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ON THE SPECTRA OF PISOT NUMBERS

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Abstract. Let θ be a real number greater than 1, and let $((\ ))$ be the fractional part function. Then, θ is said to be a Z-number if there is a non-zero real number λ *such that* $((\lambda\theta^n)) < \frac{1}{2}$ *for all* $n \in \mathbb{N}$. *Dubickas* (A. Dubickas, Even and odd integral parts of powers of a real number, *Glasg. Math. J.*, **48** (2006), 331–336) *showed that strong Pisot numbers are Z-numbers. Here it is proved that* θ *is a strong Pisot number if and only if there exists* $\lambda \neq 0$ *such that* $((\lambda\alpha)) < \frac{1}{2}$ *for all* $\alpha \in \{\theta^n \mid n \in \mathbb{N}\} \cup \{\sum_{n=0}^{N} \theta^n \mid N \in \mathbb{N}\}$. Also, the following characterisation of Pisot numbers among real numbers greater than 1 is shown: θ *is a Pisot number* $\Leftrightarrow \exists\, \lambda \neq 0$ *such that* $\|\lambda\alpha\| < \frac{1}{3}$ *for all* $\alpha \in \{\sum_{n=0}^{N} a_n\theta^n \mid a_n \in \{0, 1\},\ N \in \mathbb{N}\}$, *where* $\|\lambda\alpha\| = \min\{((\lambda\alpha)), 1 - ((\lambda\alpha))\}$.

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1. Introduction. For a point t of the real line $\mathbb{R}$ we denote by $[t]$ the largest element of the ring $\mathbb{Z}$ of rational integers, not exceeding t. We also denote by $((t))$ and $\|t\|$ the difference $t - [t]$ and the minimum of the set $\{((\lambda\alpha)), 1 - ((\lambda\alpha))\}$, respectively. Namely, $[t]$ is the integer part of t, $((\ ))$ is the fractional part function and $\|t\|$ is the usual distance from t to $\mathbb{Z}$.

Let throughout $\theta \in (1, \infty)$, $\lambda \in \mathbb{R}\backslash\{0\}$ and $n \in \mathbb{N} := \mathbb{Z} \cap [0, \infty)$. Dubickas in [**3**] defined a subset Z of $(1, \infty)$, with the property that for each $\theta \in Z$, there is $\lambda = \lambda(\theta)$ such that $((\lambda\theta^n)) < \frac{1}{2}$ for all n. An element of Z is called a Z-number. A result due to Tijdeman, and cited in [**3**] gives immediately that $[3, \infty) \subset Z$. Set $Y := (1, \infty)\setminus Z$. Some classes of algebraic integers, which belong to $Z \cap (1, 3)$, or to $Y \cap (1, 2)$, are exhibited in [**3**], and from this one can easily deduce that 2 is a left-hand limit point of Z, and 1 is a limit point of Y. Dubickas proved in particular that strong Pisot numbers are Z-numbers. Recall that a Pisot number is a real algebraic integer greater than 1 whose other conjugates are of modulus less than 1. The set of Pisot numbers is usually noted S. A Pisot number θ of degree d is called a strong Pisot number if $d = 1$, or if $d \geq 2$ and θ has a conjugate belonging to the interval $(0, 1)$, which is greater than the absolute values of $d - 2$ remaining conjugates of θ [**2**]. We denote by S_{st} the set of strong Pisot numbers.

Let

$$A_0 = A_0(\theta) := \{\theta^n \mid n \in \mathbb{N}\} \cup \left\{\sum_{n=0}^{N} \theta^n \mid N \in \mathbb{N}\right\},$$

$$A_m = A_m(\theta) := \left\{\sum_{n=0}^{N} a_n\theta^n \ \mid a_n \in \{0, \dots, m\}, N \in \mathbb{N}\right\},$$

where $m \in \mathbb{N}\backslash\{0\}$, and

$$B_m = B_m(\theta) := A_m(\theta) - A_m(\theta).$$

The first aim of this note is to show that strong Pisot numbers are a kind of 'strong Z-numbers' :

THEOREM 1. *The following are equivalent.*
(i) $\theta \in S_{st}$.
(ii) *For any* $\varepsilon > 0$ *and any* m, *there is* λ *such that* $((\lambda\alpha)) < \varepsilon$ *for all* $\alpha \in A_m(\theta)$.
(iii) *There exist* m *and* λ *such that* $((\lambda\alpha)) < \frac{1}{2}$ *for all* $\alpha \in A_m(\theta)$.
(iv) *There is* λ *such that* $((\lambda\alpha)) < \frac{1}{2}$ *for all* $\alpha \in A_0(\theta)$.

In terms of fractional part function, Theorem 1 may be viewed as a characterisation of strong Pisot numbers among real numbers greater than 1. This contrasts with the famous characterisation of Pisot numbers among real numbers, due to Pisot [**4**], which says: *If there is* λ *such that* $\sum_{n\in\mathbb{N}} \|\lambda\theta^n\|^2 < \infty$, *then* $\theta \in S$. The important question whether there is a transcendental number θ satisfying $\lim_{n\to\infty} \|\lambda\theta^n\| = 0$ for some λ, is still unsolved [**1**]. We shall mainly use this last mentioned result of Pisot to prove Theorem 1 and the result below:

THEOREM 2. *The following are equivalent.*
(i) $\theta \in S$.
(ii) *For any* $\varepsilon > 0$ *and any* $m \in \mathbb{N}$, *there is* λ *such that* $||\lambda\beta|| < \varepsilon$ *for all* $\beta \in B_m(\theta)$.
(iii) *For any* $\varepsilon > 0$ *and any* $m \in \mathbb{N}$, *there is* λ *such that* $||\lambda\alpha|| < \varepsilon$ *for all* $\alpha \in A_m(\theta)$.
(iv) *There is* λ *such that* $\|\lambda\alpha\| < \frac{1}{3}$ *for all* $\alpha \in A_1(\theta)$.

In these pages when we speak about conjugates, minimal polynomial and degree of an algebraic number we mean over the field of the rationals $\mathbb{Q}$. For a Pisot number θ of degree d, we denote by $\theta := \theta_1, \ldots, \theta_d$, the conjugates of θ, and by $\sigma_1, \ldots, \sigma_d$, the embeddings of $\mathbb{Q}(\theta)$ into the complex field $\mathbb{C}$, where σ_1 is the identity of $\mathbb{Q}(\theta)$. As usual, for an element α of the field $\mathbb{Q}(\theta)$, we denote by $Trace(\alpha)$ the sum $\sigma_1(\alpha) + \cdots + \sigma_d(\alpha)$, namely the trace of α for the extension $\mathbb{Q}(\theta)/\mathbb{Q}$. The proofs of Theorems 1 and 2 appear in the following sections, consecutively. It is interesting to determine whether the constant $1/3$ in Theorem 2(iv) is optimal, or whether we may replace $A_1(\theta)$ by one of its proper subsets without affecting the conclusion. Analogue questions may be posed for Theorem 1. Distribution in $\mathbb{R}$ of the elements of the set S_{st} is another problem related to Theorem 1. Some computations suggest the following conjecture: $\min S_{st} = 2$, $\min(S_{st}\backslash\{2\}) = (3+\sqrt{5})/2$ and $\min S'_{st} = 3$, where S'_{st} is the derived set of S_{st}. From the proof of the result below, one can easily deduce that 3 is a left-hand limit point of S_{st}.

PROPOSITION. *The set* S'_{st} *contains* $\mathbb{N} \cap [3, \infty)$.

Proof. Let b be a rational integer greater than 2 and let $P_n(x) := x^n(x-b)+1$, where $n \geq b$. Since $|bz^n| = b > 2 \geq |z^{n+1}+1|$ when the complex number z runs through the unit circle, Rouché's theorem gives that P_n has n roots with modulus less than 1, and so the polynomial P_n has a unique root, say $\theta_{(n)}$, of modulus greater than 1, as $P_n(0) = 1$. Hence, P_n is irreducible over $\mathbb{Q}$ and is the minimal polynomial of $\theta_{(n)}$. Notice also that the real function $P_n(t)$ is decreasing on the interval $(0, nb/(n+1))$ and is increasing on $(nb/(n+1), \infty)$ because its formal derivative is $(n+1)t^{n-1}(t - nb/(n+1))$. It follows by the relations $P_n(0) = 1$, $P_n(1) = 2-b$, $b-1 < nb/(n+1)$ and $P_n(b) = 1$ that P_n

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has a unique root, say ρ, in the interval $(0, 1)$ and $\theta_{(n)} \in (b-1, b)$. Consequently, $\theta_{(n)}$ is a Pisot number, and

$$\lim_{n\to\infty} \theta_{(n)} = b,$$

since $0 < b - \theta_{(n)} = \frac{1}{\theta_{(n)}^n} < \frac{1}{(b-1)^n} \leq \frac{1}{2^n}$. If α is a conjugate of $\theta_{(n)}$ such that $\alpha \neq \theta_{(n)}$, then $b|\alpha|^n = |\alpha^{n+1} + 1| \leq |\alpha|^{n+1} + 1$; hence $P(|\alpha|) = |\alpha|^{n+1} - b|\alpha|^n + 1 \geq 0$ and so $|\alpha| \leq \rho$, as $P_n(\rho) = 0$ and $P_n(t)$ is decreasing on $(0, 1)$. Moreover, the equality $|\alpha| = \rho$ holds only if $|\alpha^{n+1} + 1| = |\alpha|^{n+1} + 1$ that is when α^{n+1} is a positive real number. It follows in this case by the equality $\alpha^{n+1} = b\alpha^n - 1$ that $\alpha^n > 1/b > 0$, and so $\alpha = \alpha^{n+1}/\alpha^n$ is also a positive real number; thus, $\alpha = \rho$ and so $\theta_{(n)} \in S_{st}$. □

REMARK. A simple computation shows that any polynomial of the form $x^2 - bx + k$, where $b \in \mathbb{N} \cap [3, \infty[$ and $k \in \{1, \ldots, b-2\}$, is the minimal polynomial of a quadratic strong Pisot number, say θ_k, satisfying $b - 1 < \theta_k < b$. Similarly, as in the above proof, by considering the sequence of polynomials $x^n(x^2 - bx + k) + 1$, we easily obtain that θ_k is a left-hand limit point of the set S_{st}, when $b \geq 4$ and $k \leq b - 3$. Consequently, each interval of the form $[n, n+1]$, where $n \geq 3$, contains at least n elements of the set S'_{st}.

2. Proof of Theorem 1. To make clear the proof of Theorem 1, let us recall some results on Pisot numbers. The first two results are due to Pisot [**4**] and Smyth [**6**].

LEMMA 1. ([**4**]) *If* $\sum_{n=0}^{\infty} \|\lambda\theta^n\|^2 < \infty$ *for some* λ, *then* $\theta \in S$ *and* $\lambda \in \mathbb{Q}(\theta)$.

LEMMA 2. ([**6**]) *Two distinct conjugates of a Pisot number having the same modulus are complex conjugates.*

Theorem (ii) and Lemma 2 of [**7**] yield the following :

LEMMA 3. *If* λ *satisfies* $\lim_{n\to\infty} \|\lambda\theta^n\| = 0$ *for some* $\theta \in S$, *then* $\lambda \in \mathbb{Q}(\theta)$ *and there is* $N \in \mathbb{N}$ *such that* $Trace(\lambda\theta^n) \in \mathbb{Z}$ *for all* $n \geq N$.

Finally, let us show a simple argument on the conjugates of a Pisot number.

LEMMA 4. *Let* θ *be a Pisot number of degree* d. *Then for any positive rational integer* p, θ^p *is a Pisot number of degree* d. *If* $\rho e^{ia\pi}$ *is a conjugate of* θ, *where* $i^2 = -1$ *and* $(\rho, a) \in (0, 1) \times (0, 1)$, *then for any* $b \in \mathbb{R}$, *the sequence* $(((na + b)))_n$ *is dense in* $[0, 1]$.

Proof. Let p be a positive rational integer. Then, $\theta^p \in \mathbb{Q}(\theta)$, and the conjugates of θ^p are among the numbers $\theta^p, \theta_2^p, \ldots, \theta_d^p$. Since $|\theta_k^p| < 1$ for all $k \in \{2, \ldots, p\}$, θ^p is not repeated by the action of embeddings $\sigma_1, \ldots, \sigma_d$; thus $\mathbb{Q}(\theta) = \mathbb{Q}(\theta^p)$ and θ^p is a Pisot number of degree d. Let $\rho e^{ia\pi}$ be a non-real conjugate of θ, then $\rho e^{-ia\pi}$ is also another conjugate of θ, and so (by the first part of Lemma 4) $\rho^p e^{ipa\pi}$ and $\rho^p e^{-ipa\pi}$ are two distinct conjugates of θ^p. Hence, $a \notin \mathbb{Q}$, and the result follows immediately by Kronecker's theorem (see for instance Appendix 8 in [**5**]). □

Proof of Theorem 1. Let θ be a strong Pisot number with degree d, and let $\varepsilon > 0$. If $d = 1$, then $A_m \subset \mathbb{N}$ and so $((\alpha)) = 0 < \varepsilon$ for all $\alpha \in A_m$. Now, suppose $d \geq 2$, and $\theta_2 > |\theta_3| \geq \ldots \geq |\theta_d|$. Then, $t_n := Trace(\theta^n) = \theta^n + \theta_2^n + \cdots + \theta_d^n \in \mathbb{Z}$, $\theta_2^n + \cdots + \theta_d^n = t_n - \theta^n \in \mathbb{R}$, $\theta_2^n + \cdots + \theta_d^n < d\theta_2^n$ and $\lim_{n\to\infty} \sum_{k=2}^{d} (\frac{\theta_k}{\theta_2})^n = 1$. Let n_1 be a positive rational integer such that for all $n \geq n_1$ we have

$$0 < \theta_2^n + \cdots + \theta_d^n \tag{1}$$

and

$$\frac{md\theta_2^n}{1-\theta_2} < \min\{1, \varepsilon\}. \tag{2}$$

Setting $\lambda := -\theta^{n_1}$, we have $\lambda\theta^n = -\theta^{n+n_1} = -t_{n+n_1} + \theta_2^{n+n_1} + \cdots + \theta_d^{n+n_1}$, and the relations (1) and (2) give $0 < \theta_2^{n+n_1} + \cdots + \theta_d^{n+n_1} < d\theta_2^{n+n_1} < (1-\theta_2)/m < 1$ for all n. Hence, $-t_{n+n_1} = [\lambda\theta^n]$, $((\lambda\theta^n)) = \theta_2^{n+n_1} + \cdots + \theta_d^{n+n_1}$, and so $((\lambda\theta^n)) < (1-\theta_2)\varepsilon/m < \varepsilon$ for all n. Similarly, if $\alpha = \sum_{n=0}^{N} a_n\theta^n$, where $a_n \in \{0, 1, \ldots, m\}$ and $N \in \mathbb{N}$, then $\lambda\alpha = \sum_{n=0}^{N} a_n\lambda\theta^n = -\sum_{n=0}^{N} a_n t_{n+n_1} + \sum_{n=0}^{N} a_n(\theta_2^{n+n_1} + \cdots + \theta_d^{n+n_1})$, and the inequalities (1) and (2) again yield $0 < \sum_{n=0}^{N} a_n(\theta_2^{n+n_1} + \cdots + \theta_d^{n+n_1}) < \sum_{n=0}^{N} a_n(d\theta_2^{n+n_1}) \leq md\theta_2^{n_1}\sum_{n=0}^{N}\theta_2^n < \frac{md\theta_2^{n_1}}{1-\theta_2} < \min\{1, \varepsilon\}$; thus, $((\lambda\alpha)) = \sum_{n=0}^{N} a_n(\theta_2^{n+n_1} + \cdots + \theta_d^{n+n_1}) < \varepsilon$, as $-\sum_{n=0}^{N} a_n t_{n+n_1} \in \mathbb{Z}$, and so Theorem 1(ii) holds. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) in Theorem 1 are trivially true, since $A_0 \subset A_1 \subset A_m$. To show that the proposition (iv) $\Longrightarrow$ (i), is true, let us first verify the equalities

$$\sum_{n=0}^{N}((\lambda\theta^n)) = \left(\left(\lambda\sum_{n=0}^{N}\theta^n\right)\right), \tag{3}$$

where λ satisfies $((\lambda\alpha)) < \frac{1}{2}$ for all $\alpha \in A_0$, and $N \in \mathbb{N}$. It is clear that (3) holds for $N = 0$. By the relations $\lambda\sum_{n=0}^{N+1}\theta^n = [\lambda\sum_{n=0}^{N}\theta^n] + ((\lambda\sum_{n=0}^{N}\theta^n)) + [\lambda\theta^{N+1}] + ((\lambda\theta^{N+1}))$ and $0 \leq ((\lambda\sum_{n=0}^{N}\theta^n)) + ((\lambda\theta^{N+1})) < \frac{1}{2} + \frac{1}{2}$, where $N \in \mathbb{N}$, we have $((\lambda\sum_{n=0}^{N+1}\theta^n)) = ((\lambda\sum_{n=0}^{N}\theta^n)) + ((\lambda\theta^{N+1}))$, and a simple induction gives (3). Letting N tends to infinity in (3), we obtain

$$\sum_{n=0}^{\infty}((\lambda\theta^n)) \leq \frac{1}{2}$$

and so $\sum_{n=0}^{\infty}\|\lambda\theta^n\| \leq \frac{1}{2}$. It follows by Lemma 1 that $\theta \in S$ and $\lambda \in \mathbb{Q}(\theta)$. The last inequality also gives $\lim_{n\to\infty}\|\lambda\theta^n\| = 0$, and so by Lemma 3, there is $n_2 \in \mathbb{N}$ such that $t_n := Trace(\lambda\theta^n) \in \mathbb{Z}$ for all $n \geq n_2$. Let d be the degree of θ. If $d = 1$, then $\theta \in S_{st}$. Suppose $d \geq 2$, and $|\theta_2| \geq \ldots \geq |\theta_d|$. Lemma 2 says that we have to prove that θ_2 is a positive real number. Assume that $n \geq n_2$. Then, $t_n = \lambda\theta^n + \lambda_2\theta_2^n + \cdots + \lambda_d\theta_d^n$, where $\lambda_k = \sigma_k(\lambda)$ for $k \in \{1, \ldots, d\}$, $\lambda_2\theta_2^n + \cdots + \lambda_d\theta_d^n = t_n - \lambda\theta^n \in \mathbb{R}$, and

$$t_n - [\lambda\theta^n] = ((\lambda\theta^n)) + \lambda_2\theta_2^n + \cdots + \lambda_d\theta_d^n. \tag{4}$$

Let n_3 be the smallest element of $\mathbb{N}$ satisfying

$$d\,|\theta_2|^{n_3}\max_{2\leq j\leq d}|\lambda_j| < 1/2.$$

Then, (4) gives, for $n \geq \max\{n_2, n_3\}$,

$$-1/2 < t_n - [\lambda\theta^n] < 1/2 + 1/2,$$

since $|\lambda_2\theta_2^n + \cdots + \lambda_d\theta_d^n| < d\,|\theta_2|^n \max_{1\le j\le d} |\lambda_j|$ (recall that $0 \le ((\lambda\theta^n)) < 1/2$ for all n); thus $t_n - [\lambda\theta^n] = 0$ and so

$$\lambda_2\theta_2^n + \cdots + \lambda_d\theta_d^n = -((\lambda\theta^n)) \le 0. \tag{5}$$

Now we claim that the result follows directly by (5) and Lemmas 2 and 4. Indeed, if $\theta_2 \notin \mathbb{R}$, then $d \ge 3$, $\theta_3 = \overline{\theta_2}$ and $\lambda_3 = \overline{\lambda_2}$. Set $\theta_2 := \rho e^{ia\pi}$ and $\lambda_2 = \eta e^{ib\pi}$, where $i^2 = -1$, $(|a|, \rho) \in (0,1)\times(0,1)$, $b \in (-1,1]$ and $\eta > 0$. Then, Lemma 4 states that there are infinitely many n such that the corresponding quantities $2\eta\cos((na+b)\pi) + \sum_{k=4}^{d}\lambda_k\frac{\theta_k^n}{\rho^n}$ are all positive because $\lim_{n\to\infty}\frac{\theta_k^n}{\rho^n} = 0$ for all $k \in \{4,\ldots,d\}$; this leads to a contradiction since by (5) we have $\rho^n(2\eta\cos((na+b)\pi) + \sum_{k=4}^{d}\lambda_k\frac{\theta_k^n}{\rho^n}) = \lambda_2\theta_2^n + \lambda_3\theta_3^n + \cdots + \lambda_d\theta_d^n \le 0$. Finally, if $\theta_2 \in \mathbb{R}$, then the relation (5), together with Lemma 2, again gives $\lim_{n\to\infty}\frac{-((\lambda\theta^n))}{\theta_2^n} = \lambda_2$, and so $\theta_2 > 0$ and $\lambda_2 < 0$, as $\lambda = \sigma_2^{-1}(\lambda_2) = 0$ when $\lambda_2 = 0$; thus $\theta \in S_{st}$. □

3. Proof of Theorem 2. Let $\varepsilon > 0$, and let $\beta = \sum_{n=0}^{N} b_n\theta^n$, where θ is a Pisot number of degree d, $N \in \mathbb{N}$ and $b_n \in \{-m, -m+1, \ldots, m\}$. If $d = 1$, then $B_m \subset \mathbb{N}$ and so $||\beta|| = 0 < \varepsilon$. Suppose $d \ge 2$. It is clear that β is an integer of the field $\mathbb{Q}(\theta)$, the conjugates of β are among the numbers $\beta_k := \sigma_k(\beta) = \sum_{n=0}^{N} b_n\theta_k^n$, where $k \in \{1,\ldots,d\}$, and

$$|\beta_k| \le m\sum_{n=0}^{N}|\theta_k|^n < \frac{m}{1-|\theta_k|} \text{ for } k \in \{2,\ldots,d\}. \tag{6}$$

Set $\lambda := \theta^p$, where $p \in \mathbb{N}$ and satisfies $|\theta_k|^p < \frac{\varepsilon(1-|\theta_k|)}{m(d-1)}$ for all $k \in \{2,\ldots,d\}$. Then, $t := Trace(\lambda\beta) = \theta^p\beta + \theta_2^p\beta_2 + \cdots + \theta_d^p\beta_d \in \mathbb{Z}$, and by the relation (6) we obtain

$$||\lambda\beta|| \le |\lambda\beta - t| = \left|\theta_2^p\beta_2 \cdots + \theta_d^p\beta_d\right| < \varepsilon;$$

thus, Theorem 2(ii) holds. The implications (ii) ⇒ (iii) ⇒ (iv) in Theorem 2 are trivially true because $A_1 \subset A_m \subset B_m$. Now assume that there is λ such that $\|\lambda\alpha\| < \frac{1}{3}$ for all $\alpha \in A_1$. We shall use Lemma 1 to prove that $\theta \in S$. Set $\lambda\theta^n := x_n + y_n$, where $x_n \in \mathbb{Z}$ and $|y_n| = \|\lambda\theta^n\|$. If $s_N = \sum_{n=0}^{N} a_n\theta^n$, where $a_n \in \{0,1\}$ and $N \in \mathbb{N}$, then $\lambda s_N = \sum_{n=0}^{N} a_n x_n + \sum_{n=0}^{N} a_n y_n$ and $\sum_{n=0}^{N} a_n x_n \in \mathbb{Z}$. Similarly, as in the proof of Theorem 1, let us show the relation

$$\left|\sum_{n=0}^{N} a_n y_n\right| < \frac{1}{3} \text{ for all } N. \tag{7}$$

If $N = 0$, then $a_0y_0 \in \{0, y_0\}$, and so $-1/3 < a_0y_0 < 1/3$, as $|y_0| = \|\lambda\|$. Suppose that (7) holds for some $N \in \mathbb{N}$, and let $s_{N+1} = \sum_{n=0}^{N+1} a_n\theta^n$, where $(a_n)_{0\le n\le N+1}$ is a sequence of elements of the set $\{0,1\}$. By the hypothesis and the induction hypothesis we have $|\sum_{n=0}^{N+1} a_n y_n| \le |\sum_{n=0}^{N} a_n y_n| + |a_{n+1}y_{n+1}| < \frac{1}{3} + \frac{1}{3}$. Since $\lambda s_{N+1} = x + y$, where $x \in \mathbb{Z}$ and $|y| = \|\lambda s_{N+1}\| < 1/3$, and $\lambda s_{N+1} = \sum_{n=0}^{N+1} a_n x_n + \sum_{n=0}^{N+1} a_n y_n$, we see that $\sum_{n=0}^{N+1} a_n y_n - y \in \mathbb{Z}$. It follows by the inequalities $|\sum_{n=0}^{N+1} a_n y_n - y| \le |\sum_{n=0}^{N+1} a_n y_n| + |y| < \frac{2}{3} + \frac{1}{3}$ that $\sum_{n=0}^{N+1} a_n y_n - y = 0$, $|\sum_{n=0}^{N+1} a_n y_n| = |y| < \frac{1}{3}$, and so (7) is true. Now fix (for a moment) a positive rational integer N, and consider the subsets, say U and

V, of $\{0, 1, \ldots, N\}$ defined as follows: $n \in U \Leftrightarrow y_n > 0$, and $n \in V \Leftrightarrow y_n < 0$. Then, $\sum_{n=0}^{N} \|\lambda\theta^n\| = \sum_{n=0}^{N} |y_n| = \sum_{n\in U} y_n + \sum_{n\in V}(-y_n)$, and so

$$\sum_{n=0}^{N} \|\lambda\theta^n\| < \frac{2}{3}, \tag{8}$$

since by (7) we have $|\sum_{n\in U} y_n| < \frac{1}{3}$ and $|\sum_{n\in V}(-y_n)| < \frac{1}{3}$. Letting N tend to infinity in (8), we obtain the result by Lemma 1. □

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