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RANK PROPERTIES IN FINITE INVERSE SEMIGROUPS

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Abstract Two possible concepts of rank in inverse semigroup theory, the intermediate I-rank and the upper I-rank, are investigated for the finite aperiodic Brandt semigroup. The so-called large I-rank is found for an arbitrary finite Brandt semigroup, and the result is used to obtain the large rank of the inverse semigroup of all proper subpermutations of a finite set.

Keywords: inverse semigroup; Brandt semigroup; rank; intermediate I-rank; upper I-rank; large I-rank

AMS 1991 *Mathematics subject classification:* Primary 20M10

1. Introduction

In previous papers [**3**, **4**], Howie and I investigated various concepts of rank applied to certain known semigroups. One of the semigroups involved was the finite aperiodic Brandt semigroup B_n, whose definition is recalled below. This is of course an *inverse* semigroup, and if we choose to regard it in this way the questions regarding rank change significantly. This is because the *inverse subsemigroup* generated by a subset A of an inverse semigroup S will usually be larger than the *subsemigroup* generated by A. To avoid confusion we shall denote the inverse subsemigroup by $\langle\langle A\rangle\rangle$ and the subsemigroup by $\langle A\rangle$, and we shall say that a subset A of an inverse semigroup S is *I-independent* if, for all a in A,

$$a \notin \langle\langle A \setminus \{a\}\rangle\rangle.$$

Applying ideas due to Marczewski [**5**], we can consider the following rank functions on a finite inverse semigroup S:

(i) $r_1^I(S) = \max\{k :$ every subset U of S of cardinality k is I-independent$\}$;

(ii) $r_2^I(S) = \min\{k :$ there exists a subset U of S of cardinality k such that $\langle\langle U\rangle\rangle = S\}$;

(iii) $r_3^I(S) = \max\{k :$ there exists a subset U of S of cardinality k that is I-independent and such that $\langle\langle U\rangle\rangle = S\}$;

(iv) $r_4^I(S) = \max\{k :$ there exists a subset U of S of cardinality k that is I-independent$\}$;

(v) $r_5^I(S) = \min\{k$: every subset U of S of cardinality k has the property that $\langle\langle U\rangle\rangle = S\}$.

As in the semigroup case we can easily show that

$$r_1^I(S) \leqslant r_2^I(S) \leqslant r_3^I(S) \leqslant r_4^I(S) \leqslant r_5^I(S),$$

and for convenience we shall use the following terminology:

$r_1^I(S)$: the *small I-rank*;

$r_2^I(S)$: the *lower I-rank*;

$r_3^I(S)$: the *intermediate I-rank*;

$r_4^I(S)$: the *upper I-rank*; and

$r_5^I(S)$: the *large I-rank*.

Of the five ranks the most interesting are r_2^I, r_3^I and r_4^I, all of which coincide with the dimension when S is a vector space. The least interesting is r_1^I: for an inverse semigroup S it is easy to see that $r_1^I(S) = 1$ unless S is a semi-lattice E, and $r_1^I(E) = 2$ unless E is a chain, and $r_1^I(E) = |E|$.

From [**3**, **4**] we know that the corresponding *semigroup* ranks $r_1, \dots, r_5$, when applied to B_n, give (for $n \geqslant 2$)

$$r_1(B_n) = 1, \qquad r_2(B_n) = n, \qquad r_3(B_n) = 2n-2,$$
$$r_4(B_n) = \lfloor n^2/4 \rfloor + n, \qquad r_5(B_n) = n^2 - n + 3.$$

As remarked above, $r_1^I(B_n) = 1$, and, from [**1**], we have that $r_2^I(B_n) = n-1$. In this note we investigate $r_3^I(B_n)$, $r_4^I(B_n)$ and $r_5^I(B_n)$.

For unexplained terms in semigroup theory see [**2**].

2. The intermediate I-rank

Recall that $B_n = (\{1,2,\dots,n\} \times \{1,2,\dots,n\}) \cup \{0\}$, and that

$$(i,j)(k,l) = \begin{cases} (i,l), & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases}$$
$$(i,j)0 = 0(i,j) = 00 = 0.$$

Theorem 2.1. *With the above definitions, and with* $n \geqslant 2$,

$$r_3^I(B_n) = n-1.$$

Proof. It is easy to see that $r_3^I(B_n) \geqslant n-1$, for the subset

$$A = \{(1,2),(1,3),\dots,(1,n)\}$$

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is I-independent. To see that $\langle\langle A\rangle\rangle = B_n$, notice first that $0 \in \langle\langle A\rangle\rangle$, that $(1,1) = (1,2)(1,2)^{-1} \in \langle\langle A\rangle\rangle$, and that $(i,1) = (1,i)^{-1} \in \langle\langle A\rangle\rangle$ for all $i \in \{1,2,\dots,n\}$. Then, finally, $(i,j) = (i,1)(1,j) \in \langle\langle A\rangle\rangle$ for all $i,j \in \{1,2,\dots,n\}$.

To show that no larger I-independent generating set can be found, we use a graphical technique. To each subset A of $B_n \setminus \{0\}$ we associate an (undirected) graph $\Gamma(A)$ whose vertices are labelled $1,2,\dots,n$, and where there is an edge, $i \longleftrightarrow j$, between i and j if and only if $(i,j) \in A$.

Within a graph Γ we shall say that a sequence of vertices

$$(v_0, v_1, \dots, v_{m-1}, v_m)$$

is a *path between* v_0 *and* v_m if $v_i \longleftrightarrow v_{i+1}$ is an edge in Γ for $i = 0,1,\dots,m-1$. It is a *proper* path if $m \geqslant 2$ and if $v_1,\dots,v_{m-1}$ are all distinct from v_0 and v_m.

It is easy to see that $\langle\langle A\rangle\rangle = B_n$ if and only if the graph $\Gamma(A)$ is *connected*, that is, if and only if there is a path in $\Gamma(A)$ linking any two vertices. Also, A is I-independent if and only if the graph $\Gamma(A)$ is I*-independent*, in the sense that, for all i,j in $\{1,2,\dots,n\}$, there cannot exist both an edge and a proper path between i and j.

We therefore require to prove that, for any connected I-independent graph Γ with no repeated edges,

$$|V(\Gamma)| = n \implies |E(\Gamma)| \leqslant n-1.$$

We do this by induction on n, it being clear by direct verification that a graph with two vertices and at least two edges cannot be both I-independent and connected: if there are just two edges, the possibilities are

$$\{(1,1),(1,2)\}, \quad \{(1,1),(2,2)\}, \quad \{(1,2),(2,2)\},$$

and of these the first and the third are not I-independent, and the second is not connected; if there are three edges the only possibility is

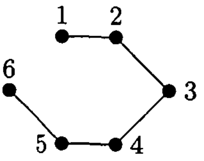
$$\{(1,1),(2,2),(1,2)\},$$

and this set is not I-independent.

Let Γ be both connected and I-independent, and suppose that $|V(\Gamma)| = n$. Notice first that Γ cannot contain any loops $x \longleftrightarrow x$, since there is an edge $x \longleftrightarrow y$ for some $y \neq x$, and then we would have both an edge $x \longleftrightarrow x$ and a proper path (x,y,x). Choose a vertex x at random, and suppose that there are edges $x \longleftrightarrow y_j$ $(j = 1,2,\dots,m)$. Notice that there cannot be any edges $y_j \longleftrightarrow y_k$ $(j,k \in \{1,\dots,m\})$, because any such edge would give rise to a proper path

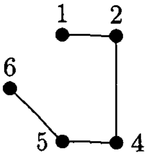
$$(x, y_j, y_k).$$

We form a new graph Γ' whose set of vertices is $V(\Gamma) \setminus \{x\}$, and whose edges are those of Γ, except that all the edges $x \longleftrightarrow y_j$ $(j = 2,\dots,m)$ are replaced by $y_1 \longleftrightarrow y_j$. So, for example, if $n = 6$ and Γ is



(2.1)

then, taking $x = 3$ and $y_1 = 2$, we see that Γ' is



(2.2)

We show first that Γ' is connected. Certainly, any two vertices i and j of Γ' are linked by a path in Γ. If this path does not pass through x, then it is a path in Γ', and there is nothing to prove. If the path

$$(i, \dots, z, x, t, \dots, j)$$

passes through x (and we may assume that the path is without loops, so that this happens only once) then $z, t \in \{y_1, \dots, y_m\}$. If $z \neq y_1$ and $t \neq y_1$, then $(i, \dots, z, y_1, t, \dots, j)$ is a path in Γ'. If $z = y_1$, then $t \neq y_1$, and $(i, \dots, y_1, t, \dots, j)$ is a path in Γ'. If $t = y_1$, then $z \neq y_1$, and $(i, \dots, z, y_1, \dots, j)$ is a path in Γ'. Thus Γ' is connected.

To show that Γ' is I-independent, suppose that $i \longleftrightarrow j$ is an edge in Γ' and that there is also a proper path $P : (i, \dots, j)$ in Γ'. If P does not pass through y_1, then it is in fact a proper path in Γ, and we have a contradiction. Suppose, therefore, that P passes through y_1,

$$(i, \dots, u, y_1, v, \dots, j),$$

where at least one of the edges $u \longleftrightarrow y_1$, $y_1 \longleftrightarrow v$ is absent from Γ. From the construction of Γ' it follows that if $u \longleftrightarrow y_1$ is absent, then $u \longleftrightarrow x$ is an edge of Γ; and, similarly, if $y_1 \longleftrightarrow v$ is absent, then $x \longleftrightarrow v$ is an edge in Γ. Hence, we have one of the following proper paths in Γ:

$$(i, \dots, u, x, y_1, v, \dots, j), \quad (i, \dots, u, y_1, x, v, \dots, j), \quad (i, \dots, u, x, v, \dots, j).$$

By induction we may assume that $|E(\Gamma')| \leqslant n - 2$. Hence,

$$|E(\Gamma)| = |E(\Gamma')| + 1 \leqslant n - 1.$$

This completes the proof. □

3. The upper I-rank

In fact it is fairly easy to modify the proof of Theorem 2.1 to also deal with the upper I-rank $r_4^I(B_n)$.

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Theorem 3.1. *Let $n \geqslant 2$. Then $r_4^I(B_n) = n$.*

Proof. It is easy to see that $r_4^I(B_n) \geqslant n$, for the subset

$$A = \{(1,1), (2,2), \dots, (n,n)\}$$

is I-independent.

To show that no larger I-independent set can be found, we use the same graphical technique, and now we need consider only the I-independent property of the graph.

We require to prove that, for any I-independent graph Γ with no repeated edges,

$$|V(\Gamma)| = n \implies |E(\Gamma)| \leqslant n.$$

We do this by induction on n, it being clear by direct verification that no graph with two vertices and three or more edges can be I-independent: there cannot in fact be more than three edges, and if there are just three edges the only possibility is

$$\{(1,1), (1,2), (2,2)\},$$

which is certainly not I-independent.

Let Γ be I-independent, and suppose that $|V(\Gamma)| = n$. Choose a vertex x at random, and form a new graph Γ^* such that $V(\Gamma^*) = V(\Gamma) \setminus \{x\}$ and whose edges are all the edges of Γ except $x \longleftrightarrow x$ if the loop $x \longleftrightarrow x$ is present, all the edges of Γ if the loop $x \longleftrightarrow x$ is not present and if there does not exist y, with $y \neq x$, such that $x \longleftrightarrow y$ is an edge in Γ and, if there are edges $x \longleftrightarrow y_j$ $(j = 1, 2, \dots, m)$, Γ^* has the same set of edges of Γ except that all the edges $x \longleftrightarrow y_j$ $(j = 2, \dots, m)$ are replaced by $y_1 \longleftrightarrow y_j$. Suppose first that the loop $x \longleftrightarrow x$ is present. Then there can be no edges $x \longleftrightarrow y$ with $y \neq x$, for otherwise we have a proper path (x, y, x). In this case, $E(\Gamma^*)$ consists of all the edges of $E(\Gamma)$ except $x \longleftrightarrow x$, and Γ^* is certainly independent. By induction, since $|V(\Gamma^*)| = n - 1$, we must have $|E(\Gamma^*)| \leqslant n - 1$, and so it follows that

$$|E(\Gamma)| = |E(\Gamma^*)| + 1 \leqslant n,$$

as required.

Suppose now that the loop $x \longleftrightarrow x$ is not present. If there does not exist y such that $x \longleftrightarrow y$ is an edge in Γ, then Γ^* is certainly independent, and $|E(\Gamma)| = |E(\Gamma^*)| \leqslant n-1$ by the induction hypothesis. So suppose that there are edges $x \longleftrightarrow y_j$ $(j = 1, 2, \dots, m)$. Notice there cannot be any edges $y_j \longleftrightarrow y_k$ $(j, k \in \{1, \dots, m\})$, because any such edge would give rise to a proper path

$$(x, y_j, y_k).$$

Now, let Γ^* have the same set of edges as Γ, except that all the edges $x \longleftrightarrow y_j$ $(j = 2, \dots, m)$ are replaced by $y_1 \longleftrightarrow y_j$. The diagrams (2.1) and (2.2) are as relevant here as in the earlier proof. Indeed, the proof that Γ^* is I-independent is in effect identical to the proof in the last section that Γ' is I-independent, and is omitted.

By induction we may assume that $|E(\Gamma^*)| \leqslant n - 1$. Hence,

$$|E(\Gamma)| = |E(\Gamma^*)| + 1 \leqslant n.$$

This completes the proof. □

4. The large I-rank

We now investigate $r_5^I(B_n)$. In fact, we consider something more general, and examine the finite Brandt semigroup $B(G,n)$ for an arbitrary finite group G. Recall that

$$B(G,n) = (\{1,2,\dots,n\} \times G \times \{1,2,\dots,n\}) \cup \{0\},$$

and that the multiplication is given by

$$(i,a,j)(k,b,l) = \begin{cases} (i,ab,l), & \text{if } j=k, \\ 0, & \text{otherwise,} \end{cases}$$
$$0(i,a,j) = (i,a,j)0 = 00 = 0.$$

By analogy with [**4**, Theorem 3], we have the following theorem.

Theorem 4.1. *Let S be a finite inverse semigroup, and let V be the largest proper inverse subsemigroup of S. Then $r_5^I(S) = |V| + 1$.*

We now have the following theorem.

Theorem 4.2. *Let $n \geqslant 2$. Then $r_5^I B(G,n) = (n^2 - 2n + 2)|G| + 2$.*

Proof. We begin by describing all of the inverse subsemigroups of $S = B(G,n)$. Let V be an inverse subsemigroup of S. Denote $\{1,\dots,n\}$ by $[n]$.

Define a relation ρ_V on $[n]$ by

$$(i,j) \in \rho_V \;\Leftrightarrow\; (\exists g \in G)(i,g,j) \in V.$$

It is easy to see that the relation ρ_V is symmetric and transitive, but, in general it is not reflexive. Let $M = \{i \in [n] : (i,i) \in \rho_V\}$.

In fact, $V \setminus 0 \subseteq M \times G \times M$. For suppose that $i \notin M$; then there cannot exist j such that $(i,j) \in \rho_V$, since then we would have $(j,i) \in \rho_V$ by symmetry and so $(i,i) \in \rho_V$ by transitivity. Now, the relation $\bar{\rho}_V = \rho_V \cap (M \times M)$ is an equivalence on M. Let $C \subseteq M$ be a $\bar{\rho}_V$-class. For all $i,j \in C$, let $H_{i,j}$ be the (non-empty) subset of G defined by $H_{i,j} = \{g \in G : (i,g,j) \in V\}$. For all $i,j,k \in C$ we have $H_{i,j}H_{j,k} \subseteq H_{i,k}$, and, in particular, $H_{i,i}^2 \subseteq H_{i,i}$ for all $i \in C$.

Let $i,j,k,l \in C$ and let $p \in H_{i,k}$, $q \in H_{l,j}$. Then $p^{-1} \in H_{k,i}$, $q^{-1} \in H_{j,l}$ and $pH_{k,l}q \subseteq H_{i,j}$, $p^{-1}H_{i,j}q^{-1} \subseteq H_{k,l}$.

Hence,

$$|H_{k,l}| = |pH_{k,l}q| \leqslant |H_{i,j}| = |p^{-1}H_{i,j}q^{-1}| \leqslant |H_{k,l}|,$$

and, for all $i,j,k,l \in C$, we have $|H_{k,l}| = |H_{i,j}|$.

Now we choose and fix i in C and denote the group $H_{i,i}$ by H_V^C, and choose for each $j \in C$ an element p_j in $H_{i,j}$, arbitrarily.

Then, every $H_{j,k}$ with $j,k \in C$ is the double coset $p_j^{-1}H_V^C p_k$, and

$$V = \{0\} \cup \bigcup_{C \in M/\bar{\rho}_V} \bigcup_{j,k \in C} \{j\} \times p_j^{-1}H_V^C p_k \times \{k\}.$$

The choice of i and of the elements p_j is arbitrary, but $p_j^{-1}H_V^C p_k$ will always coincide with $H_{j,k}$.

Conversely, choose a subset M of $[n]$, and let ρ be an equivalence on M. For each ρ-class C, choose and fix an element i in C and let H^C be a subgroup of G. For each j in C choose p_j in G arbitrarily. Then

$$V = \{0\} \cup \bigcup_{C \in M/\rho} \bigcup_{j,k \in C} \{j\} \times p_j^{-1} H^C p_k \times \{k\} \tag{4.1}$$

is an inverse subsemigroup of S. To see this, consider two non-zero elements x, y of this set. Then $xy = 0$ unless $x = (j, p_j^{-1}ap_k, k)$, $y = (k, p_k^{-1}bp_l, l)$ for some C in M/ρ, some $j,k,l \in C$ and some $a, b \in H^C$. In this case, $xy = (j, p_j^{-1}abp_l, l)$, and so $xy \in V$. Also,

$$x^{-1} = (k, p_k^{-1}a^{-1}p_j, j) \in V.$$

Thus, V is an inverse subsemigroup of S. We have described all the inverse subsemigroups of S.

Remark 4.3. If in (4.1) $M = \emptyset$, then $V = \{0\}$. If $M = [n]$, $\rho = [n] \times [n]$ and $H^{[n]} = G$, then $V = S$.

From (4.1) we have

$$|V \setminus 0| = \sum_{C \in M/\rho} |C|^2 |H^C|.$$

Suppose that ρ has classes $C_1, C_2, \ldots, C_r$ containing $m_1, m_2, \ldots, m_r$ elements, respectively, and let us write H^{C_i} as H_i. Then

$$|V \setminus 0| = \sum_{i=1}^{r} m_i^2 |H_i|. \tag{4.2}$$

Lemma 4.4. *Let $m = m_1 + m_2 + \cdots + m_r$, with $m_i \geqslant 1$ for all i. Then*

(1) *if $r \geqslant 1$, then $m^2 \geqslant m_1^2 + \cdots + m_r^2$;*

(2) *if $r \geqslant 2$, then $(m-1)^2 + 1 \geqslant m_1^2 + \cdots + m_r^2$.*

Proof. Part (1) is a standard inequality. As for part (2), notice that $1 \leqslant m_r \leqslant m - 1$ and that

$$(m - m_r)^2 \geqslant m_1^2 + \cdots + m_{r-1}^2.$$

Hence,

$$\begin{aligned}[(m-1)^2 + 1] - [m_1^2 + \cdots + m_r^2] &\geqslant [(m-1)^2 + 1] - [(m - m_r)^2 + m_r^2] \\ &= 2(m_r - 1)[(m-1) - m_r] \geqslant 0,\end{aligned}$$

as required. Looking at (4.2), suppose first that $|M| = m \leqslant n - 1$. Then

$$|V \setminus 0| = \sum_{i=1}^{r} m_i^2 |H_i| \leqslant |G| \sum_{i=1}^{r} m_i^2 \leqslant |G| m^2 \leqslant |G|(n-1)^2. \tag{4.3}$$

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Next, suppose that $M = [n]$, and that $\rho = [n] \times [n]$, the universal relation. Then we have only one equivalence class and $r = 1$, $m_1 = n$ in (4.2). So $|V \setminus 0| = n^2|H|$, where H is a subgroup of G. If $H = G$, then $V = S$. If H is a proper subgroup of G, then $|H| \leqslant \frac{1}{2}|G|$, and so

$$|V \setminus 0| \leqslant \tfrac{1}{2}|G|n^2. \tag{4.4}$$

Finally, suppose that $M = [n]$ and that ρ has at least two classes. Suppose, in fact, that there are r ρ-classes $C_1, C_2, \dots, C_r$, where $|C_i| = m_i$ $(i = 1, \dots, r)$ and $m_1 + m_2 + \cdots + m_r = n$, then

$$\begin{aligned} |V \setminus 0| &= \sum_{i=1}^{r} m_i^2 |H_i| \leqslant |G| \sum_{i=1}^{r} m_i^2 \leqslant |G|[(m-1)^2 + 1] \\ &= |G|[(n-1)^2 + 1] = |G|(n^2 - 2n + 2). \end{aligned} \tag{4.5}$$

From (4.3), (4.4) and (4.5) we see that if V is a proper inverse subsemigroup of S then

$$|V| \leqslant (n^2 - 2n + 2)|G| + 1.$$

If we take $M = [n]$, the equivalence ρ with two classes, $\{1, \dots, n-1\}$ and $\{n\}$, and $H^{\{1,\dots,n-1\}} = H^{\{n\}} = G$, we obtain an inverse subsemigroup

$$\{0\} \cup \{(i, g, j) : i, j = 1, \dots, n-1, g \in G\} \cup \{(n, g, n) : g \in G\}$$

of order $(n^2 - 2n + 2)|G| + 1$, and we now know that this is the best possible.

We deduce that $r_5^I B(G, n) = (n^2 - 2n + 2)|G| + 2$. □

Corollary 4.5. $r_5^I(B_n) = n^2 - 2n + 4$.

We end the paper by applying Theorem 4.2 to a standard inverse semigroup of permutations. First, however, we consider $r_5^I(\mathcal{I}_n)$, where $\mathcal{I}_n$ is the symmetric inverse semigroup on $X = \{1, 2, \dots, n\}$.

Theorem 4.6. *For* $n \geqslant 3$,

$$r_5^I(\mathcal{I}_n) = \sum_{r=0}^{n} \binom{n}{r}^2 r! - \tfrac{1}{2}n! + 1.$$

Proof. The inverse semigroup $\mathcal{I}_n$ has $n+1$ $\mathcal{J}$-classes, $\mathcal{J}_0, \mathcal{J}_1, \dots, \mathcal{J}_n$, where $\mathcal{J}_r = \{\alpha \in \mathcal{I}_n : |\operatorname{dom}\alpha| = r(= |\operatorname{im}\alpha|)\}$. The $\mathcal{J}$-class $\mathcal{J}_0$ consists of the empty map usually denoted by $\{0\}$ and $\mathcal{J}_n$ coincides with the symmetric group S_n.

Consider the proper inverse semigroup of $\mathcal{I}_n$, $V = \mathcal{J}_0 \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{A}_n$, where A_n is the alternating group.

In fact, V is the largest proper inverse subsemigroup of $\mathcal{I}_n$. It is clear that it is the largest proper inverse subsemigroup whose intersection with S_n is a proper subgroup. So now consider U, a proper inverse subsemigroup of $\mathcal{I}_n$ such that $S_n \subseteq U$.

Now, if $U \cap \mathcal{J}_{n-1} \neq \emptyset$, then $U = \mathcal{I}_n$ (see [**2**]).

So let $U \subseteq \mathcal{J}_0 \cup \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_{n-2} \cup S_n$. Now

$$|U| \leqslant |\mathcal{I}_n| - |\mathcal{J}_{n-1}| = |\mathcal{I}_n| - n^2(n-1)! \leqslant |\mathcal{I}_n| - \tfrac{1}{2}n! = \sum_{r=0}^{n} \binom{n}{r}^2 r! - \tfrac{1}{2}n! = |V|.$$

Consequently,

$$r_5^I(I_n) = \sum_{r=0}^{n} \binom{n}{r}^2 r! - \tfrac{1}{2}n! + 1,$$

as required. □

Consider now $SP_n = \{\alpha \in \mathcal{I}_n : |\operatorname{dom}\alpha| < n\}$, the inverse semigroup of proper subpermutations of $X = \{1, 2, \ldots, n\}$, where $n \geqslant 3$. The inverse semigroup SP_n has n $\mathcal{J}$-classes. The top $\mathcal{J}$-class in SP_n is $\mathcal{J}_{n-1}$, and the associated principal factor $P_{n-1} = SP_n/(\mathcal{J}_0 \cup \mathcal{J}_1 \cdots \cup \mathcal{J}_{n-2})$ may be thought of in the usual way as $\mathcal{J}_{n-1} \cup \{0\}$, where the product in P_{n-1} of two elements of $\mathcal{J}_{n-1}$ is the product in SP_n if it lies in $\mathcal{J}_{n-1}$ and is 0 otherwise.

Since the principal factor P_{n-1} is a Brandt semigroup $B(G, n)$, where $G = S_{n-1}$ (see [**2**, **6**]), the symmetric group on $\{1, \ldots, n-1\}$ and $I = \{1, 2, \ldots, n\}$, we have, as a consequence of Theorem 4.2, an inverse subsemigroup $S \cup \mathcal{J}_{n-2} \cup \cdots \cup \mathcal{J}_0$, where S is of order $(n-1)!(n^2 - 2n + 2)$. This is certainly the largest proper inverse subsemigroup not containing J_{n-1}, and, indeed, is the largest proper inverse subsemigroup of SP_n, for it is well known that $\langle\langle J_{n-1} \rangle\rangle = SP_n$ (see [**1**]). Hence we obtain the following theorem.

Theorem 4.7. *Let $SP_n = \{\alpha \in I_n : |\operatorname{dom}\alpha| < n\}$ be the inverse semigroup of proper subpermutations of $X = \{1, 2, \ldots, n\}$. Then, if $n \geqslant 3$,*

$$r_5^I(SP_n) = \sum_{r=0}^{n-2} \binom{n}{r}^2 r! + (n-1)!(n^2 - 2n + 2) + 1.$$

□

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