = N^* \sum_{k=0}^{\infty} c_k \lambda^{-k} < 0$$

which is absurd.

So, the equation $(*)$ has no root $\lambda \geq 1$.

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