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INJECTIVE AND WEAKLY INJECTIVE RINGS

BY

B. J. GARDNER AND P. N. STEWART

ABSTRACT. Let V be a variety of rings and let $A \in V$. The ring A is injective in V if every triangle

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & C \arrow[dl, dashed] \\ A & \end{tikzcd}$$

with $C \in V$, m a monomorphism and f a homomorphism has a commutative completion as indicated. A ring which is injective in some variety (equivalently, injective in the variety it generates) is called *injective*. When only triangles with f surjective are considered we obtain the notion of *weak injectivity*. Directly indecomposable injective and weakly injective rings are classified.

Introduction. All rings considered in this paper are associative, but we do not assume that rings have an identity.

A ring A which is in a variety of rings V is *injective in* V if every triangle

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & C \arrow[dl, dashed] \\ A & \end{tikzcd}$$

with $C \in V$, m a monomorphism and f a homomorphism has a commutative completion as indicated. If A is injective in some variety of rings (equivalently, A is injective in Var(A), the variety generated by A), we shall say that A *injective*. When only triangles with f onto are considered, we obtain the notion of *weak-injectivity* and if we further restrict the triangles by insisting that $B = A$ and f be the identity map, we have *absolute subretracts*. This terminology differs from that of [3] where larger varieties are considered and where it is shown, among other things, that no nonzero ring is injective (or even is an absolute subretract) in the variety of all rings or in the variety of all commutative rings.

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The next section contains basic results about absolute subretracts. Directly indecomposable weakly injective and injective rings are classified in the third section, and in the final section an example is presented which shows that our results on weakly injective and injective rings cannot be extended to absolute subretracts.

This paper was inspired by the very general universal algebraic theorems in [1], and our Propositions 2, 3 and 6 are simply ring theoretic specializations of results from [1].

Absolute Subretracts. The notation $I \lhd A$ means that I is a (two-sided) ideal of A. A subring $S \subseteq A$ is *essential* if $S \cap I \neq 0$ whenever $0 \neq I \lhd A$.

PROPOSITION 1. *Let A be a ring and suppose that $B \in \mathrm{Var}(A)$ and $m:A \to B$ is a monomorphism. If $m(A)$ is essential in B and $m(A) \neq B$, then A is not an absolute subretract.*

PROOF. Any homomorphism $g:B \to A$ which is such that $gm = 1_A$ must have nonzero kernel because $m(A) \neq B$. But then $\ker g \cap m(A) \neq 0$, so no such g can exist. □

A ring A is *uniform* if every nonzero ideal of A is essential.

PROPOSITION 2 [1]. *A directly indecomposable absolute subretract is uniform.*

PROOF. Let A be directly indecomposable and suppose that A is not uniform. Using Zorn's lemma we obtain nonzero ideals I and J such that $I \cap J = 0$, $I \subsetneq K \lhd A$ implies $K \cap J \neq 0$ and $J \subsetneq L \lhd A$ implies $I \cap L \neq 0$.

Now one checks, in order, the following: (i) $((J + I)/I) \oplus (A/J)$ is essential in $(A/I) \oplus (A/J)$, (ii) similarly, $(A/I) \oplus ((I + J)/J)$ is essential in $(A/I) \oplus (A/J)$, (iii) $((J \oplus I)/I) \oplus ((I \oplus J)/J)$ is essential in $(A/I) \oplus (A/J)$, (iv) the image of A in $(A/I) \oplus (A/J)$, under the usual embedding, is essential.

Since A is directly indecomposable, the image of A does not equal $(A/I) \oplus (A/J)$, so A is not an absolute subretract by Proposition 1. □

PROPOSITION 3 [1]. *If A is directly indecomposable absolute subretract and $A^2 \neq 0$, then the (two-sided) annihilator of A is zero.*

PROOF. Let A be directly indecomposable, $A^2 \neq 0$, and suppose that $I = \{x \in A: xA = Ax = 0\} \neq 0$.

Let $J = \{(i, -i): i \in I\}$. Clearly $J \lhd A \oplus A$ and it is straightforward to check that the map $c:A \to (A \oplus A)/J$ defined by $c(a + i) = (a, i) + J$ for all $a \in A$, $i \in I$ is a well-defined monomorphism. Also, since $A^2 \neq 0$, $I \neq A$ and if $x \in A \backslash I$, then $(x, x) + J$ is not in the image of c. Thus c is not onto.

In view of Proposition 1 it suffices to show that the image of A under c is essential in $(A \oplus A)/J$. Let $J \subsetneq D \lhd A \oplus A$. Clearly $c(A) \cap (D/J) \neq 0$ unless

$(a, i) \in D$, $i \in I$, implies $a = -i$ and so we now assume this. Since $D \neq J$ there is an $x \in A \backslash I$ such that $(a, x) \in D$ for some $a \in A$. From Proposition 2 we know that the ideal $Ax + xA + AxA$ contains a nonzero $i \in I$. Thus $(0, i) \in D$ and so $c(A) \cap (D/J) \neq 0$. $\square$

PROPOSITION 4. *A prime absolute subretract is finite.*

PROOF. Suppose that A is a prime infinite absolute subretract. If A does not satisfy a proper identity, then $\mathrm{Var}(A)$ is either the variety of all rings (if the characteristic of A is zero) or the variety of all $\mathbf{Z}_p$ algebras (if the characteristic of A is $p \neq 0$). Now suppose that A does satisfy a proper identity. From [2, Corollary 4.2], A is a *PI* algebra and so Posner's theorem [2, Theorem 5.7] implies that the total ring of left and right fractions of A, $Q(A)$, is in $\mathrm{Var}(A)$ and $A \subseteq Q(A)$. Since $Q(A)$ is simple and A is an absolute subretract, $A = Q(A)$. Thus A is an infinite prime algebra over a field and so from [4, Corollary 2.3.36], $\mathrm{Var}(A)$ contains every central extension of A (recall that B is a central extension of A if $A \subseteq B$ and $B = AC$ where C is the centre of B). To sum up: whether or not A satisfies a property identity, $\mathrm{Var}(A)$ contains all central extensions of A.

Let A' be the usual unital extension of A and T the central extension of A' considered by Raphael in [3]: T is the localization of the polynomial ring $A'[x]$ obtained by inverting those monic polynomials whose coefficients are in the centre of A'. Now, T is a central extension of A and A is an absolute subretract, so there is a homomorphism $g: T \to A$ which extends the identity map $1: A \to A$. Since x is in the centre of T, $g(x) = a$ is in the centre of A. Also, $g(x - a) = 0$. This is a contradiction since $x - a$ is invertible in T. $\square$

If A is a ring, the underlying additive group of A will be denoted by A^+, and if G is an abelian group, the ring with trivial multiplication and underlying additive group G will be denoted by G^0. For any prime p, the cyclic p-groups will be denoted by $\mathbf{Z}_{p^n}$, $n = 1, 2, \ldots$ and the p-primary component of the rationals modulo 1 will be denoted by $\mathbf{Z}_{p^\infty}$. The additive group of the rationals will be denoted by $\mathbf{Q}$, and $\mathbf{Z}$ will denote the additive group of rational integers.

PROPOSITION 5. *If A is a directly indecomposable absolute subretract and $A^2 = 0$, then $A \cong (\mathbf{Z}_{p^i})^0$, for some prime p and some i, $1 \leqq i \leqq \infty$, or $A \cong \mathbf{Q}^0$. Moreover, all of these rings are injective.*

PROOF. Let A be a directly indecomposable absolute subretract with $A^2 = 0$. Since A^+ is directly indecomposable, either A^+ is torsion free or $A^+ \cong \mathbf{Z}_{p^i}$ for some prime p and some i, $1 \leqq i \leqq \infty$. In the torsion free case, $\mathrm{Var}(A)$ is the class of all zerorings and so $D^0 \in \mathrm{Var}(A)$ where D is the injective hull of A^+. Since A is an absolute subretract, there is a homomorphism from D^0 to A

extending the identity map on A. Thus A^+ is divisible and therefore, being directly indecomposable, is isomorphic to $\mathbf{Q}$.

Now, $\mathrm{Var}(\mathbf{Q}) = \mathrm{Var}(\mathbf{Z}_{p^\infty})$ is the variety of all abelian groups and $\mathbf{Q}$ and $\mathbf{Z}_{p^\infty}$ are injective in this variety. Hence $\mathbf{Q}^0$ and $(\mathbf{Z}_{p^\infty})^0$ are injective in $\mathrm{Var}(\mathbf{Q}^0) = \mathrm{Var}((\mathbf{Z}_{p^\infty})^0)$ which is the variety of all zerorings.

Finally we consider $(\mathbf{Z}_{p^i})^0$, $1 \leqq i < \infty$. Let E denote the injective hull of $\mathbf{Z}_{p^i}$ in the category $\mathrm{Mod}(\mathbf{Z}_{p^i})$ of unital $\mathbf{Z}_{p^i}$-modules (for the moment, $\mathbf{Z}_{p^i}$ denotes the ring of integers modulo p^i). Then E, as an abelian group, is an essential extension of $(\mathbf{Z}_{p^i})^+$ and we may assume that $(\mathbf{Z}_{p^i})^+ \subseteq E \subseteq \mathbf{Z}_{p^\infty}$. Now, since $p^iE = 0$, we have $E = \mathbf{Z}_{p^i}$. Thus $\mathbf{Z}_{p^i}$ is injective in $\mathrm{Mod}(\mathbf{Z}_{p^i})$ and so $(\mathbf{Z}_{p^i})^0$ is injective in $\mathrm{Var}((\mathbf{Z}_{p^i})^0)$. □

Injectivity and Weak Injectivity.

PROPOSITION 6 [1]. *A directly indecomposable weakly injective ring A such that $A^2 \neq 0$ is prime.*

PROOF. Let A be a directly indecomposable weakly injective ring such that $A^2 \neq 0$. In view of Proposition 2 we need only show that A is semiprime.

Suppose that $I \lhd A$ and $I^2 = 0$. Let B be the subring of $A \oplus A \oplus A$ defined by $B = \{(a, b, c): a - b, b - c \in I\}$ and define $f: B \to A$ by $f(a, b, c) = a - b + c$. The map f is clearly additive and

$$
\begin{aligned}
f(a, b, c)f(a', b', c') &= (a - b + c)(a' - b' + c) \\
&= aa' + a(-b' + c') + (-b + c)(a' - b') + (-b + c)c' \\
&= aa' - ab' + ac' - bc' + cc' \\
&= aa' - bb' + bb' - ab' + ac' - bc' + cc' \\
&= aa' - bb' + cc' + (b - a)(b' - c') \\
&= aa' - bb' + cc' = f((a, b, c)(a', b', c')),
\end{aligned}
$$

so f is a homomorphism. Since $f(a, a, a) = a$, f is onto.

Because A is weakly injective there is a homomorphism $g: A \oplus A \oplus A \to A$ such that g extends f. Since A is uniform by Proposition 2, and g is one-to-one on $I \oplus 0 \oplus 0$, $\ker g \cap (A \oplus 0 \oplus 0) = 0$. Similarly, $\ker g \cap (0 \oplus A \oplus 0) = 0$ and $\ker g \cap (0 \oplus 0 \oplus A) = 0$. It follows that $(\ker g)(A \oplus A \oplus A) = 0 = (A \oplus A \oplus A)(\ker g)$ and so Proposition 3 implies that $\ker g = 0$. However, $\{(0, x, x): x \in I\} \subseteq \ker f \subseteq \ker g$ and so $I = 0$ as is required. □

THEOREM 1. *A directly indecomposable ring A is weakly injective if and only if $A \cong (\mathbf{Z}_{p^i})^0$ for some prime p and some i, $1 \leqq i \leqq \infty$, $A \cong \mathbf{Q}^0$ or A is isomorphic to a complete ring of finite matrices with entries from a finite field.*

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PROOF. First assume that A is weakly injective and $A^2 = 0$. Then Proposition 5 implies that $A \cong (\mathbf{Z}_{p^i})^0$ for some prime p and some i, $1 \leqq i \leqq \infty$ or $A \cong \mathbf{Q}^0$. If $A^2 \neq 0$, then combining Propositions 4 and 6 we see that A is a finite prime ring and so A is isomorphic to a complete ring of finite matrices with entries from a finite field by the Artin-Webberburn theorem.

In view of Proposition 5, to prove the converse we need only prove that complete rings of finite matrices with entries from a finite field are weakly injective. Let A be such a ring and suppose we are given

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & C \in \mathrm{Var}(A) \\ A & \end{tikzcd}$$

where m is an embedding and f is onto. From Tarski's theorem we know that C is a homomorphic image of a subring D of a product $P = \Pi\{A_i : i \in I, A_i \cong A\}$. Let $p : D \rightarrow C$ be the homomorphism of D onto C and let $B' = p^{-1}(B)$. We now have the following diagram where p' is the restriction of p to B'.

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "p'"'] & D \arrow[r] \arrow[d, "p"'] & P \\ B \arrow[r] \arrow[d, "f"'] & C & \\ A & & \end{tikzcd}$$

Let $K =$ the kernel of $f \circ p'$. Since p' and f are both onto, $B'/K \cong A$. We now use Zorn's lemma to choose M maximal in $\{J : J \lhd P$ and $J \cap B' \subseteq K\}$. Since A is a simple idempotent ring, M is a prime ideal of P. Let $A = \{a_1, \ldots, a_n\}$ and, for each $j = 1, \ldots, n$, define $v_j \in P$ by $v_j(i) = a_j$ for all $i \in I$.

Let $v \in P$. Define, for each $k = 1, \ldots, n$, $I_k = \{i \in I : v(i) = a_k\}$ and $P_k = \{w \in P : w(i) = 0$ for all $i \in I_k\}$. For each $k = 1, \ldots, n$, $P_k \lhd P$ and since $\cap\{P_k : k = 1, \ldots, n\} = 0$, $P_t \subseteq M$ for some t, $1 \leqq t \leqq n$. Since $v - v_t \in P_t$, $v + M = v_t + M$. This shows that P/M is a homomorphic image of A and so, since A is simple, $P/M \cong A$. Now $B' \cap M \subseteq K$ and so $B'/(B' \cap M)$ has at least as many elements as does B'/K. Since $B'/(B' \cap M)$ is isomorphic to a subring of P/M we must have $B' \cap M = K$ and $B' + M = P$ because P/M and B'/K have the same number of elements.

Define $g:C \to A$ as follows: If $c \in C$ and $c = p(d)$ for $d \in D$ where $d + M = b' + M$ with $b' \in B'$, set $g(c) = (f \circ p)(b')$. It is routine to check that g is a well-defined extension of f. Thus A is weakly injective. □

THEOREM 2. *A directly indecomposable ring A is injective if and only if $A \cong (\mathbf{Z}_{p^i})^0$ for some prime p and some i, $1 \leqq i \leqq \infty$, $A \cong \mathbf{Q}^0$ or A is isomorphic to a finite field.*

PROOF. We first note that if A is the ring of $n \times n$ matrices, $n > 1$, with entries from a field F, then A is not injective. To see this consider the diagram

$$\begin{tikzcd} F \arrow[r, "m"] \arrow[d, "f"'] & A \\ A & \end{tikzcd}$$

where $(m(x))_{ij} = 0$ unless $i = j$ and $(m(x))_{ii} = x$, and $(f(x))_{ij} = 0$ unless $i = j = 1$ and $(f(x))_{11} = x$, for all i, j, $1 \leqq i, j, \leqq n$. Since it is clear that the above triangle cannot be completed, A is not injective.

In view of Proposition 5 and Theorem 1, the proof will be complete if we show that finite fields are injective.

Let F be a finite field and suppose that we have

$$\begin{tikzcd} B \arrow[r, "m"] \arrow[d, "f"'] & C \in \mathrm{Var}(F) \\ F & \end{tikzcd}$$

where m is an embedding and $f \neq 0$. As in the last part of the proof of Theorem 1 we obtain a diagram

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "p'"'] & D \arrow[r] \arrow[d, "p"'] & P \\ B \arrow[r, "m"] \arrow[d, "f"'] & C & \\ F & & \end{tikzcd}$$

where P is a product of copies of F, p is onto, $B' = \{d \in D:p(d) \in B\}$ and p' is the restriction of p to B'. Let K be the kernel of $f \circ p'$. Again, as in the proof of

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Theorem 1, we obtain an ideal M of P such that $M \cap B' \subseteq K$ and $P/M \cong F$. Now $B'/(M \cap B')$ is isomorphic to a subring of F and is thus a field; since $K/(M \cap B') \lhd B'/(M \cap B')$ and $f \neq 0$, $K = M \cap B'$. However, $(B' + M)/M$ may not equal P/M because f may not be onto.

We now have the following diagram,

$$\begin{array}{ccc} (B' + M)/M & \longrightarrow & P/M \cong F \\ \Big\downarrow{\scriptstyle\theta} & & \\ F & & \end{array}$$

where $\theta(b' + M) = (f \circ p')(b')$ for all $b' \in B$. The map θ is a monomorphism of the subfield $(B' + M)/M$ of F into F and since F is a normal extension of $(B' + M)/M$, θ extends to an isomorphism $\bar{\theta}: P/M \to F$.

Define $g: C \to F$ by $g(c) = \bar{\theta}(s + M)$ where $s \in S$ and $p(s) = c$. It is straightforward to check that g is a well-defined extension of f and so F is injective. □

Absolute Subretracts Revisited. Proposition 5 characterizes directly indecomposable absolute subretracts A such that $A^2 = 0$.

Let A be a directly indecomposable absolute subretract with $A^2 \neq 0$. We know that A is uniform (Proposition 2) and so if A is finite, in particular, if A is prime (Proposition 4), then A is subdirectly irreducible. Moreover, the two-sided annihilator of A is zero (Proposition 3). However, A need not be semiprime.

The prime radical of a ring S will be denoted by $\beta(S)$.

PROPOSITION 7. *Let* $R = \mathbf{Z}_2[x]/(x^2)$.

(a) *If* $B \in \mathrm{Var}(R)$, *then* $(\beta(B))^2 = 0$ *and* $B/\beta(B)$ *is a Boolean ring.*

(b) *If* $B \in \mathrm{Var}(R)$ *is subdirectly irreducible, then* $B \cong R$, $\mathbf{Z}_2$ *or* $(\mathbf{Z}_2)^0$.

(c) R *is an absolute subretract.*

PROOF. (a) Let $B \in \mathrm{Var}(R)$. Tarski's theorem implies that there is a product $\Pi\{R_i : i \in I, R_i \cong R\}$ with a subring S such that B is a homomorphic image of S. We shall use the notation ΠR for $\Pi\{R_i : i \in I, R_i \cong R\}$.

Since R is commutative, $\beta(S) = (\beta(\Pi R)) \cap S$ and $(\Pi R)/\beta(\Pi R) \cong (\Pi R)/\Pi\beta(R)$ is a Boolean ring. Since $(\beta(\Pi R))^2 = 0$, $\beta(S)^2 = 0$ and since $S/\beta(S) \cong (S + \beta(\Pi R))/\beta(\Pi R) \subseteq (\Pi R)/\beta(\Pi R)$, $S/\beta(S)$ is a Boolean ring.

Because B is a homomorphic image of S, there is an ideal K of S such that $B \cong S/K$.

If $\beta(B) = B$, then $\beta(S/K) = S/K$ and since $S/\beta(S)$ is Boolean, $S/(K + \beta(S))$ is both nil and Boolean. Hence $K + \beta(S) = S$ and so $B \cong S/K = (K + \beta(S))/K \cong \beta(S)/(K \cap \beta(S))$ is such that $B^2 = 0$.

If $\beta(B) = 0$, $\beta(S) \subseteq K$ and so B, a homomorphic image of a Boolean ring, is Boolean.

The result now follows because Var(R) is closed under ideals and homomorphic images.

(b) Let $B \in$ Var(R) be subdirectly irreducible with heart $0 \neq H \lhd B$. If $\beta(B) = 0$, B is Boolean and hence $B = H \cong \mathbf{Z}_2$. If $\beta(B) = B$, then, since $B^2 = 0$ and B has characteristic 2, $B = H \cong (\mathbf{Z}_2)^0$.

Suppose now that $0 \neq \beta(B) \neq B$. Since B satisfies the identity $x^2 = x^4$ and B is not nil, B contains nonzero idempotents. Let $0 \neq e = e^2 \in B$ and suppose that $b \in B$. If $eb - b \neq 0$, then $H \subseteq B(eb - b) \cup \{eb - b\}$ and $H \subseteq Be \cup \{e\}$. The first containment implies that $He = 0$ and the second implies that $He = H$. Thus $eb - b = 0$ and so e is an identity for B. If $0 \neq f = f^2 \in B$, then $(B(e - f) \cup \{e - f\}) \cap (Bf \cup \{f\}) = 0$ and hence $e = f$. Since idempotents can be lifted modulo $\beta(B)$ and $B/\beta(B)$ is Boolean, this shows that $B/\beta(B) \cong \mathbf{Z}_2$. Let $0 \neq w \in H$ and suppose that there is a $0 \neq v \in \beta(B)$ such that $v \neq w$. Then $Bv \cup \{v\} \supseteq H$, so $w = bv$ for some $b \in B$. Also, since $B/\beta(B) \cong \mathbf{Z}_2$, $b = e + z$ for some $z \in \beta(B)$ or $b \in \beta(B)$. Because $\beta(B)^2 = 0$, $b \notin \beta(B)$. Thus $w = (e + z)v = ev + zv = v + 0 = v$. This contradiction shows that $\beta(B) = H = \{0, w\}$ and so $B \cong R$.

(c) Suppose that $R \subseteq C \in$ Var(R). Choose $M \lhd C$, M maximal with respect to $M \cap R = 0$. Since R is subdirectly irreducible, so is C/M. Now $R \cong (R + M)/M \subseteq C/M$ and so from (b) we see that $C/M \cong R$. Thus $R + M = C$ and it follows that R is an absolute subretract. □

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MATHEMATICS DEPARTMENT
UNIVERSITY OF TASMANIA
G.P.O. BOX 252-C
HOBART, TASMANIA 7001
AUSTRALIA

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTING SCIENCE
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA B3H 3J5
CANADA