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ZETA INVARIANTS OF MORSE FORMS

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Abstract Let η be a closed real 1-form on a closed Riemannian n-manifold (M,g). Let d_z, δ_z and Δ_z be the induced Witten's type perturbations of the de Rham derivative and coderivative and the Laplacian, parametrized by $z = \mu + i\nu \in \mathbb{C}$ ($\mu,\nu \in \mathbb{R}$, $i = \sqrt{-1}$). Let $\zeta(s,z)$ be the zeta function of $s \in \mathbb{C}$, defined as the meromorphic extension of the function $\zeta(s,z) = \operatorname{Str}(\eta \wedge \delta_z \Delta_z^{-s})$ for $\Re s \gg 0$. We prove that $\zeta(s,z)$ is smooth at $s = 1$ and establish a formula for $\zeta(1,z)$ in terms of the associated heat semigroup. For a class of Morse forms, $\zeta(1,z)$ converges to some $\mathbf{z} \in \mathbb{R}$ as $\mu \to +\infty$, uniformly on ν. We describe $\mathbf{z}$ in terms of the instantons of an auxiliary Smale gradient-like vector field X and the Mathai–Quillen current on TM defined by g. Any real 1-cohomology class has a representative η satisfying the hypothesis. If n is even, we can prescribe any real value for $\mathbf{z}$ by perturbing g, η and X and achieve the same limit as $\mu \to -\infty$. This is used to define and describe certain tempered distributions induced by g and η. These distributions appear in another publication as contributions from the preserved leaves in a trace formula for simple foliated flows, giving a solution to a problem stated by C. Deninger.

1. Introduction

1.1. Witten's perturbed operators

Let M be a closed n-manifold. For any smooth function h on M, Witten [74] introduced a perturbed de Rham differential operator $d_\mu = d + \mu\, dh\wedge$, depending on a parameter

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$\mu \in \mathbb{R}$. Endowing M with a Riemannian metric g, we have a corresponding perturbed codifferential operator $\delta_\mu = \delta - \mu\, dh \lrcorner$, and a perturbed Laplacian $\Delta_\mu = d_\mu \delta_\mu + \delta_\mu d_\mu$. Since $d_\mu = e^{-\mu h}\, d\, e^{\mu h}$, it defines the same Betti numbers as d. However, Δ_μ and the usual Laplacian Δ have different spectrum in general. In fact, if h is a Morse function and g is Euclidean with respect to Morse coordinates around the critical points, then the spectrum of Δ_μ develops a long gap as $\mu \to +\infty$, giving rise to the small and large spectrum. The eigenforms of the small/large eigenvalues generate the small/large subcomplex, $(E_{\mu,\mathrm{sm/la}}, d_\mu)$. When h is a Morse function, Witten gave a beautiful analytic proof of the Morse inequalities by analyzing the small spectrum. This was refined by subsequent work of Helffer and Sjöstrand [35] and Bismut and Zhang [10, 11], showing that, if moreover $X := -\operatorname{grad} h$ is a Smale vector field, then the Morse complex $(\mathbf{C}^\bullet, \mathbf{d})$ of X can be considered as the limit of $(E_{\mu,\mathrm{sm}}, d_\mu)$. More precisely, for certain perturbed Morse complex $(\mathbf{C}^\bullet, \mathbf{d}_\mu)$, isomorphic to $(\mathbf{C}^\bullet, \mathbf{d})$, there is a quasi-isomorphism $\Phi_\mu : (E_{z,\mathrm{sm}}, d_\mu) \to (\mathbf{C}^\bullet, \mathbf{d}_\mu)$, defined by integration on the unstable cells of the zero points of X, which becomes an isomorphism for $\mu \gg 0$ and almost isometric as $\mu \to +\infty$ (after rescaling at every degree).

We can replace dh with any closed real 1-form η, obtaining a generalization of the Witten's perturbations, d_μ, δ_μ and Δ_μ. Now, d_μ need not be gauge equivalent to d, obtaining new twisted Betti numbers β^k_μ. However, the numbers β^k_μ have well-defined ground values β^k_{No}, called the Novikov numbers, which depend upon the de Rham cohomology class $[\eta] \in H^1(M, \mathbb{R})$. Assume that:

(a) η is a Morse form (it has Morse-type zeros), and g is Euclidean with respect to Morse coordinates around the zero points of η.

(Some concepts used in this section are recalled in Sections 4.1 and 6.1.) Then Δ_μ also develops a long gap separating a small spectrum and a large spectrum, and the analysis of the small spectrum gives Morse inequalities for the Novikov numbers. Take any auxiliary vector field X such that:

(b) X has Morse-type zeros and is gradient-like and Smale; and

(c) η is Lyapunov for X, and η and g are in standard form with respect to X.

Then the small complex approaches a perturbed Morse complex of X. We refer to work by Novikov [55, 56], Pajitnov [58], Braverman and Farber [14], Burghelea and Haller [17, 18, 20] and Harvey and Minervini [34, 52].

We can similarly define the perturbation $d_z = d + z\eta\wedge$ with parameter $z = \mu + i\nu \in \mathbb{C}$ ($\mu, \nu \in \mathbb{R}$ and $i = \sqrt{-1}$). Its adjoint is $\delta_z = \delta - \bar{z}\eta \lrcorner$, and we have a corresponding perturbed Laplacian $\Delta_z = d_z \delta_z + \delta_z d_z$. As a first step in our study, we prove extensions of the above results to this case, taking limits as $|\mu| \to +\infty$, uniformly on ν. First, assuming (a), we get the long gap in the spectrum of Δ_z separating the small and large spectrum, which depends only on μ (Theorem 4.10). Second, assuming (a)–(c), we show that the quasi-isomorphism $\Phi_z : (E_{z,\mathrm{sm}}, d_z) \to (\mathbf{C}^\bullet, \mathbf{d}_z)$ becomes an isomorphism for $|\mu| \gg 0$ and almost isometric as $|\mu| \to +\infty$ (Theorem 6.3). To get that the convergence is uniform on ν, the key ingredient is a version of a Sobolev inequality for integers $m > n/2$: on smooth complex differential forms,

$$\| \ \|_{L^\infty} \le C_m \| \ \|_{m, i\nu}\,, \tag{1.1}$$

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where $C_m > 0$ is independent of ν and $\|\alpha\|_{m,i\nu} = \sum_{k=0}^{m} \langle \Delta_{i\nu}^k \alpha, \alpha\rangle^{1/2}$ (Proposition 2.2). (The analogous property for Δ_μ is wrong.) Then we adapt the arguments of Bismut and Zhang [10, 11] (see also [75]).

The indicated properties of Δ_z, holding uniformly on μ, depend on remarkable differences between $\Delta_{i\nu}$ and Δ_μ. For instance, if η is exact, all operators $\Delta_{i\nu}$ are gauge equivalent, whereas this is not true for the operators Δ_μ when $\eta \neq 0$. If η is not exact, the operators $\Delta_{i\nu}$ are not gauge equivalent either. Moreover, $\Delta_{i\nu} - \Delta$ is of order one when $\nu \neq 0$, whereas $\Delta_\mu - \Delta$ is of order zero.

1.2. Zeta invariants of Morse forms

To begin with, η is only assumed to be an arbitrary closed real 1-form. Let $\Pi_z^\perp$ and Π_z^1 be the orthogonal projections to the images of Δ_z and d_z. We consider a zeta function $\zeta(s,z)$ associated with η and the parameter $z \in \mathbb{C}$. As a function of $s \in \mathbb{C}$, it is the meromorphic extension of the holomorphic function

$$\zeta(s,z) = \operatorname{Str}(\eta \wedge \delta_z \Delta_z^{-s} \Pi_z^\perp) = \operatorname{Str}(\eta \wedge d_z^{-1} \Delta_z^{-s+1} \Pi_z^1)$$

defined for $\Re s \gg 0$, where Str stands for the supertrace. We are interested in the zeta invariant $\zeta(1,z)$ that can be interpreted as a renormalization of the supertrace of $\eta \wedge d_z^{-1} \Pi_z^1$, which is not of trace class by the Weyl's law. According to the general theory of zeta functions of elliptic operators, $\zeta(s,z)$ might have a simple pole at $s = 1$. However, our first main theorem states that $\zeta(s,z)$ is smooth at $s = 1$ and gives a formula for $\zeta(1,z)$ in terms of the associated heat semigroup.

Theorem 1.1. *Let $M \equiv (M,g)$ be a closed Riemannian n-manifold, and let η be a closed real 1-form on M. If n is even (resp., odd), then, for any $z \in \mathbb{C}$, $s \mapsto \zeta(s,z)$ is smooth on the half-plane $\Re s > 0$ (resp., $\Re s > 1/2$). Furthermore,*

$$\zeta(1,z) = \lim_{t \downarrow 0} \operatorname{Str}\left(\eta \wedge d_z^{-1} e^{-t\Delta_z} \Pi_z^1\right).$$

The existence of the limit of Theorem 1.1 is surprising because $\eta \wedge d_z^{-1} e^{-t\Delta_z} \Pi_z^1$ is weakly convergent to $\eta \wedge d_z^{-1} \Pi_z^1$. An expression similar to $\operatorname{Str}(\eta \wedge d_z^{-1} e^{-t\Delta_z} \Pi_z^1)$ was used by Mrowka, Ruberman and Saveliev to define a cyclic eta invariant [53].

Next, we additionally assume that η is a Morse form and use the results described in the previous section. The zeta-function decomposes as the sum of terms defined by the contributions from the small/large spectrum, $\zeta_{\mathrm{sm/la}}(s,z) = \zeta_{\mathrm{sm/la}}(s,z,\eta)$, where $\zeta_{\mathrm{sm}}(s,z)$ is an entire function of s. Our second main theorem describes the asymptotic behavior of $\zeta(1,z)$ as $\mu \to \pm\infty$, uniformly on ν. In fact, since

$$\zeta(s,z,\eta) = -\zeta(s,-z,-\eta), \quad \zeta_{\mathrm{sm/la}}(s,z,\eta) = -\zeta_{\mathrm{sm/la}}(s,-z,-\eta), \tag{1.2}$$

it is enough to consider the case where $\mu \gg 0$ and take the limit as $\mu \to +\infty$.

We use the current $\psi(M,\nabla^M)$ of degree $n-1$ on TM constructed by Mathai and Quillen in [44], depending on the Levi–Civita connection ∇^M. This current is smooth on the complement of the zero section, where it is given by the solid angle. It is also locally integrable, and its wave front set is contained in the conormal bundle in T^*TM of the zero

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section of TM. Since this set does not meet the conormal bundle of the map $X:M\to TM$ (assuming (b)), $(-X)^*\psi(M,\nabla^M)$ is well defined as a current on M. Assuming also (a)–(c), consider the real number

$$\mathbf{z}_{\mathrm{la}}=\mathbf{z}_{\mathrm{la}}(M,g,\eta)=\int_M \eta\wedge(-X)^*\psi(M,\nabla^M)\,,$$

which is known to be independent of X [10, Proposition 6.1].

Now, suppose also that:

(d) for every zero point p of X with Morse index k, the maximum value of the integrals of η along the instantons of X with α-limit p only depends on k.

This maximum value is denoted by $-a_k$ for some $a_k>0$. Let $m_k^1=\dim d_z(E_{z,\mathrm{sm}}^{k-1})$ for $\mu\gg 0$, which is independent of z. Consider also the real number

$$\mathbf{z}_{\mathrm{sm}}=\mathbf{z}_{\mathrm{sm}}(M,g,\eta,X)=\sum_{k=1}^{n}(-1)^k\big(1-e^{a_k}\big)m_k^1\,,$$

and let $\mathbf{z}=\mathbf{z}(M,g,\eta,X)=\mathbf{z}_{\mathrm{sm}}+\mathbf{z}_{\mathrm{la}}$.

Recall that we write $z=\mu+i\nu$.

Theorem 1.2. *Let $M\equiv(M,g)$ be a closed Riemannian n-manifold, let η be a closed real 1-form on M satisfying (a) and let X be a vector field on M satisfying (b)–(c).*

(i) *We have*

$$\zeta_{\mathrm{la}}(1,z)=\mathbf{z}_{\mathrm{la}}+O(\mu^{-1})$$

as $\mu\to+\infty$, uniformly on ν.

(ii) *If moreover (d) holds, then*

$$\zeta_{\mathrm{sm}}(1,z)=\mathbf{z}_{\mathrm{sm}}+O(\mu^{-1})$$

as $\mu\to+\infty$, uniformly on ν.

Theorem 1.2 (ii) shows that $\mathbf{z}_{\mathrm{sm}}$ and $\mathbf{z}$ are also independent of X. Thus, X will be omitted in their notation. In the notation of $\mathbf{z}_{\mathrm{sm/la}}$ and $\mathbf{z}$, we may also omit M or g if they are fixed.

By Equation (1.2), if we take $\mu\to-\infty$ in Theorem 1.2, we have to replace $\mathbf{z}_{\mathrm{sm/la}}(\eta)$ with $-\mathbf{z}_{\mathrm{sm/la}}(-\eta)$. Descriptions of $-\mathbf{z}_{\mathrm{sm/la}}(-\eta)$ are given in Equations (7.9) and (8.1).

Our third main theorem is about the prescription of $\mathbf{z}=\mathbf{z}(M,g,\eta)$ without changing the cohomology class of η.

Theorem 1.3. *Let M be a smooth closed n-manifold. If n is even (resp., odd), for all $\xi\in H^1(M,\mathbb{R})$ and $\tau\in\mathbb{R}$ (resp., $\tau\gg 0$), there is some $\eta\in\xi$, a Riemannian metric g and a vector field X satisfying (a)–(d) such that $\pm\mathbf{z}(M,g,\pm\eta)=\tau$ (resp., $\mathbf{z}(M,g,\eta)=\tau$).*

1.3. A distribution associated to some Morse forms

A trace formula for simple foliated flows on closed foliated manifolds was conjectured by C. Deninger (see, e.g., [24]). He was motivated by analogies with Weil's explicit formulas

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in arithmetics and previous work of Guillemin and Sternberg [32]. This trace formula is an expression for a Lefschetz distribution in terms of infinitesimal data of the flow at the fixed points and closed orbits. This Lefschetz distribution should be an analogue of the Lefschetz number for the action induced by the flow on some leafwise cohomology, whose value is a distribution on $\mathbb{R}$—the precise definition of these notions is part of the problem. In [4, 5], the first two authors proved such a trace formula when the flow has no preserved leaves; see also the contributions [42, 43] by the third author. The general case is considerably more involved. In [6], we propose a solution to this problem using a few additional ingredients. One of them is the b-trace introduced by Melrose [46]. Since the b-trace is not really a trace, it produces an extra term, denoted by Z, in the same way as the eta invariant shows up in Index Theory on manifolds with boundary. In our trace formula, the term Z is a contribution from the compact leaves preserved by the flow, which depends on the choice of a form defining the foliation and a metric on the ambient manifold. But Z may not be well defined in general; it will be proved that appropriate choices of the form and the metric guarantee its existence.

Precisely, we would like to define

$$Z = Z(M,g,\eta) = \lim_{\mu\to+\infty} Z_\mu\,, \tag{1.3}$$

in the space of tempered distributions on $\mathbb{R}$, where $Z_\mu = Z_\mu(M,g,\eta)$ ($\mu \gg 0$) should be a tempered distribution defined by

$$\langle Z_\mu, f\rangle = -\frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty \operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\hat{f}(\nu)\,d\nu\,du\,, \tag{1.4}$$

for any Schwartz function f, where $\hat{f}$ stands for the Fourier transform of f.

Let δ_0 denote the Dirac distribution at 0 on $\mathbb{R}$. The problem about the definition of Z is solved in our fourth main theorem for the same class of Morse forms as before.

Theorem 1.4. *Let $M \equiv (M,g)$ be a closed Riemannian n-manifold. Let η be a closed 1-form on M satisfying (a), (c) and (d) with some vector field satisfying (b). Then Equations (1.3) and (1.4) define the tempered distribution $Z = \mathbf{z}\delta_0$.*

According to Theorems 1.3 and 1.4, we can choose η and g in the trace formula for foliated flows so that $Z(M,g,\pm\eta) = 0$ if n is even, achieving the original expression of Deninger's conjecture.

It looks clear that extensions of Theorems 1.1 to 1.4 with coefficients in flat vector bundles could be similarly proved. We only consider complex coefficients for the sake of simplicity since this is enough for our application.

1.4. Some ideas of the proofs of Theorems 1.1 to 1.4

As mentioned before, the inequality (1.1) is essential to obtain the uniformity on ν of our estimates. To prove it, we can take $\nu = 1$ by considering an arbitrary closed real 1-form η (Proposition 2.2). Let $\|\ \|_{m,i\eta}$ be the mth Sobolev norm defined with the perturbed Laplacian $\Delta_{i\eta}$ induced by $i\eta$ as above. By ellipticity, $\|\ \|_{L^\infty} \le C_{m,i\eta}\|\ \|_{m,i\eta}$ for some $C_{m,i\eta} > 0$ depending on η, which can be chosen to be optimal. For two such forms,

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η and η', the cohomology class $[\eta - \eta']$ is in the lattice $2\pi H^1(M,\mathbb{Z})$ of $H^1(M,\mathbb{R})$ just when $\eta - \eta' = h^*d\theta$ for some smooth map $h : M \to \mathbb{S}^1$, where θ is the multivalued angle function on the circle $\mathbb{S}^1$. This gives the gauge equivalence $\Delta_{i\eta'} = e^{-ih^*\theta}\Delta_{i\eta}e^{ih^*\theta}$, where $e^{\pm ih^*\theta}$ is well defined on M. It follows that $\eta \mapsto C_{m,i\eta}$ induces a function on the torus $H^1(M,\mathbb{R})/2\pi H^1(M,\mathbb{Z})$. On the other hand, every $C_{m,i\eta}$ can be estimated in terms of the C^m norm of η (Proposition 2.1). Hence, by compactness of $H^1(M,\mathbb{R})/2\pi H^1(M,\mathbb{Z})$, the values $C_{m,i\eta}$ have an upper bound C_m, which satisfies the desired inequality $\| \ \|_{L^\infty} \leq C_m\| \ \|_{m,i\eta}$.

For an arbitrary closed real 1-form η and for all $t > 0$ and $z \in \mathbb{C}$, a supersymmetric argument shows that (Proposition 3.7)

$$\partial_z \operatorname{Str}\left(\mathsf{N}e^{-t\Delta_z}\right) = -t\operatorname{Str}\left(\eta\wedge D_z e^{-t\Delta_z}\right), \tag{1.5}$$

where N is the number operator on $\Omega(M)$ (Section 2.1.1). Then we apply that the coefficients of the asymptotic expansion of $\operatorname{Str}(\mathsf{N}e^{-t\Delta_z})$ as $t \downarrow 0$ (the derived heat trace invariants) are independent of z up to order n [10, Theorem 7.10] (see also [3]). Thus, by Equation (1.5), the coefficients of the asymptotic expansion of $\operatorname{Str}(\eta\wedge D_z e^{-t\Delta_z})$ as $t \downarrow 0$ vanish up to order n. Now, Theorem 1.1 follows by the general theory of zeta functions of operators (Section 3.6).

The theta function $\theta(s,z)$ is defined like $\zeta(s,z)$ by using $-\operatorname{Str}(\mathsf{N}\Delta_z^{-s}\Pi_z^{\perp})$ instead of $\operatorname{Str}(\eta\wedge\delta_z\Delta_z^{-s}\Pi_z^{\perp})$. Assuming the hypotheses of Theorem 1.2, write $\theta(s,z)$ as the sum of contributions from the small/large spectrum, $\theta_{\mathrm{sm/la}}(s,z)$, as before. Thus, $e^{\theta'(0,z)/2}$ is the factor used to define the Ray–Singer metric on $\det H_z^\bullet(M)$ [10], where the prime denotes ∂_s. We obtain (Corollary 5.10)

$$\zeta_{\mathrm{la}}(1,z) = \partial_z\theta'_{\mathrm{la}}(0,z). \tag{1.6}$$

This equality allows us to use the deep relation between the Ray–Singer metric and the Milnor metric on $\det H_z^\bullet(M)$, proved by Bismut and Zhang [10, 11]. To apply this result, we have to make involved computations concerning derivatives with respect to z of the orthogonal projection to $E_{z,\mathrm{sm}}$ and of other operators related with the isomorphism $\Phi_z : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$, as well as estimates of the asymptotic behavior as $\mu \to +\infty$ of these operators and their derivatives (Sections 4.4, 4.5, 6.3, 6.4 and 7.2). In this way, we obtain that $\zeta_{\mathrm{la}}(1,z)$ is asymptotic to $\mathbf{z}_{\mathrm{la}}$ as $\mu \to +\infty$ (Section 7.2). This proves Theorem 1.2 (i).

When η is exact, we show this asymptotic expression of $\zeta_{\mathrm{la}}(1,z)$ assuming only (a) (Section 5.5), without using Equation (1.6) and the indicated strong result of Bismut and Zhang. Instead, we apply that the index density of Δ_z is independent of z, also proved by Bismut and Zhang [10, Theorem 13.4]; see also [1, Theorem 1.5] and [6].

On the other hand, given any $\xi \in H^1(M,\mathbb{R})$ and a vector field X satisfying (b), we prove that there is some $\eta \in \xi$ and a metric g satisfying (a), (c) and (d) (Theorem 8.1). This can be considered as an extension of a theorem of Smale stating the existence of nice Morse functions [69, Theorem B] (the case where $\xi = 0$). Its proof is relegated to Appendix A because of its different nature.

The properties (a)–(d) are used to give an asymptotic description of $\mathbf{d}_z$ as $\mu \to +\infty$ (Section 8.2). From this asymptotic description and using that $\Phi_z : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$ is an isomorphism for $\mu \gg 0$, we get upper and lower bounds of the nonzero small spectrum of Δ_z (Theorem 8.4), which are independent of ν. This is a partial extension of accurate descriptions of the nonzero small eigenvalues achieved in the case where η is exact and the parameter is real [41, 48]. With the same procedure and using the bounds of the nonzero small spectrum, it also follows that $\zeta_{\mathrm{sm}}(1,z) = \mathbf{z}_{\mathrm{sm}} + O(\mu^{-1})$ as $\mu \to +\infty$ (Section 8.4), showing Theorem 1.2 (ii).

Next, by modifying η and X around its zero points of index 0 and n, without changing the cohomology class of η, we can achieve any real number as $\pm\mathbf{z}(\pm\eta)$ if n is even, or any large enough real number as $\mathbf{z}(\eta)$ if n is odd (Section 9). This shows Theorem 1.3.

If it is possible to switch the order of integration in Equation (1.4),

$$\begin{aligned} \langle Z_\mu, f\rangle &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\hat{f}(\nu)\,du\,d\nu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{t\downarrow 0}\operatorname{Str}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\hat{f}(\nu)\,d\nu\,, \end{aligned} \tag{1.7}$$

then Theorem 1.4 is an easy consequence of Theorem 1.1. Thus, it only remains to prove that both Equations (1.4) and (1.7) define the same tempered distribution Z_μ. This follows from the Lebesgue's dominated convergence theorem and Fubini's theorem (Section 10). The verification of the hypothesis of the Fubini's theorem requires the above lower estimate of the nonzero spectrum.

For the readers convenience, we recall the needed preliminaries about the many topics involved: Witten's perturbations, Morse forms, asymptotic expansions of heat kernels, zeta functions of operators, Morse and Smale vector fields, the Morse complex and Quillen metrics (Reidemeister, Milnor and Ray–Singer metrics).

2. Witten's perturbations

2.1. Preliminaries on the Witten's perturbations

2.1.1. Basic notation. Let $M \equiv (M,g)$ be a closed Riemannian n-manifold. For any smooth Euclidean/Hermitian vector bundle E over M, let $C^m(M;E)$, $C^\infty(M;E)$, $L^2(M;E)$, $L^\infty(M;E)$ and $H^m(M;E)$ denote the spaces of distributional sections that are C^m, C^∞, L^2, L^∞ and of Sobolev order m, respectively; as usual, E is removed from this notation if it is the trivial line bundle. Consider the induced scalar product $\langle\ ,\ \rangle$ and norm $\|\ \|$ on $L^2(M;E)$, and the induced norm $\|\ \|_{L^\infty}$ on $L^\infty(M;E)$. Fix also norms, $\|\ \|_m$ on every $H^m(M;E)$ and $\|\ \|_{C^m}$ on $C^m(M;E)$. If P is the orthogonal projection of $L^2(M;E)$ to some closed subspace V, then $P^\perp$ denotes the orthogonal projection to $V^\perp$. Let $o(E)$ denote the flat real orientation line bundle of E. It is said that E is orientable when $o(E)$ is trivial. In this case, an orientation of E is described by a (necessarily smooth) nonvanishing flat section $\mathcal{O}_E$ of $o(E)$; for simplicity, it will be said that $\mathcal{O}_E$ itself is an orientation. In particular, an orientation of M is described using $o(M) := o(TM)$. The flat line bundle $o(E)\otimes o(E)$ is always trivial.

Let $T_{\mathbb{C}}M = TM\otimes\mathbb{C}$ and $T_{\mathbb{C}}^*M = T^*M\otimes\mathbb{C}$. The exterior bundle with coefficients in $\mathbb{K} = \mathbb{R},\mathbb{C}$ is denoted by $\Lambda_{\mathbb{K}} = \Lambda_{\mathbb{K}}M$, and let $\Omega(M,\mathbb{K}) = C^\infty(M;\Lambda_{\mathbb{K}})$; in particular, $C^\infty(M,\mathbb{K}) = \Omega^0(M,\mathbb{K})$. The Levi–Civita connection is denoted by $\nabla = \nabla^M$. As usual, d and δ denote the de Rham derivative and coderivative, and let $D = d+\delta$ and

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$\Delta = D^2 = d\delta + \delta d$ (the Laplacian). Let $Z(M,\mathbb{K})$ and $B(M,\mathbb{K})$ denote the kernel and image of d in $\Omega(M,\mathbb{K})$. Thus, $H^\bullet(M,\mathbb{K}) = Z(M,\mathbb{K})/B(M,\mathbb{K})$ is the de Rham cohomology with coefficients in $\mathbb{K}$. We typically consider complex coefficients, so we will omit $\mathbb{K}$ from all of the above notation just when $\mathbb{K} = \mathbb{C}$. Take $\|\ \|_m$ and $\|\ \|_{C^m}$ given on $\Omega(M)$ by

$$\|\alpha\|_m = \sum_{k=0}^{m} \|D^k\alpha\|\,, \quad \|\alpha\|_{C^m} = \sum_{k=0}^{m} \|\nabla^k\alpha\|_{L^\infty}.$$

In particular, we take $\|\ \| = \|\ \|_0$ and $\|\ \|_{C^0} = \|\ \|_{L^\infty}|_{C^0(M;E)}$.

On any graded vector space $V^\bullet$, let w and N be the degree involution and number operator; that is, $\mathsf{w} = (-1)^k$ and $\mathsf{N} = k$ on V^k. For any homogeneous linear operator between graded vector spaces, $T : V^\bullet \to W^\bullet$, the notation T_k means its precomposition with the canonical projection of $V^\bullet$ to V^k. If T is of degree l ($T(V^k) \subset W^{k+l}$ for all k), then

$$\mathsf{w}T = (-1)^l T\mathsf{w}\,, \quad \mathsf{N}T = T(\mathsf{N}+l). \tag{2.1}$$

For any $\eta \in \Omega^1(M,\mathbb{R})$ with $\eta^\sharp = X \in \mathfrak{X}(M) := C^\infty(M;TM)$ ($\eta = g(X,\cdot)$), let $\mathcal{L}_X$ and ι_X denote the Lie derivative and interior product with respect to X, and let $\eta\lrcorner = -(\eta\wedge)^* = -\iota_X$. Using the identity $\mathrm{Cl}(T^*M) \equiv \Lambda_{\mathbb{R}}M$ defined by the symbol of filtered algebras, the left Clifford multiplication by η is $c(\eta) = \eta\wedge + \eta\lrcorner$, and the composition of w with the right Clifford multiplication by η is $\hat{c}(\eta) = \eta\wedge - \eta\lrcorner$; in particular, $c(\eta)^* = -c(\eta)$ and $\hat{c}(\eta)^* = \hat{c}(\eta)$. Recall that, for any $h \in C^\infty(M,\mathbb{R})$,

$$[D,h] = \hat{c}(dh). \tag{2.2}$$

In the whole paper, unless otherwise indicated, we will use the following notation without further comment. We use constants $C,c > 0$ without even mentioning their existence, and their precise values may change from line to line. We may add subindices or primes to these constants if needed. We also use a complex parameter $z = \mu + i\nu \in \mathbb{C}$ ($\mu,\nu \in \mathbb{R}$ and $i = \sqrt{-1}$). Recall that $\partial_z = (\partial_\mu - i\partial_\nu)/2$ and $\partial_{\bar z} = (\partial_\mu + i\partial_\nu)/2$.

2.1.2. Perturbations defined by a closed real 1-form. For any $\omega \in Z^1(M)$, we have the Witten's type perturbations d_ω, δ_ω, D_ω and Δ_ω of d, δ, D and Δ. Given $\eta \in Z^1(M,\mathbb{R})$ and $z \in \mathbb{C}$, we write $d_z = d_{z\eta}$, $\delta_z = \delta_{z\eta}$, $D_z = D_{z\eta}$ and $\Delta_z = \Delta_{z\eta}$. These operators have the following expressions:

$$\left.\begin{aligned} d_z &= d + z\eta\wedge\,, \quad \delta_z = d_z^* = \delta - \bar{z}\eta\lrcorner\,,\\ D_z &= d_z + \delta_z = D + \mu\hat{c}(\eta) + i\nu c(\eta) = D_{i\nu} + \mu\hat{c}(\eta)\,,\\ \Delta_z &= D_z^2 = d_z\delta_z + \delta_z d_z = \Delta + \mu\mathsf{H}_\eta + i\nu\mathsf{J}_\eta + |z|^2|\eta|^2\\ &= \Delta_{i\nu} + \mu\mathsf{H}_\eta + \mu^2|\eta|^2\,, \end{aligned}\right\} \tag{2.3}$$

where, for $X = \eta^\sharp$,

$$\mathsf{H}_\eta = D\hat{c}(\eta) + \hat{c}(\eta)D = \mathcal{L}_X^* + \mathcal{L}_X\,, \quad \mathsf{J}_\eta = Dc(\eta) + c(\eta)D = \mathcal{L}_X^* - \mathcal{L}_X.$$

Note that H_η is of order zero and J_η of order one.

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As families of operators, d_z and δ_z are holomorphic and antiholomorphic functions of z, respectively. More precisely, it follows from Equation (2.3) that

$$\left.\begin{aligned} &\partial_z d_z = \eta\wedge\,, \quad \partial_z\delta_z = 0\,, \qquad \partial_z\Delta_z = \eta\wedge\delta_z + \delta_z\,\eta\wedge\,,\\ &\partial_{\bar z} d_z = 0\,, \quad \partial_{\bar z}\delta_z = -\eta\lrcorner\,, \quad \partial_{\bar z}\Delta_z = -\eta\lrcorner\, d_z - d_z\,\eta\lrcorner. \end{aligned}\right\} \tag{2.4}$$

The operator d_z defines an elliptic complex on $\Omega(M)$, whose cohomology is denoted by $H_z^\bullet(M)$. Since d_z has the same principal symbol as d, it is a generalized Dirac complex and Δ_z a self-adjoint generalized Laplacian [7, Definition 2.2]. If $\theta = \eta + dh$ for some $h \in C^\infty(M,\mathbb{R})$, then the multiplication operator

$$e^{zh} : (\Omega(M), d_{z\theta}) \to (\Omega(M), d_{z\eta}) \tag{2.5}$$

is an isomorphism of differential complexes, and therefore it induces an isomorphism $H_{z\theta}^\bullet(M) \cong H_{z\eta}^\bullet(M)$. Thus, the isomorphism class of $H_z^\bullet(M)$ only depends on $\xi := [\eta] \in H^1(M,\mathbb{R})$ and $z \in \mathbb{C}$. By ellipticity, D_z and Δ_z have a discrete spectrum, and there is a decomposition, equalities and isomorphism of Hodge type,

$$\left.\begin{gathered} \Omega(M) = \ker\Delta_z \oplus \operatorname{im} d_z \oplus \operatorname{im}\delta_z\,,\\ \ker\Delta_z = \ker D_z = \ker d_z \cap \ker\delta_z\,, \quad \operatorname{im}\Delta_z = \operatorname{im} D_z = \operatorname{im} d_z \oplus \operatorname{im}\delta_z\,,\\ H_z^\bullet(M) \cong \ker\Delta_z\,, \end{gathered}\right\} \tag{2.6}$$

as topological vector spaces. The orthogonal projections of $\Omega(M)$ to $\ker\Delta_z$, $\operatorname{im} d_z$ and $\operatorname{im}\delta_z$ are denoted by $\Pi_z = \Pi_z^0$, Π_z^1 and Π_z^2, respectively; thus, $\Pi_z^\perp = \Pi_z^1 + \Pi_z^2$. The restrictions $d_z : \operatorname{im}\delta_z \to \operatorname{im} d_z$, $\delta_z : \operatorname{im} d_z \to \operatorname{im}\delta_z$ and $D_z : \operatorname{im} D_z \to \operatorname{im} D_z$ are topological isomorphisms, and therefore the compositions $d_z^{-1}\Pi_z^1$, $\delta_z^{-1}\Pi_z^2$ and $D_z^{-1}\Pi_z^\perp$ are defined and continuous on $\Omega(M)$. For every degree k, the diagram

$$\begin{array}{ccc} \operatorname{im}\delta_{z,k+1} & \xrightarrow{d_{z,k}} & \operatorname{im} d_{z,k}\\ {\scriptstyle\Delta_{z,k}}\big\downarrow & & \big\downarrow{\scriptstyle\Delta_{z,k+1}}\\ \operatorname{im}\delta_{z,k+1} & \xrightarrow{d_{z,k}} & \operatorname{im} d_{z,k} \end{array} \tag{2.7}$$

is commutative. The twisted Betti numbers $\beta_z^k = \beta_z^k(M,\xi) = \dim H_z^k(M)$ give rise to the usual Euler characteristic [28, Proposition 1.40],

$$\sum_k (-1)^k \beta_z^k = \chi(M). \tag{2.8}$$

(This is also a consequence of the index theorem.) For every degree k, β_z^k is independent of z outside a discrete subset of $\mathbb{C}$, where β_z^k jumps (Mityagin and Novikov [57, Theorem 1]). This ground value of β_z^k is called the k-th *Novikov Betti number*, denoted by $\beta_{\mathrm{No}}^k = \beta_{\mathrm{No}}^k(M,\xi)$. It will be shown in Section 6.2.4 that

$$\beta_z^k = \beta_{\mathrm{No}}^k \quad \text{for} \quad |\mu| \gg 0. \tag{2.9}$$

(When z is real, this is proved in [27, Theorem 2.8], [14, Lemma 1.3], [18, Proposition 4].) Thus, the discrete set of parameters $z \in \mathbb{C}$ with $\beta_z^k(M,\xi) > \beta_{\mathrm{No}}^k(M,\xi)$ for some degree k is contained in a strip $|\mu| \le C$.

By Equation (2.3) and since η is real, for all $\alpha \in \Omega(M)$,

$$\overline{d_z\alpha} = d_{\bar z}\bar\alpha\,, \quad \overline{\delta_z\alpha} = \delta_{\bar z}\bar\alpha\,, \quad \overline{D_z\alpha} = D_{\bar z}\bar\alpha\,, \quad \overline{\Delta_z\alpha} = \Delta_{\bar z}\bar\alpha. \tag{2.10}$$

So conjugation induces $\mathbb{C}$-antilinear isomorphisms

$$H_z^k(M) \cong H_{\bar z}^k(M)\,, \quad \ker\Delta_{z,k} \cong \ker\Delta_{\bar z,k}\,,$$

yielding $\beta_z^k = \beta_{\bar z}^k$.

2.1.3. Case of an exact form. When $\eta = dh$ for some $h \in C^\infty(M,\mathbb{R})$, we have the original Witten's perturbations, which satisfy

$$\left.\begin{aligned} d_z = e^{-zh}\,d\,e^{zh} = e^{-i\nu h}\,d_\mu\,e^{i\nu h}\,, &\quad \delta_z = e^{\bar z h}\,\delta\,e^{-\bar z h} = e^{-i\nu h}\,\delta_\mu\,e^{i\nu h}\,,\\ D_z = e^{-i\nu h}\,D_\mu\,e^{i\nu h}\,, &\quad \Delta_z = e^{-i\nu h}\,\Delta_\mu\,e^{i\nu h}. \end{aligned}\right\} \tag{2.11}$$

Thus, the multiplication operator

$$e^{zh} : (\Omega(M),d_z) \to (\Omega(M),d) \tag{2.12}$$

is an isomorphism of differential complexes. Therefore, $H_z^\bullet(M) \cong H^\bullet(M)$, yielding $\beta_z^k = \beta^k = \beta^k(M)$ (the kth Betti number) in this case. Moreover multiplication by $e^{i\nu h}$ defines a unitary isomorphism $\ker\Delta_z \cong \ker\Delta_\mu$.

2.1.4. Interpretation of the closed form as a flat connection. There is a unique flat connection $\nabla^{M\times\mathbb{C}}$ on the trivial complex line bundle $M\times\mathbb{C}$ so that $\nabla^{M\times\mathbb{C}}1 = \eta$. The corresponding flat complex line bundle is denoted by $\mathcal{L} = \mathcal{L}_\eta$. Note that $\mathcal{L}_{z\eta} = \mathcal{L}^z$. Let $(\Omega(M,\mathcal{L}^z) = (\Omega(M),d^{\mathcal{L}^z})$ be the de Rham complex with coefficients in $\mathcal{L}^z$. It is well known that $d_z = d^{\mathcal{L}^z}$ on $\Omega(M) = \Omega(M,\mathcal{L}^z)$, and therefore $H^\bullet(M,\mathcal{L}^z) = H_z^\bullet(M)$. Since every $\mathcal{L}^z$ is canonically trivial as a line bundle, it has a canonical Hermitian structure $g^{\mathcal{L}^z}$. An easy local computation shows that (see the example given in [10, pp. 11–12])

$$\nabla^{\mathcal{L}^z} g^{\mathcal{L}^z} = -2\mu\eta\otimes g^{\mathcal{L}^z}. \tag{2.13}$$

2.1.5. Perturbed operators on oriented manifolds. The mappings $(\alpha,\beta)\mapsto\alpha\wedge\beta$ and $(\alpha,\beta)\mapsto\alpha\wedge\bar\beta$ induce respective bilinear and sesquilinear maps,

$$H_z^k(M)\times H_{-z}^l(M) \to H^{k+l}(M)\,, \quad H_z^k(M)\times H_{-\bar z}^l(M) \to H^{k+l}(M)\,,$$

as follows from the interpretation of d_z given in Section 2.1.4, or by a direct check.

Now, assume M is oriented. Then the above maps and integration on M define respective nondegenerate bilinear and sesquilinear pairings

$$H_z^k(M)\times H_{-z}^{n-k}(M) \to \mathbb{C}\,, \quad H_z^k(M)\times H_{-\bar z}^{n-k}(M) \to \mathbb{C}.$$

Thus

$$\beta_z^k = \beta_{-z}^{n-k} = \beta_{-\bar z}^{n-k} = \beta_{\bar z}^k. \tag{2.14}$$

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Let $\star$ and $\bar\star$ denote the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear extensions to ΛM of the Hodge operator $\star$ on $\Lambda_{\mathbb{R}}M$, respectively. These operators are determined by the conditions

$$\alpha\wedge\overline{\star\beta}=g(\alpha,\beta)\,\mathrm{dvol}=\alpha\wedge\bar\star\beta$$

for $\alpha,\beta\in\Omega(M)$, where $\mathrm{dvol}=\star 1$ is the volume form. The following equalities on $\Omega^k(M)$ follow from Equation (2.3) and the usual equalities relating $\star$, d, δ, $\eta\wedge$ and $\eta\lrcorner$ (see, e.g., [63, Chapters 1 and 3], [31, Section 1.5.2], [7, Section 3.6]):

$$\left.\begin{aligned}d_z\star&=(-1)^k\star\delta_{-\bar z}\,, &\delta_z\star&=(-1)^{k+1}\star d_{-\bar z}\,, &\Delta_z\star&=\star\Delta_{-\bar z}\,,\\ d_z\bar\star&=(-1)^k\,\bar\star\,\delta_{-z}\,, &\delta_z\,\bar\star&=(-1)^{k+1}\,\bar\star\,d_{-z}\,, &\Delta_z\,\bar\star&=\bar\star\,\Delta_{-z}.\end{aligned}\right\}\tag{2.15}$$

Then we get a linear isomorphism $\star:\ker\Delta_z\to\ker\Delta_{-\bar z}$ and an antilinear isomorphism $\bar\star:\ker\Delta_z\to\ker\Delta_{-z}$, inducing a linear isomorphism $H^k_z(M)\cong H^{n-k}_{-\bar z}(M)$ and an antilinear isomorphism $H^k_z(M)\cong H^{n-k}_{-z}(M)$ by Equation (2.6).

2.2. Perturbation of the Sobolev norms

For $m\in\mathbb{N}_0$ and $\omega\in Z^1(M)$, define the norm $\|\ \|_{m,\omega}$ on $H^m(M;\Lambda)$ by

$$\|\alpha\|_{m,\omega}=\sum_{k=0}^m\left\|D^k_\omega\alpha\right\|.$$

Proposition 2.1. *For all $\omega\in Z^1(M)$ and $\alpha\in H^m(M;\Lambda)$,*

$$\|\alpha\|_{m,\omega}\le C_m\sum_{k=0}^m\|\omega\|^{m-k}_{C^k}\|\alpha\|_k\,,\quad \|\alpha\|_m\le C_m\sum_{k=0}^m\|\omega\|^{m-k}_{C^k}\|\alpha\|_{k,\omega}.$$

Proof. We proceed by induction on m. We have $\|\ \|_{0,\omega}=\|\ \|$. Now, take $m>0$ and assume these inequalities hold for $m-1$. For $\eta\in Z^1(M,\mathbb{R})$ and $\alpha\in\Omega(M)$, we have

$$\|\hat c(\eta)\alpha\|_m,\|c(\eta)\alpha\|_m\le C'_m\|\eta\|_{C^m}\|\alpha\|_m.\tag{2.16}$$

Applying these inequalities to the real and imaginary parts of ω and using the induction hypothesis and Equation (2.3), we get

$$\begin{aligned}\|\alpha\|_{m,\omega}=\|\alpha\|+\|D_\omega\alpha\|_{m-1,\omega}&\le\|\alpha\|+C_{m-1}\sum_{k=0}^{m-1}\|\omega\|^{m-1-k}_{C^k}\|D_\omega\alpha\|_k\\ &\le\|\alpha\|+C_{m-1}\sum_{k=0}^{m-1}\|\omega\|^{m-1-k}_{C^k}\big(\|D\alpha\|_k+C'_k\|\omega\|_{C^k}\|\alpha\|_k\big)\\ &\le\|\alpha\|+C_{m-1}\sum_{k=0}^{m-1}\|\omega\|^{m-1-k}_{C^k}\big(\|\alpha\|_{k+1}+C'_k\|\omega\|_{C^k}\|\alpha\|_k\big)\\ &\le C_m\sum_{l=0}^m\|\omega\|^{m-l}_{C^l}\|\alpha\|_l\,,\end{aligned}$$

$$\begin{aligned}\|\alpha\|_m &= \|\alpha\| + \|D\alpha\|_{m-1} \le \|\alpha\| + \|D_\omega\alpha\|_{m-1} + C'_{m-1}\|\omega\|_{C^{m-1}}\|\alpha\|_{m-1}\\ &\le \|\alpha\| + C_{m-1}\sum_{k=0}^{m-1}\left(\|\omega\|_{C^k}^{m-1-k}\|D_\omega\alpha\|_{k,\omega} + C'_{m-1}\|\omega\|_{C^k}^{m-k}\|\alpha\|_{k,\omega}\right)\\ &\le \|\alpha\| + C_{m-1}\sum_{k=0}^{m-1}\left(\|\omega\|_{C^k}^{m-1-k}\|\alpha\|_{k+1,\omega} + C'_{m-1}\|\omega\|_{C^k}^{m-k}\|\alpha\|_{k,\omega}\right)\\ &\le C_m\sum_{l=0}^{m}\|\omega\|_{C^l}^{m-l}\|\alpha\|_{l,\omega}.\end{aligned}$$

□

Let $Z(M,\mathbb{Z}) \subset Z(M,\mathbb{R})$ denote the graded additive subgroup of forms that represent cohomology classes in the image of the canonical homomorphism $H^\bullet(M,\mathbb{Z}) \to H^\bullet(M,\mathbb{R})$. Recall that we can consider $H^1(M,\mathbb{Z})$ as a lattice in $H^1(M,\mathbb{R})$ by the universal coefficient theorem for cohomology. Let θ be the multivalued angle function on $\mathbb{S}^1$. Then $d\theta$ is the angular form on $\mathbb{S}^1$ with $\int_{\mathbb{S}^1} d\theta = 2\pi$. For $\eta \in Z^1(M,\mathbb{R})$, we have $\eta \in 2\pi Z^1(M,\mathbb{Z})$ if and only if there is some smooth map $h : M \to \mathbb{S}^1$ such that $\eta = h^* d\theta$ (see, e.g., [28, Lemma 2.1]).

In Proposition 2.1, the dependence of the constants on ω cannot be avoided. For instance, for $M = \mathbb{S}^1$ with the standard metric $g = (d\theta)^2$, we have $\|1\|_m = \sqrt{2\pi}$, whereas $\|1\|_{m,i\eta} = \sqrt{2\pi}\sum_{k=0}^m |\nu|^k$ for $\eta = \nu\, d\theta$ ($\nu \in \mathbb{R}$). However, the following version of a Sobolev inequality for $\|\ \|_{m,i\eta}$ involves a constant independent of η.

Proposition 2.2. *If $m > n/2$, for all $\eta \in Z^1(M,\mathbb{R})$ and $\alpha \in H^m(M;\Lambda)$,*

$$\|\alpha\|_{L^\infty} \le C_m\|\alpha\|_{m,i\eta}.$$

Proof. By the Sobolev embedding theorem, we have

$$C_{m,i\eta} := \sup_{0\ne\alpha\in\Omega(M)} \frac{\|\alpha\|_{L^\infty}}{\|\alpha\|_{m,i\eta}} > 0.$$

Take any $\eta \in Z^1(M,\mathbb{R})$ and $\omega \in 2\pi Z^1(M,\mathbb{Z})$, and let $\eta' = \eta + \omega$. Then $\omega = h^* d\theta$ for some smooth function $h : M \to \mathbb{S}^1$. Since the difference between the multiple values of θ at every point of $\mathbb{S}^1$ are in $2\pi\mathbb{Z}$, the functions $e^{\pm ih^*\theta}$ are well defined and smooth on M. Moreover, applying Equation (2.11) locally, we get $D_{i\eta'} = e^{-ih^*\theta} D_{i\eta} e^{ih^*\theta}$. So, for $0 \ne \alpha \in \Omega(M)$,

$$\begin{aligned}\|\alpha\|_{L^\infty} &= \|e^{ih^*\theta}\alpha\|_{L^\infty} \le C_{m,i\eta}\|e^{ih^*\theta}\alpha\|_{m,i\eta}\\ &= C_{m,i\eta}\sum_{k=0}^{m}\|D_{i\eta}^k e^{ih^*\theta}\alpha\| = C_{m,i\eta}\sum_{k=0}^{m}\|e^{-ih^*\theta} D_{i\eta}^k e^{ih^*\theta}\alpha\|\\ &= C_{m,i\eta}\sum_{k=0}^{m}\|D_{i\eta'}^k\alpha\| = C_{m,i\eta}\|\alpha\|_{m,i\eta'}.\end{aligned}$$

This shows that

$$\eta - \eta' \in 2\pi Z^1(M,\mathbb{Z}) \Rightarrow C_{m,i\eta} = C_{m,i\eta'}. \tag{2.17}$$

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Since $2\pi H^1(M,\mathbb{Z})$ is a lattice in $H^1(M,\mathbb{R})$, there is a compact subset $K \subset H^1(M,\mathbb{R})$ such that

$$K + 2\pi H^1(M,\mathbb{Z}) = H^1(M,\mathbb{R}). \tag{2.18}$$

Take a linear subspace $V \subset Z^1(M,\mathbb{R})$ such that the canonical projection $V \to H^1(M,\mathbb{R})$ is an isomorphism, and let $L \subset V$ be the compact subset that corresponds to K. By Equation (2.18),

$$L + 2\pi Z^1(M,\mathbb{Z}) = Z^1(M,\mathbb{R}). \tag{2.19}$$

Moreover, L is bounded with respect to $\| \ \|_{C^m}$. Therefore, by Proposition 2.1, for all $\eta \in L$ and $\alpha \in \Omega(M)$,

$$\|\alpha\|_{L^\infty} \leq C_{m,0}\|\alpha\|_m \leq C_m\|\alpha\|_{m,i\eta}\,,$$

yielding

$$\sup_{\eta\in L} C_{m,i\eta} \leq C_m. \tag{2.20}$$

The result follows from Equations (2.17), (2.19) and (2.20). □

Given $\eta \in Z^1(M,\mathbb{R})$, we write $\| \ \|_{m,z} = \| \ \|_{m,z\eta}$. Proposition 2.1 has the following direct consequence.

Corollary 2.3. *For all $\alpha \in H^m(M;\Lambda)$ and $z \in \mathbb{C}$,*

$$\|\alpha\|_{m,z} \leq C_m \sum_{k=0}^{m} |z|^{m-k}\|\alpha\|_k\,, \quad \|\alpha\|_m \leq C_m \sum_{k=0}^{m} |z|^{m-k}\|\alpha\|_{k,z}.$$

Proposition 2.4. *For all $\alpha \in H^1(M;\Lambda)$ and $z \in \mathbb{C}$,*

$$\|\alpha\|_{1,z} \leq C\big(\|\alpha\|_{1,i\nu} + |\mu|\|\alpha\|\big)\,, \quad \|\alpha\|_{1,i\nu} \leq C\big(\|\alpha\|_{1,z} + |\mu|\|\alpha\|\big).$$

Proof. By Equations (2.3) and (2.16),

$$\begin{aligned}\|\alpha\|_{1,z} &= \|\alpha\| + \|D_z\alpha\| \leq \|\alpha\| + \|D_{i\nu}\alpha\| + C'|\mu|\|\alpha\| \leq C\big(\|\alpha\|_{1,i\nu} + |\mu|\|\alpha\|\big)\,,\\ \|\alpha\|_{1,i\nu} &= \|\alpha\| + \|D_{i\nu}\alpha\| \leq \|\alpha\| + \|D_z\alpha\| + C'|\mu|\|\alpha\| \leq C\big(\|\alpha\|_{1,z} + |\mu|\|\alpha\|\big).\end{aligned}$$

□

3. Zeta invariants of closed real 1-forms

3.1. Preliminaries on asymptotic expansions of heat kernels

Let A be a positive semidefinite symmetric elliptic differential operator of order a, and B a differential operator of order b; both of them are defined in $C^\infty(M;E)$ for some Hermitian vector bundle E over M. Then Be^{-tA} is a smoothing operator with Schwartz kernel $K_t(x,y)$ in $C^\infty(M^2;E \boxtimes E^*)$ (omitting the Riemannian density $\operatorname{dvol}(y)$ of the second factor). On the diagonal, there is an asymptotic expansion (as $t \downarrow 0$) with respect to the

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seminorms $\|\ \|_{C^m}$ $(m \in \mathbb{N}_0)$ on $C^\infty(M; E \otimes E^*)$ [31, Lemma 1.9.1], [7, Theorem 2.30, Proposition 2.46 and the paragraph that follows],

$$K_t(x,x) \sim \sum_{l=0}^{\infty} e_l(x) t^{(l-n-b)/a}, \tag{3.1}$$

with $e_l \in C^\infty(M; E \otimes E^*)$. Moreover, using a local system of coordinates, a local trivialization of E and standard multi-index notation, if $B = \sum_\alpha b_\alpha(x) D_x^\alpha$, then $e_l(x) = \sum_\alpha b_\alpha(x) e_{l,\alpha}(x)$, where the $e_{l,\alpha}(x)$ are smooth local invariants of the symbol of A which are homogeneous of degree $l + |\alpha| - b$. They vanish if $l + b$ is odd or if $l + |\alpha| - b < 0$. Hence, the function

$$h(t) = \operatorname{Tr}\left(Be^{-tA}\right) = \int_M \operatorname{tr} K_t(x,x)\, d\mathrm{vol}(x)$$

has an asymptotic expansion

$$h(t) \sim \sum_{l=0}^{\infty} a_l t^{(l-n-b)/a}, \tag{3.2}$$

where

$$a_l = \int_M \operatorname{tr} e_l(x)\, d\mathrm{vol}(x), \tag{3.3}$$

which vanishes if $l + b$ is odd.

The case of truncated heat kernels, in the following sense, is also needed. Given any $\lambda \geq 0$, let $P_{A,\lambda}$ be the spectral projection of A corresponding to $[0,\lambda]$; thus, $P_{A,\lambda}^\perp$ is the spectral projection corresponding to (λ,∞). By ellipticity, $P_{A,\lambda}$ is of finite rank, and $Be^{-tA}P_{A,\lambda}$ is a smoothing operator defined for all $t \in \mathbb{R}$. Take any orthonormal frame $\phi_1,\dots,\phi_\kappa$ of $\operatorname{im} P_{A,\lambda}$, consisting of eigensections with corresponding eigenvalues $0 \leq \lambda_1 \leq \cdots \leq \lambda_\kappa \leq \lambda$. Then the Schwartz kernel $H_t(x,y)$ of $Be^{-tA}P_{A,\lambda}$ $(t \geq 0)$ is given by

$$H_t(x,y) = \sum_{j=1}^{\kappa} e^{-t\lambda_j} (B\phi_j)(x) \otimes \phi_j(y),$$

using the isomorphism $E \cong E^*$ given by the Hermitian structure. Thus, $H_t(x,y)$ is defined for all $t \in \mathbb{R}$ and smooth. So

$$\operatorname{Tr}(Be^{-tA}P_{A,\lambda}) = \int_M \operatorname{tr} H_t(x,x)\, d\mathrm{vol}(x).$$

In particular, for $t = 0$, we have

$$H_0(x,x) = \sum_{j=1}^{\kappa} (B\phi_j)(x) \otimes \phi_j(x), \tag{3.4}$$

$$\operatorname{Tr}(BP_{A,\lambda}) = \int_M \operatorname{tr} H_0(x,x)\, d\mathrm{vol}(x). \tag{3.5}$$

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The Schwartz kernel of $Be^{-tA}P^{\perp}_{A,\lambda}$ is $\widetilde{K}_t(x,y) = K_t(x,y) - H_t(x,y)$ $(t > 0)$, which has an asymptotic expansion

$$\widetilde{K}_t(x,x) \sim \sum_{l=0}^{\infty} \tilde{e}_l(x) t^{(l-n-b)/a} , \tag{3.6}$$

where the first $n+b$ sections $\tilde{e}_l$ are given by

$$\tilde{e}_l(x) = \begin{cases} e_l(x) & \text{if } l < n+b \\ e_l(x) - H_0(x,x) & \text{if } l = n+b. \end{cases}$$

Then the function

$$\tilde{h}_\lambda(t) = \operatorname{Tr}\left(Be^{-tA}P^{\perp}_{A,\lambda}\right) = \operatorname{Tr}\left(Be^{-tA}\right) - \operatorname{Tr}(Be^{-tA}P_{A,\lambda}) \tag{3.7}$$

has an asymptotic expansion

$$\tilde{h}_\lambda(t) = \int_M \widetilde{K}_t(x,x)\,\mathrm{dvol}(x) \sim \sum_{l=0}^{\infty} \tilde{a}_l t^{(l-n-b)/a} , \tag{3.8}$$

where the first $n+b$ coefficients $\tilde{a}_l$ are given by

$$\tilde{a}_l = \begin{cases} a_l & \text{if } l < n+b \\ a_l - \operatorname{Tr}(BP_{A,\lambda}) & \text{if } l = n+b. \end{cases} \tag{3.9}$$

Consider also smooth families of such operators, $\{A_\epsilon\}$ and $\{B_\epsilon\}$, for ϵ in some parameter space. Then $\operatorname{Tr}(B_\epsilon e^{-tA_\epsilon})$ is smooth in (t,ϵ), and we add ϵ to the above notation, writing for instance $K_t(x,y,\epsilon)$, $e_l(x,\epsilon)$, $h(t,\epsilon)$, $a_l(\epsilon)$, $\widetilde{K}_t(x,y,\epsilon)$, $\tilde{e}_l(x,\epsilon)$, $\tilde{h}(t,\epsilon)$ and $\tilde{a}_l(\epsilon)$ in Equations (3.1), (3.2), (3.6) and (3.8). The operator $B_\epsilon P_{A_\epsilon,\lambda}$ may not be smooth in ϵ when some nonconstant spectral branch of $\{A_\epsilon\}$ reaches the value λ. If the values of all nonconstant spectral branches of $\{A_\epsilon\}$ stay away from some neighborhood of λ, then $\tilde{h}_\lambda(t,\epsilon)$ is smooth in (t,ϵ).

3.2. Preliminaries on zeta functions of operators

Proposition 3.1 (See [31, Theorems 1.12.2 and 1.12.5], [7, Propositions 9.35–9.37]). *The following holds:*

(i) *For every $\lambda \in \mathbb{R}$, there is a meromorphic function $\zeta(s,A,B,\lambda)$ on $\mathbb{C}$ such that, for $\Re s \gg 0$,*

$$\zeta(s,A,B,\lambda) = \operatorname{Tr}\left(BA^{-s}P^{\perp}_{A,\lambda}\right) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\tilde{h}_\lambda(t)\,dt. \tag{3.10}$$

(ii) *The meromorphic function $\Gamma(s)\zeta(s,A,B,\lambda)$ has simple poles at the points $s = (n+b-l)/a$, for $l \in \mathbb{N}_0$ with $\tilde{a}_l \neq 0$. The corresponding residues are $\tilde{a}_l$, and $\zeta(s,A,B,\lambda)$ is smooth away from these exceptional values of s.*

(iii) *For $\mu > \lambda \geq 0$, let $\lambda_1 \leq \cdots \leq \lambda_k$ denote the eigenvalues of A in $(\lambda,\mu]$, taking multiplicities into account, and let $\psi_1,\ldots,\psi_k$ be corresponding orthonormal*

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eigensections. Then, for all s,

$$\zeta(s,A,B,\mu)-\zeta(s,A,B,\lambda)=\sum_{j=1}^{k}\lambda_k^{-s}\langle B\psi_j,\psi_j\rangle.$$

(iv) *For smooth families $\{A_\epsilon\}$ and $\{B_\epsilon\}$ of such operators, if the values of all nonconstant branches of eigenvalues of $\{A_\epsilon\}$ stay away from some neighborhood of λ, then $\zeta(s,A_\epsilon,B_\epsilon,\lambda)$ is smooth in (s,ϵ) away from the exceptional values of s given in (ii).*

(v) *Consider the conditions of (iv) for ϵ in some open neighborhood of 0 in $\mathbb{R}$. If A_0 and B_0 commute, then*

$$\partial_\epsilon\zeta(s,A_\epsilon,B_\epsilon,\lambda)\big|_{\epsilon=0}=\zeta(s,A_0,\dot{B}_0,\lambda)-s\zeta(s+1,A_0,\dot{A}_0B_0,\lambda)\,,$$

where the dot denotes ∂_ϵ.

The last expression of Equation (3.10) is the Mellin transform of the function $\tilde{h}_\lambda(t)$ divided by $\Gamma(s)$. This function $\zeta(s,A,B,\lambda)$ is called the *zeta function* of (A,B,λ). If $B=1$ or $\lambda=0$, they may be omitted from the notation.

We will also use $\zeta(s,A,B,\lambda)$ when B is not a differential operator, with the same definition. Then the asymptotic expansion (3.8) and the properties stated in Proposition 3.1 need to be checked. With this generality, we can write

$$\zeta(s,A,B,\lambda)=\zeta(s,A,BP_{A,\lambda}^{\perp})=\zeta(s,A,P_{A,\lambda}^{\perp}B)\,,$$
$$\zeta(s,A,B)=\zeta(s,A,BP_{A,\lambda})+\zeta(s,A,B,\lambda).$$

Since $P_{A,\lambda}$ is of finite rank, $\zeta(s,A,BP_{A,\lambda})$ is always defined and holomorphic on $\mathbb{C}$.

3.3. Zeta invariants of closed real 1-forms

According to Proposition 3.1 (i), let

$$\zeta(s,z)=\zeta(s,z,\eta)=\zeta(s,\Delta_z,\eta\wedge D_z\mathsf{w})\,,$$

which is a meromorphic function of $s\in\mathbb{C}$. For $\Re s\gg 0$,

$$\begin{aligned}\zeta(s,z)&=\operatorname{Str}\left(\eta\wedge D_z\Delta_z^{-s}\Pi_z^{\perp}\right)=\operatorname{Str}\left(\eta\wedge\delta_z\Delta_z^{-s}\Pi_z^{1}\right)\\&=\operatorname{Str}\left(\eta\wedge D_z^{-1}\Delta_z^{-s+1}\Pi_z^{\perp}\right)=\operatorname{Str}\left(\eta\wedge d_z^{-1}\Delta_z^{-s+1}\Pi_z^{1}\right)\,,\end{aligned}$$

using that $\eta\wedge d_z$ and $\eta\wedge\delta_z^{-1}$ change the degree of homogeneous forms. So, when $\zeta(s,z)$ is regular at $s=1$, the value $\zeta(1,z)$ is a renormalized version of the super-trace of $\eta\wedge d_z^{-1}\Pi_z^1$, which is called the *zeta invariant* of (M,g,η,z) for the scope of this paper. According to Proposition 3.1 (ii) and since $\Gamma(s)$ is regular at $s=1$, $\zeta(s,z)$ might have a simple pole at $s=1$. But it will be shown that $\zeta(s,z)$ is regular at $s=1$ for all $\eta\in Z^1(M,\mathbb{R})$ and $z\in\mathbb{C}$ (Corollary 3.9).

3.4. Heat invariants of perturbed operators

Consider the notation of Section 2.1.2. For $k=0,\dots,n$, let $K_{z,k,t}(x,y)$ denote the Schwartz kernel of $e^{-t\Delta_{z,k}}$. By Equation (3.1), its restriction to the diagonal has an asymptotic expansion (as $t\downarrow 0$),

$$K_{z,k,t}(x,x)\sim\sum_{l=0}^{\infty}e_{k,l}(x,z)t^{(l-n)/2}\,,$$

where every $e_{k,l}(x,z)$ is a smooth local invariant of z and the jets of the local coefficients of g and η, which is homogeneous of degree l, and vanishes if l is odd. According to Equations (3.2) and (3.3),

$$h_k(t,z):=\operatorname{Tr}\left(e^{-t\Delta_{z,k}}\right)\sim\sum_{l=0}^{\infty}a_{k,l}(z)t^{(l-n)/2}\,,$$

where

$$a_{k,l}(z)=\int_M \operatorname{str}e_{k,l}(x,z)\,\operatorname{dvol}(x).$$

The Schwartz kernel of $e^{-t\Delta_z}\mathsf{w}$ is

$$K_{z,t}(x,y)=\sum_{k=0}^{n}(-1)^kK_{z,k,t}(x,y).$$

We have induced asymptotic expansions,

$$K_{z,t}(x,x)\sim\sum_{l=0}^{\infty}e_l(x,z)t^{(l-n)/2}\,,$$
$$h(t,z):=\operatorname{Str}\left(e^{-t\Delta_z}\right)\sim\sum_{l=0}^{\infty}a_l(z)t^{(l-n)/2}\,,$$

where

$$e_l(x,z)=\sum_{k=0}^{n}(-1)^ke_{k,l}(x,z)\,,\quad a_l(z)=\sum_{k=0}^{n}(-1)^ka_{k,l}(z).$$

Theorem 3.2 ([10, Theorem 13.4]; see also [1, Theorem 1.5] and [6]). *We have:*

(i) $e_l(x,z)=0$ *for* $l<n$*; and,*

(ii) *if n is even, then* $e_n(x,z)=e(M,\nabla^M)(x)$.

Remark 3.3. The analog of Theorem 3.2 fails for Witten's type perturbations of the Dolbeault complex on Kähler manifolds [2].

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3.5. Derived heat invariants of perturbed operators

The following are sometimes called the *derived heat density* and *derived heat invariant* of order l of d_z or Δ_z [33], [61], [31, page 181], [3]:

$$\mathfrak{e}_l(x,z) = \sum_{k=0}^{n} (-1)^k k e_{k,l}(x,z)\,,$$

$$\mathfrak{a}_l(z) = \sum_{k=0}^{n} (-1)^k k a_{k,l}(z) = \int_M \operatorname{str} \mathfrak{e}_l(x,z)\, \operatorname{dvol}(x).$$

We have

$$\operatorname{Str}\left(\mathsf{N}e^{-t\Delta_z}\right) \sim \sum_{l=0}^{\infty} \mathfrak{a}_l(z) t^{(l-n)/2}. \tag{3.11}$$

Theorem 3.4 [10, Theorem 7.10]. *For $l \leq n$, $\mathfrak{a}_l(z)$ is independent of z.*

Remark 3.5. [10, Theorem 7.10] gives Theorem 3.4 for real z. But, since the functions $\mathfrak{e}_l(x,z)$ have local expressions, we can assume η is exact. Then the result can be extended to nonreal z using Equation (2.11). The exactness of η in [10, Theorem 7.10] is irrelevant because a general flat vector bundle is considered. Moreover, [10, Theorem 7.10] gives an explicit expression of $\mathfrak{a}_l(z)$ for $l \leq n$.

Remark 3.6. A refinement of Theorem 3.4 is given in [3, Theorem 1.3 (1b)], where $\mathfrak{e}_l(x,z)$ is described for $l \leq n$, showing its independence of z.

3.6. Regularity

By Equations (3.2) and (3.3), we have an asymptotic expansion of the form

$$\operatorname{Str}\left(\eta \wedge D_z e^{-t\Delta_z}\right) \sim \sum_{l=0}^{\infty} b_l(z) t^{(l-n-1)/2}\,, \tag{3.12}$$

where $b_l(z) = 0$ if l is even.

Proposition 3.7. *For all $t > 0$ and $z \in \mathbb{C}$, the equality (1.5) is true.*

Proof. For all k, we have [7, Corollary 2.50]

$$\partial_z \operatorname{Tr}\left(e^{-t\Delta_{z,k}}\right) = -t \operatorname{Tr}\left((\partial_z \Delta_{z,k}) e^{-t\Delta_{z,k}}\right).$$

So, by Equations (2.1) and (2.4),

$$\begin{aligned}
\partial_z \operatorname{Str}\left(\mathsf{N}e^{-t\Delta_z}\right) &= -t \operatorname{Str}\left(\mathsf{N}(\partial_z \Delta_z) e^{-t\Delta_z}\right) \\
&= -t \operatorname{Str}\left(\mathsf{N}\, \eta \wedge \delta_z e^{-t\Delta_z}\right) - t \operatorname{Str}\left(\mathsf{N}\delta_z\, \eta \wedge e^{-t\Delta_z}\right) \\
&= -t \operatorname{Str}\left(\mathsf{N}\, \eta \wedge \delta_z e^{-t\Delta_z}\right) - t \operatorname{Str}\left(\delta_z (\mathsf{N}-1)\, \eta \wedge e^{-t\Delta_z}\right) \\
&= -t \operatorname{Str}\left(\mathsf{N}\, \eta \wedge \delta_z e^{-t\Delta_z}\right) + t \operatorname{Str}\left((\mathsf{N}-1)\, \eta \wedge \delta_z e^{-t\Delta_z}\right) \\
&= -t \operatorname{Str}\left(\eta \wedge D_z e^{-t\Delta_z}\right).
\end{aligned}$$

□

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Corollary 3.8. *For $l \le n-1$, $b_l(z) = 0$.*

Proof. By Equations (3.11) and (3.12); Theorem 3.4; and Proposition 3.7, for $l \le n-1$,

$$b_l(z) = -\partial_z \mathfrak{a}_{l+1}(z) = 0.$$

□

Corollary 3.9. *If n is even and $\Re s > 0$, or n is odd and $\Re s > 1/2$, then*

$$\zeta(s,z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Str}\left(\eta \wedge D_z e^{-t\Delta_z}\right) dt\,,$$

where the integral is absolutely convergent, and therefore $\zeta(s,z)$ is smooth in this half-plane.

Proof. By Equation (3.12) and Corollary 3.8,

$$\operatorname{Str}\left(\eta \wedge D_z e^{-t\Delta_z}\right) = \begin{cases} O(1) & \text{if } n \text{ is even} \\ O\left(t^{-1/2}\right) & \text{if } n \text{ is odd} \end{cases} \quad (t \downarrow 0). \tag{3.13}$$

On the other hand, there is some $c > 0$ such that

$$\operatorname{Str}\left(\eta \wedge D_z e^{-t\Delta_z}\right) = O(e^{-ct}) \quad (t \uparrow +\infty). \tag{3.14}$$

So the stated integral is absolutely convergent for $\Re s > 0$ if n is even, or for $\Re s > 1/2$ if n is odd, defining a holomorphic function of s on this half-plane. Then the stated equality is true because it holds for $\Re s \gg 0$. □

Remark 3.10. From Proposition 3.1 (ii) and Corollary 3.8, it also follows that, if n is even (resp., odd), then $\zeta(s,z)$ is smooth on $\mathbb{C}$ (resp., on $\mathbb{C} \setminus ((1-\mathbb{N}_0)/2)$). But this additional regularity is not needed in this work.

Corollary 3.11. *For all $z \in \mathbb{C}$,*

$$\zeta(1,z) = \lim_{t \downarrow 0} \operatorname{Str}\left(\eta \wedge D_z^{-1} e^{-t\Delta_z} \Pi_z^\perp\right).$$

Proof. By Corollary 3.9, Equation (3.13) and Equation (3.14), and since

$$\operatorname{Str}\left(\eta \wedge D_z^{-1} e^{-t\Delta_z} \Pi_z^\perp\right) = O(e^{-ct}) \quad (t \uparrow +\infty)\,,$$

we get

$$\begin{aligned} \zeta(1,z) &= \int_0^\infty \operatorname{Str}\left(\eta \wedge D_z e^{-u\Delta_z} \Pi_z^\perp\right) du = \lim_{t \downarrow 0} \int_t^\infty \operatorname{Str}\left(\eta \wedge D_z e^{-u\Delta_z} \Pi_z^\perp\right) du \\ &= \lim_{t \downarrow 0} \operatorname{Str}\left(\eta \wedge D_z^{-1} e^{-t\Delta_z} \Pi_z^\perp\right). \end{aligned}$$

□

Corollaries 3.9 and 3.11 give Theorem 1.1.

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3.7. The case of the differential of a function

Let us consider the special case where $\eta = dh$ for a smooth real-valued function h.

Lemma 3.12. *We have*

$$\operatorname{Str}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right) = -\operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^{\perp}\right).$$

Proof. Since $\eta\wedge = [d,h]$,

$$\begin{aligned}\operatorname{Str}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right) &= \operatorname{Str}\left([d_z,h]\,d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\\ &= \operatorname{Str}\left(d_z\,h\,d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right) - \operatorname{Str}\left(h\,d_z d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\\ &= -\operatorname{Str}\left(h\,d_z^{-1}e^{-t\Delta_z}\Pi_z^1 d_z\right) - \operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^1\right)\\ &= -\operatorname{Str}\left(h\,d_z^{-1}d_z e^{-t\Delta_z}\Pi_z^2\right) - \operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^1\right)\\ &= -\operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^2\right) - \operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^1\right)\\ &= -\operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^{\perp}\right).\end{aligned}$$

□

Corollary 3.13. *We have*

$$\zeta(1,z) = -\lim_{t\downarrow 0}\operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^{\perp}\right).$$

Proof. Apply Corollary 3.11 and Lemma 3.12. □

Corollary 3.14. *We have $\zeta(1,z)\in\mathbb{R}$.*

Proof. By Corollary 3.13, it is enough to prove that $\operatorname{Str}(h\,e^{-t\Delta_z}\Pi_z^{\perp})\in\mathbb{R}$. But, taking adjoints,

$$\operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^{\perp}\right) = \overline{\operatorname{Str}\left(\Pi_z^{\perp}e^{-t\Delta_z}\,h\right)} = \overline{\operatorname{Str}\left(h\,\Pi_z^{\perp}e^{-t\Delta_z}\right)} = \overline{\operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^{\perp}\right)}.$$

□

Corollary 3.15. *If M is oriented, then*

$$\zeta(1,z) = \zeta(1,-\bar z) = \zeta(1,-z) = \zeta(1,\bar z).$$

Proof. By Equation (2.15),

$$\begin{aligned}\operatorname{Str}\left(h\,e^{-t\Delta_z}\Pi_z^{\perp}\right) &= \operatorname{Str}\left(\star\star^{-1}h\,e^{-t\Delta_z}\Pi_z^{\perp}\right) = \operatorname{Str}\left(\star^{-1}\,h\,e^{-t\Delta_z}\Pi_z^{\perp}\,\star\right)\\ &= \operatorname{Str}\left(\star^{-1}\star h\,e^{-t\Delta_{-\bar z}}\Pi_{-\bar z}^{\perp}\right) = \operatorname{Str}\left(h\,e^{-t\Delta_{-\bar z}}\Pi_{-\bar z}^{\perp}\right).\end{aligned}$$

Thus, the first equality of the statement holds by Corollary 3.13. The second equality follows with a similar argument, using $\bar\star$ instead of $\star$. The third equality is equivalent to the first one. □

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4. Small and large complexes of Morse forms

4.1. Preliminaries on Morse forms

Recall that a critical point p of any $h \in C^\infty(M,\mathbb{R})$ is called *nondegenerate* if the symmetric bilinear form $\operatorname{Hess}_p h$ on T_pM is nondegenerate; then the index of $\operatorname{Hess}_p h$ is denoted by $\operatorname{ind}(p)$. By the Morse lemma [49, Lemma 2.2], this means that

$$h - h(p) = \frac{1}{2}\sum_{j=1}^{n} \epsilon_{p,j}(x_p^j)^2 = \frac{1}{2}\left(|x_p^+|^2 - |x_p^-|^2\right), \tag{4.1}$$

where

$$\epsilon_{p,j} = \begin{cases} -1 & \text{if } j \le \operatorname{ind}(p) \\ 1 & \text{if } j > \operatorname{ind}(p), \end{cases} \tag{4.2}$$

on some chart $(U_p, x_p = (x_p^1,\dots,x_p^n))$ (centered) at p (*Morse coordinates*), where $x_p^- = (x_p^1,\dots,x_p^{\operatorname{ind}(p)})$ and $x_p^+ = (x_p^{\operatorname{ind}(p)+1},\dots,x_p^n)$.

Recall that h is called a *Morse function* when all of its critical points are nondegenerate. Then its critical points form a finite set denoted by $\operatorname{Crit}(h)$. The Morse functions form an open and dense subset of $C^\infty(M,\mathbb{R})$ [36, Theorem 6.1.2]. On every U_p, we can assume the metric is Euclidean with respect to Morse coordinates:

$$g = \sum_{j=1}^{n}(dx_p^j)^2. \tag{4.3}$$

Now, take any $\eta \in Z^1(M,\mathbb{R})$. We can show that if p is a zero of η, then $(\nabla\eta)_p$ is independent of the choice of the connection ∇, and is symmetric. The zero p is called *nondegenerate* of *index* k if $(\nabla\eta)_p$ is nondegenerate of index k. In this case, any local primitive $h_{\eta,p}$ of η near p is a Morse function, and we can choose it so that $h_{\eta,p}(p) = 0$. On a domain U_p of Morse coordinates $x_p = (x_p^1,\dots,x_p^n)$ for $h_{\eta,p}$ at p, also called *Morse coordinates* for η at p, $h_{\eta,p}$ is given by the center and right-hand side of Equation (4.1), and

$$\eta = \sum_{j=1}^{n} \epsilon_{p,j}\, x_p^j\, dx_p^j. \tag{4.4}$$

If all zeros are nondegenerate, then η is called a *Morse form*. In this case, its zeros form a finite set, $\mathcal{X} = \operatorname{Zero}(\eta)$; subsets of $\mathcal{X}$ defined by conditions on the index are denoted by writing the conditions as subscripts; for instance, $\mathcal{X}_k$, $\mathcal{X}_+$ and $\mathcal{X}_{<k}$ are the subsets of zeros of index k, of positive index and of index $< k$, respectively. For any $\xi \in H^1(M,\mathbb{R})$, the Morse representatives of ξ form a dense open subset of ξ, considering $\xi \subset \Omega^1(M,\mathbb{R})$ with the C^∞ topology (see, e.g., [59, Theorem 2.1.25]). If $\xi = 0$, this is just the classical property of Morse functions mentioned before.

From now on, unless otherwise stated, we will use some $\eta \in Z^1(M,\mathbb{R})$ and a Riemannian metric g on M satisfying (a) (Section 1.1).

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The Hopf index of $\eta^\sharp$ at any $p \in \mathcal{X}_k$ is $(-1)^k$ (Section 6.1.1). Thus, by the Hopf index theorem,

$$\sum_{k=0}^{n}(-1)^k|\mathcal{X}_k| = \chi(M). \tag{4.5}$$

4.2. The small and large spectrum

Consider the perturbed operators (2.3) defined by η and g. We can suppose the closures $\overline{U_p}$ ($p \in \mathcal{X}$) are disjoint from each other, and $x_p(U_p) = (-4r, 4r)^n$ for some $r > 0$ independent of p with $4r < 1$. Let $U = \bigcup_{p\in\mathcal{X}} U_p$.

Denoting also the coordinates of $\mathbb{R}^n$ by $(x_p^1, \dots, x_p^n)$, consider the function $h_p \in C^\infty(\mathbb{R}^n)$ defined by the center and right-hand side of Equation (4.1). Let $d'_{p,z}$, $\delta'_{p,z}$, $D'_{p,z}$ and $\Delta'_{p,z}$ ($z \in \mathbb{C}$) denote the corresponding Witten's operators on $\mathbb{R}^n$, whose restrictions to $(-4r, 4r)^n$ agree via x_p with d_z, δ_z, D_z and Δ_z on U_p.

Proposition 4.1 (See, e.g., [63, Chapters 9 and 14], [75, Sections 4.5 and 4.7]). *The following holds for $\mu \in \mathbb{R}$:*

(i) *We have*

$$\Delta'_{p,\mu} = \sum_{j=1}^{n}\left(-\left(\frac{\partial}{\partial x_p^j}\right)^2 + \mu^2 (x_p^j)^2 + \mu\epsilon_{p,j}[dx_p^j \lrcorner, dx_p^j \wedge]\right). \tag{4.6}$$

Here, $[\cdot,\cdot]$ stands for the commutator of operators. Using multi-index notation, we can write

$$[dx_p^j \lrcorner, dx_p^j \wedge] dx_p^J = \begin{cases} dx_p^J & \text{if } j \in J \\ -dx_p^J & \text{if } j \notin J. \end{cases}$$

(ii) *$\Delta'_{p,\mu}$ is a nonnegative selfadjoint operator in $L^2(\mathbb{R}^n; \Lambda)$ with a discrete spectrum, which consists of the eigenvalues*

$$\mu \sum_{j=1}^{n}(1 + 2u_j + \epsilon_{p,j} v_j), \tag{4.7}$$

where $u_j \in \mathbb{N}_0$ and $v_j = \pm 1$. For the restriction of $\Delta'_{p,\mu}$ to k-forms, the spectrum has the additional requirement that exactly k of the numbers v_j are equal to 1. In particular, 0 is an eigenvalue of $\Delta'_{p,\mu}$ with multiplicity 1 (choosing $u_j = 0$ and $v_j = -\epsilon_{p,j}$ for all j), and the nonzero eigenvalues are of order μ as $\mu \to +\infty$. $D'_{p,\mu}$ is also a self-adjoint operator in $L^2(\mathbb{R}^n; \Lambda)$ with a discrete spectrum, which consists of the positive and negative square roots of Equation (4.7).

(iii) *The kernel of $D'_{p,\mu}$ and $\Delta'_{p,\mu}$ is generated by the normalized form*

$$e'_{p,\mu} = \left(\frac{\mu}{\pi}\right)^{n/4} e^{-\mu|x_p|^2/2} dx_p^1 \wedge \cdots \wedge dx_p^{\mathrm{ind}(p)}.$$

For any $z \in \mathbb{C}$ with $\mu > 0$, let $\Delta'_{p,z} = e^{-i\nu h_p}\Delta'_{p,\mu}e^{i\nu h_p}$. Since the operator of multiplication by $e^{-i\nu h_p}$ is unitary, $\Delta'_{p,z}$ is also selfadjoint and nonnegative in $L^2(\mathbb{R}^n; \Lambda)$, it has a

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discrete spectrum with the same eigenvalues and multiplicities as $\Delta'_{p,\mu}$, and its kernel is generated by the normalized form $e'_{p,z} := e^{-i\nu h_p} e'_{p,\mu}$. We will also use the notation

$$e'_{p,z} = x_p^* e'_{p,z} \in C^\infty\big(U_p; \Lambda^{\mathrm{ind}(p)}\big).$$

The function $x_p^* h_p \in C^\infty(U_p)$ agrees with $h_{\eta,p}$, which is also denoted by h_p in this section.

Fix an even C^∞ function $\rho : \mathbb{R} \to [0,1]$ such that $\rho = 1$ on $[-r,r]$ and $\operatorname{supp}\rho \subset [-2r,2r]$. For every $p \in \mathcal{X}$, let

$$\rho_p = \rho(x_p^1) \cdots \rho(x_p^n) \in C_c^\infty(U_p)\,, \tag{4.8}$$

$$e_{p,\mu} = \frac{\rho_p}{a_\mu} e'_{p,\mu} \in C_c^\infty\big(U_p; \Lambda^{\mathrm{ind}(p)}\big)\,, \tag{4.9}$$

$$e_{p,z} = e^{-i\nu h_p} e_{p,\mu} = \frac{\rho_p}{a_\mu} e'_{p,z} \in C_c^\infty\big(U_p; \Lambda^{\mathrm{ind}(p)}\big)\,, \tag{4.10}$$

where

$$a_\mu = \left(\int_{-2r}^{2r} \rho(x)^2 e^{-\mu x^2}\, dx \right)^{\frac{n}{2}} = \left(\frac{\pi}{\mu}\right)^{\frac{n}{4}} + O(e^{-c\mu})\,, \tag{4.11}$$

as $\mu \to +\infty$. The extensions by zero of the forms $e_{p,z}$ to M are also denoted by $e_{p,z}$. They form an orthonormal basis of a graded subspace $E_z \subset \Omega(M)$ with $\dim E_z = |\mathcal{X}|$. Observe that d_z does not preserve E_z so that E_z is not a subcomplex of $(\Omega(M), d_z)$. Let P_z be the orthogonal projection of $L^2(M;\Lambda)$ to E_z.

Remark 4.2. For the sake of simplicity, most of our results are stated for $\mu \gg 0$ or as $\mu \to +\infty$, but they have obvious versions for $\mu \ll 0$ or as $\mu \to -\infty$, as follows by considering $-\eta$ and using that $\mathcal{X}_k(-\eta) = \mathcal{X}_{n-k}(\eta)$.

Proposition 4.3. *If $\mu \gg 0$ and $\beta \in H^1(M;\Lambda)$ with $\operatorname{supp}\beta \subset M \setminus U$, then*

$$\|D_z \beta\| \geq C\mu \|\beta\|.$$

Proof. This follows like [75, Proposition 4.7], using that H_η is of order zero in Equation (2.3). Actually, according to the statement of [75, Proposition 4.7], this inequality would hold with $\sqrt{\mu}$ instead of μ, but its proof clearly shows that using μ is fine. □

Proposition 4.4. *The following properties hold:*

(i) $P_z D_z P_z = 0$.

(ii) *If $\mu \gg 0$, $\alpha \in E_z$ and $\beta \in E_z^\perp \cap H^1(M;\Lambda)$, then*

$$\|P_z^\perp D_z \alpha\| \leq e^{-c\mu} \|\alpha\|\,, \quad \|P_z D_z \beta\| \leq e^{-c\mu} \|\beta\|.$$

(iii) *If $\mu \gg 0$ and $\beta \in E_z^\perp \cap H^1(M;\Lambda)$, then*

$$\|P_z^\perp D_z \beta\| \geq C\sqrt{\mu} \|\beta\|.$$

Proof. This follows like [75, Propositions 4.11, 4.12 and 5.6]. Property (i) is true because every $D_z e_{p,z}$ is supported in U_p and has homogeneous components of degree different

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from ind(p); therefore, it is orthogonal to $\ker\Delta_z$. The other properties are consequences of Propositions 4.1 and 4.3 and Equations (4.8)–(4.11). According to [75, Proposition 4.11], the inequalities of (ii) hold with $1/\mu$ instead of $e^{-c\mu}$, but its proof shows that indeed $e^{-c\mu}$ can be achieved. □

Proposition 4.5. *For all* $m\in\mathbb{N}_0$*, if* $\mu\gg 0$*, then*

$$\|D_z e_{p,z}\|_m \le |\nu|^m e^{-c_m\mu}\,, \quad \|D_z e_{p,z}\|_{m,i\nu} \le e^{-c_m\mu}.$$

Proof. From Proposition 4.1 (iii) and Equations (2.2), (4.9) and (4.10), we get

$$D_z e_{p,z} = D_z\Big(\frac{\rho_p}{a_\mu}e'_{p,z}\Big) = e^{-i\nu h_p}\frac{1}{a_\mu}\Big(\frac{\pi}{4}\Big)^{n/4}\hat{c}(d\rho_p)e'_{p,\mu}. \tag{4.12}$$

Thus, the stated estimate of $\|D_z e_{p,z}\|_m$ is true by Equations (4.9) and (4.11), since $d\rho_p = 0$ around p, and using the definition of h_p and the condition $4r < 1$. (When $\nu = 0$, this is indicated in [75, Eq. (6.17)].)

By Equation (2.11), for all $k\in\mathbb{N}_0$ and $p\in\mathcal{X}$, the form $D^k_{i\nu}D_z e_{p,z}$ is the extension by zero of the form $e^{-i\nu h_p}D^k D_\mu e_{p,\mu}$ on U_p. Then the stated estimate of $\|D_z e_{p,z}\|_{m,i\nu}$ follows from the case $\nu = 0$. □

Proposition 4.6. *If* $\mu\gg 0$*, then*

$$\|D_z e_{p,z}\|_{L^\infty} \le e^{-c\mu}.$$

Proof. Apply Equations (4.9) and (4.11) in Equation (4.12), and use that $d\rho_p = 0$ around p. □

Consider the partition of $\operatorname{spec}\Delta_z$ into its intersections with $[0,1]$ and $(1,\infty)$, called the *small* and *large spectrum*; the term *small/large eigenvalues* may be also used. Let $E_{z,\mathrm{sm}}\subset\Omega(M)$ denote the graded finite-dimensional subspace generated by the eigenforms of the small eigenvalues, let $E_{z,\mathrm{la}} = E^\perp_{z,\mathrm{sm}}$ in $L^2(M;\Lambda)$, and let $P_{z,\mathrm{sm/la}}$ be the orthogonal projection to $E_{z,\mathrm{sm/la}}$, called *small/large projection*. Moreover, $(\Omega(M),d_z)$ splits into a topological direct sum of the subcomplexes $E_{z,\mathrm{sm}}$ and $E_{z,\mathrm{la}}\cap\Omega(M)$, called the *small* and *large complexes*, and Equation (2.6) gives

$$H^\bullet(E_{z,\mathrm{sm}},d_z)\cong H^\bullet_z(M)\,, \quad H^\bullet(E_{z,\mathrm{la}}\cap\Omega(M),d_z) = 0. \tag{4.13}$$

For any operator B defined on $\Omega(M)$ or $L^2(M;\Lambda)$, let $B_{z,\mathrm{sm/la}} = BP_{z,\mathrm{sm/la}}$.

Proposition 4.7. *For all* $m\in\mathbb{N}_0$*,* $\mu\gg 0$ *and* $\alpha\in E_z$*,*

$$\|\alpha - P_{z,\mathrm{sm}}\alpha\|_{m,i\nu} \le e^{-c_m\mu}\|\alpha\|.$$

Proof. This follows like [75, Lemma 5.8 and Theorem 6.7], using $\|\ \|_{m,i\nu}$ instead of $\|\ \|_m$. The following are the main steps of the proof.

Let $\mathbb{S}^1 = \{\,\omega\in\mathbb{C}\mid|\omega| = 1\,\}$. With the argument of the proof of [75, Eq. (5.27)], using Proposition 4.4, we get that, for all $\alpha\in H^1(M;\Lambda)$, $w\in\mathbb{S}^1$ and $\mu\gg 0$,

$$\|(w - D_z)\alpha\| \ge C\|\alpha\|.$$

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Thus, $w-D_z : H^1(M;\Lambda)\to L^2(M;\Lambda)$ is bijective, and, for all $\beta\in L^2(M;\Lambda)$, $w\in\mathbb{S}^1$ and $\mu\gg 0$,

$$\left\|(w-D_z)^{-1}\beta\right\|\le C^{-1}\|\beta\|. \tag{4.14}$$

On the other hand, arguing like in the proof of [75, Eq. (6.18)], it follows that, for all $\gamma\in H^m(M;\Lambda)$, $w\in\mathbb{S}^1$ and $\mu\gg 0$,

$$\|\gamma\|_{m,i\nu}\le C_m\big(\left\|(w-D_z)\gamma\right\|_{m-1,i\nu}+\mu\|\gamma\|_{m-1,i\nu}+\|\gamma\|\big).$$

Continuing by induction on $m\in\mathbb{N}_0$, we obtain

$$\|\gamma\|_{m,i\nu}\le C_m\Big(\mu^m\|\gamma\|+\sum_{k=1}^{m}\mu^{k-1}\left\|(w-D_z)\gamma\right\|_{m-k,i\nu}\Big).$$

In other words, for all $\beta\in H^{m-1}(M;\Lambda)$,

$$\left\|(w-D_z)^{-1}\beta\right\|_{m,i\nu}\le C_m\Big(\mu^m\left\|(w-D_z)^{-1}\beta\right\|+\sum_{k=1}^{m}\mu^{k-1}\|\beta\|_{m-k,i\nu}\Big).$$

Applying Equation (4.14) to this inequality, we get, for $m\ge 1$,

$$\left\|(w-D_z)^{-1}\beta\right\|_{m,i\nu}\le C_m\mu^m\|\beta\|_{m-1,i\nu}. \tag{4.15}$$

From Equations (4.14) and (4.15) and Proposition 4.5, it follows that, for $m\in\mathbb{N}_0$,

$$\left\|(w-D_z)^{-1}D_z e_{p,z}\right\|_{m,i\nu}=O\big(e^{-c_m\mu}\big) \tag{4.16}$$

as $\mu\to+\infty$, uniformly on $w\in\mathbb{S}^1$. But, endowing $\mathbb{S}^1$ with the counterclockwise orientation, basic spectral theory gives (see, e.g., [25, Section VII.3])

$$\begin{aligned}P_{z,\mathrm{sm}}e_{p,z}-e_{p,z}&=\frac{1}{2\pi i}\int_{\mathbb{S}^1}\big((w-D_z)^{-1}-w^{-1}\big)e_{p,z}\,dw\\&=\frac{1}{2\pi i}\int_{\mathbb{S}^1}w^{-1}(w-D_z)^{-1}D_z e_{p,z}\,dw.\end{aligned} \tag{4.17}$$

The result follows using Equation (4.16) in Equation (4.17). □

Corollary 4.8. *For $\mu\gg 0$ and $\alpha\in E_z$,*

$$\|\alpha-P_{z,\mathrm{sm}}\alpha\|_{L^\infty}\le e^{-c\mu}\|\alpha\|.$$

Proof. Apply Propositions 2.2 and 4.7.

Alternatively, the proof of Proposition 4.7 can be modified as follows to get this result (some step of this alternative argument will be used later). Iterating Equation (4.15), we get

$$\left\|(w-D_z)^{-1}\beta\right\|_{m,i\nu}\le C'_m\mu^{(m+1)m/2}\|\beta\|\,,$$

for all $\beta\in L^2(M;\Lambda)$. Then, by Proposition 2.2,

$$\left\|(w-D_z)^{-1}\beta\right\|_{L^\infty}\le C\mu^{(m+1)m/2}\|\beta\|. \tag{4.18}$$

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Thus, by Proposition 4.5,

$$\left\|(w-D_z)^{-1}D_z e_{p,z}\right\|_{L^\infty} = O\big(e^{-c_m\mu}\big)$$

as $\mu\to+\infty$. Finally, apply this expression in Equation (4.17). □

Corollary 4.9. *If $\mu\gg 0$, then $P_{z,\mathrm{sm}}: E_z\to E_{z,\mathrm{sm}}$ is an isomorphism; in particular, $\dim E_{z,\mathrm{sm}} = |\mathcal{X}|$ and $\dim E^k_{z,\mathrm{sm}} = |\mathcal{X}_k|$.*

Proof. This follows from Propositions 4.4 and 4.7 for $m=0$ like [75, Proposition 5.5]. □

When $\mu\gg 0$, Equation (4.5) also follows from Corollary 4.9 and Equations (2.8) and (4.13).

Theorem 4.10 (Cf. [17, Theorem 3]). *We have*

$$\operatorname{spec}\Delta_z \subset \big[0,e^{-c|\mu|}\big]\cup\big[C|\mu|,\infty\big).$$

Proof. First, we establish the theorem for $|\mu|\gg 0$, and then the constants will be changed to cover all μ.

We can assume $\mu\ge 0$ according to Remark 4.2. By Propositions 4.4, 4.7 and 2.4, for all $\alpha\in E_z$,

$$\begin{aligned}\|D_zP_{z,\mathrm{sm}}\alpha\| &\le \|D_z\alpha\| + \|D_z(\alpha-P_{z,\mathrm{sm}}\alpha)\| \le \|D_z\alpha\| + \|\alpha-P_{z,\mathrm{sm}}\alpha\|_{1,z}\\ &\le \|P_z^\perp D_z\alpha\| + C(\mu\|\alpha-P_{z,\mathrm{sm}}\alpha\| + \|\alpha-P_{z,\mathrm{sm}}\alpha\|_{1,i\nu})\\ &\le \big(e^{-c\mu}+C\big(\mu e^{-c_0\mu}+e^{-c_1\mu}\big)\big)\|\alpha\|.\end{aligned}$$

Hence, by Corollary 4.9, for all $\beta\in E_{z,\mathrm{sm}}$,

$$0\le\langle\Delta_z\beta,\beta\rangle = \|D_z\beta\|^2 \le e^{-c\mu}\|\beta\|^2.$$

This shows that

$$\operatorname{spec}\Delta_z\cap[0,1]\subset\big[0,e^{-c\mu}\big]. \tag{4.19}$$

Now, let $\phi\in E_{z,\mathrm{la}}\cap H^1(M;\Lambda)$, and write $\alpha=P_z\phi\in E_z$ and $\beta=P_z^\perp\phi\in E_z^\perp\cap H^1(M;\Lambda)$. By Proposition 4.7,

$$\|\alpha\|^2 = \langle\alpha,\phi\rangle = \langle\alpha-P_{z,\mathrm{sm}}\alpha,\phi\rangle \le \|\alpha-P_{z,\mathrm{sm}}\alpha\|\|\phi\| \le e^{-c_0\mu}\|\alpha\|\|\phi\|,$$

yielding

$$\|\alpha\|\le e^{-c_0\mu}\|\phi\|.$$

So

$$\|\beta\| = \|\phi-\alpha\| \ge \|\phi\|-\|\alpha\| \ge \big(1-e^{-c_0\mu}\big)\|\phi\|.$$

Then, by Proposition 4.4,

$$\begin{aligned}\|D_z\phi\| &\ge \|D_z\beta\|-\|D_z\alpha\| \ge \|P_z^\perp D_z\beta\| - e^{-c\mu}\|\alpha\|\\ &\ge C\sqrt{\mu}\|\beta\| - e^{-c\mu}\|\phi\| \ge \big(C\sqrt{\mu}\big(1-e^{-c_0\mu}\big)-e^{-c\mu}\big)\|\phi\|.\end{aligned}$$

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Therefore, for all $\phi \in E_{z,\mathrm{la}} \cap H^1(M;\Lambda)$,

$$\langle \Delta_z \phi, \phi \rangle = \|D_z \phi\|^2 \geq C\mu \|\phi\|^2.$$

This proves that

$$\operatorname{spec} \Delta_z \cap (1,\infty) \subset [C\mu, \infty). \tag{4.20}$$

The inclusions (4.19) and (4.20) give the result for $\mu \gg 0$. But, in those inclusions, we can take c and C so small that, if one of them is not true for some $\mu \geq 0$, then $C\mu \leq e^{-c\mu}$. □

4.3. Ranks of some projections in the small complex

Recall that $(\Pi_z^\perp)_{\mathrm{sm},k}$, $\Pi^1_{z,\mathrm{sm},k}$ and $\Pi^2_{z,\mathrm{sm},k}$ denote the orthogonal projections to the images of $\Delta_{z,\mathrm{sm},k}$, $d_{z,\mathrm{sm},k-1}$ and $\delta_{z,\mathrm{sm},k+1}$, respectively. Let $m_{z,k}$, $m^1_{z,k}$ and $m^2_{z,k}$ be the corresponding ranks (or traces) of these projections. They satisfy

$$m_{z,k} = m^1_{z,k} + m^2_{z,k}\,, \quad m^1_{z,0} = m^2_{z,n} = 0\,, \quad m^2_{z,k} = m^1_{z,k+1}\,, \tag{4.21}$$

where the last equality is true because $d_z : \operatorname{im}\delta_z \to \operatorname{im} d_z$ is an isomorphism. For $\mu \gg 0$, we have $m_{z,k}, m^j_{z,k} \leq |\mathcal{X}_k|$ by Corollary 4.9 and Equation (4.21).

Lemma 4.11. *The numbers $m^j_{z,k}$ are determined by the numbers $m_{z,k}$:*

$$m^1_{z,k+1} = m^2_{z,k} = \sum_{p=0}^{k} (-1)^{k-p} m_{z,p} = \sum_{q=k+1}^{n} (-1)^{q-k-1} m_{z,q}.$$

Proof. This follows from Equation (4.21) with an easy induction argument on k. □

Lemma 4.12. *For $\mu \gg 0$, we have $m_{z,k} = |\mathcal{X}_k| - \beta^k_z$.*

Proof. This is a consequence of Equations (2.6) and (4.13) and Corollary 4.9. □

Corollary 4.13. $\operatorname{Str}((\Pi_z^\perp)_{\mathrm{sm}}) = 0$.

Proof. By Equations (2.8) and (4.5) and Lemma 4.12,

$$\operatorname{Str}\left((\Pi_z^\perp)_{\mathrm{sm}}\right) = \sum_k (-1)^k |\mathcal{X}_k| - \sum_k (-1)^k \beta^k_z = \chi(M) - \chi(M) = 0.$$

□

Lemma 4.14. *If M is oriented, then, for $k = 0, \ldots, n$,*

$$m_{z,k} = m_{-\bar{z},n-k} = m_{-z,n-k}\,, \quad m^1_{z,k} = m^2_{-\bar{z},n-k} = m^2_{-z,n-k}.$$

Proof. This is true because, by Equation (2.15),

$$(\Pi_z^\perp)_{\mathrm{sm},k} \star = \star (\Pi_{-\bar{z}}^\perp)_{\mathrm{sm},n-k}\,, \quad \Pi^1_{z,\mathrm{sm},k} \star = \star \Pi^2_{-\bar{z},\mathrm{sm},n-k}\,,$$
$$(\Pi_z^\perp)_{\mathrm{sm},k} \bar{\star} = \bar{\star} (\Pi_{-z}^\perp)_{\mathrm{sm},n-k}\,, \quad \Pi^1_{z,\mathrm{sm},k} \bar{\star} = \bar{\star} \Pi^2_{-z,\mathrm{sm},n-k}.$$

□

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Corollary 4.15. *For $\mu \gg 0$, $m_{z,k}$ and $m^j_{z,k}$ only depend on $|\mathcal{X}_k|$ and the class $\xi = [\eta] \in H^1(M,\mathbb{R})$.*

Proof. Apply Equation (2.9) and Lemmas 4.11 and 4.12. □

By Corollary 4.15, we write $m_k = m_k(\eta) = m_{z,k}$ and $m^j_k = m^j_k(\eta) = m^j_{z,k}$ for $\mu \gg 0$.

Corollary 4.16. *If M is oriented, then, for $k = 0,\dots,n$,*

$$m_k(\eta) = m_{n-k}(-\eta)\,, \quad m^1_k(\eta) = m^2_{n-k}(-\eta) = m^1_{n-k+1}(-\eta).$$

Proof. Apply Equation (4.21), Lemma 4.14 and Corollary 4.15. Alternatively, we can apply Equations (2.9), (2.14) and (4.21); Remark 4.2; and Lemma 4.12. □

Corollary 4.17. *For $\mu \gg 0$,*

$$\operatorname{Str}(\Pi^1_{z,\mathrm{sm}}) = -\operatorname{Str}(\Pi^2_{z,\mathrm{sm}}) = \sum_{k=0}^n (-1)^k k m_k.$$

If moreover M is oriented and n is even, then

$$\sum_{k=0}^n (-1)^k k m_k = \sum_{k=0}^n (-1)^k k|\mathcal{X}_k| - \frac{n}{2}\chi(M).$$

Proof. Corollary 4.13 gives the first equality. By Lemma 4.11 and Corollary 4.13,

$$\operatorname{Str}(\Pi^1_{z,\mathrm{sm}}) = \sum_{k=0}^n (-1)^k \sum_{q=k}^n (-1)^{q-k} m_q = \sum_{q=0}^n (-1)^q (q+1) m_q = \sum_{q=0}^n (-1)^q q m_q.$$

Now, assume M is oriented and n is even. Then, by Equations (2.8), (2.9) and (2.14),

$$\begin{aligned}\sum_{k=0}^n (-1)^k k \beta^k_{\mathrm{No}} &= \sum_{l=0}^n (-1)^{n-l}(n-l)\beta^{n-l}_{\mathrm{No}} = \sum_{l=0}^n (-1)^l (n-l)\beta^l_{\mathrm{No}}\\ &= n\chi(M) - \sum_{l=0}^n (-1)^l l \beta^l_{\mathrm{No}}.\end{aligned}$$

Hence, the last equality of the statement follows from Lemma 4.12. □

4.4. Asymptotic properties of the small projection

Notation 4.18. Consider a function $f(x) > 0$ $(x > 0)$. When referring to vectors in Banach spaces, the order notation $O(f(|\mu|))$ $(\mu \to \pm\infty)$ will be used for a family of vectors $v = v(z)$ $(z \in \mathbb{C})$ with $\|v(z)\| = O(f(|\mu|))$. This notation applies, for example, to bounded operators between Banach spaces. We may also consider this notation when the Banach spaces depend on z.

Proposition 4.19. *For every $\tau \in \mathbb{R}$, on $L^2(M;\Lambda)$, as $\mu \to +\infty$,*

$$P_{z,\mathrm{sm}} = P_z + O\big(e^{-c\mu}\big) = P_{z,\mathrm{sm}} P_{z+\tau,\mathrm{sm}} P_{z,\mathrm{sm}} + O\big(\mu^{-2}\big) = P_{z+\tau,\mathrm{sm}} + O\big(\mu^{-1}\big).$$

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Proof. By Corollary 4.9, for $\mu \gg 0$, the elements $P_{z,\mathrm{sm}} e_{p,z}$ $(p \in \mathcal{X})$ form a base of $E_{z,\mathrm{sm}}$. Applying the Gram–Schmidt process to this base, we get an orthonormal base $\tilde{e}_{p,z}$. By Proposition 4.7,

$$\tilde{e}_{p,z} = e_{p,z} + O\big(e^{-c\mu}\big). \tag{4.22}$$

This gives the first equality of the statement: for any $\alpha \in L^2(M;\Lambda)$,

$$P_z\alpha = \sum_{p\in\mathcal{X}} \langle \alpha, e_{p,z}\rangle e_{p,z} = \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z}\rangle \tilde{e}_{p,z} + O\big(e^{-c\mu}\big)\|\alpha\| = P_{z,\mathrm{sm}}\alpha + O\big(e^{-c\mu}\big)\|\alpha\|.$$

Since the sets U_p $(p \in \mathcal{X})$ are disjoint one another, for $p \neq q$ in $\mathcal{X}$,

$$\langle e_{p,z}, e_{q,z+\tau}\rangle = 0. \tag{4.23}$$

On the other hand, by Equations (4.8)–(4.11), we can also assume

$$\begin{aligned}
\langle e_{p,z}, e_{p,z+\tau}\rangle &= \langle e^{-i\nu h_p} e_{p,\mu}, e^{-i\nu h_p} e_{p,\mu+\tau}\rangle = \langle e_{p,\mu}, e_{p,\mu+\tau}\rangle \\
&= \frac{(\mu(\mu+\tau))^{n/4}}{\pi^{n/2}} \big\langle \rho_p e^{-\mu|x_p|^2/2}, \rho_p e^{-(\mu+\tau)|x_p|^2/2}\big\rangle + O\big(e^{-c\mu}\big) \\
&= \frac{(\mu(\mu+\tau))^{n/4}}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-(\mu+\tau/2)|x_p|^2}\, dx_p + O\big(e^{-c\mu}\big) \\
&= \frac{(\mu(\mu+\tau))^{n/4}}{(\mu+\tau/2)^{n/2}} + O\big(e^{-c\mu}\big) = 1 + O\big(\mu^{-2}\big),
\end{aligned} \tag{4.24}$$

where $dx_p = dx_p^1 \dots dx_p^n = \mathrm{dvol}(x_p)$. Combining Equation (4.22) for z and $z+\tau$ with Equations (4.23) and (4.24), we obtain

$$\begin{aligned}
P_{z+\tau,\mathrm{sm}}\tilde{e}_{p,z} &= \sum_{q\in\mathcal{X}} \langle \tilde{e}_{p,z}, \tilde{e}_{q,z+\tau}\rangle \tilde{e}_{q,z+\tau} = \sum_{q\in\mathcal{X}} \langle e_{p,z}, e_{q,z+\tau}\rangle e_{q,z+\tau} + O\big(e^{-c\mu}\big) \\
&= e_{p,z+\tau} + O\big(\mu^{-2}\big) = \tilde{e}_{p,z+\tau} + O\big(\mu^{-2}\big).
\end{aligned} \tag{4.25}$$

Repeating Equation (4.25) interchanging the roles of z and $z+\tau$, we get

$$P_{z,\mathrm{sm}} P_{z+\tau,\mathrm{sm}} \tilde{e}_{p,z} = P_{z,\mathrm{sm}} \tilde{e}_{p,z+\tau} + O\big(\mu^{-2}\big) = \tilde{e}_{p,z} + O\big(\mu^{-2}\big).$$

This gives the second equality of the statement: For any $\alpha \in L^2(M;\Lambda)$,

$$\begin{aligned}
P_{z,\mathrm{sm}}\alpha = \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z}\rangle \tilde{e}_{p,z} &= P_{z,\mathrm{sm}} P_{z+\tau,\mathrm{sm}} \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z}\rangle \tilde{e}_{p,z} + O\big(\mu^{-2}\big)\|\alpha\| \\
&= P_{z,\mathrm{sm}} P_{z+\tau,\mathrm{sm}} P_{z,\mathrm{sm}}\alpha + O\big(\mu^{-2}\big)\|\alpha\|.
\end{aligned}$$

By Equation (4.25),

$$\begin{aligned}
\big\|\tilde{e}_{p,z} - \tilde{e}_{p,z+\tau}\big\|^2 &= \|\tilde{e}_{p,z}\|^2 - 2\Re\langle \tilde{e}_{p,z}, \tilde{e}_{p,z+\tau}\rangle + \|\tilde{e}_{p,z+\tau}\|^2 = 2 - 2\Re\langle P_{z+\tau,\mathrm{sm}}\tilde{e}_{p,z}, \tilde{e}_{p,z+\tau}\rangle \\
&= 2 - 2\Re\langle \tilde{e}_{p,z+\tau}, \tilde{e}_{p,z+\tau}\rangle + O\big(\mu^{-2}\big) = O\big(\mu^{-2}\big),
\end{aligned}$$

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which means

$$\tilde{e}_{p,z} = \tilde{e}_{p,z+\tau} + O\big(\mu^{-1}\big). \tag{4.26}$$

The last stated equality follows from Equations (4.25) and (4.26): For any $\alpha \in L^2(M;\Lambda)$,

$$\begin{aligned} P_{z,\mathrm{sm}}\alpha &= \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z}\rangle \tilde{e}_{p,z} = \sum_{p\in\mathcal{X}} \langle \alpha, \tilde{e}_{p,z+\tau}\rangle \tilde{e}_{p,z+\tau} + O\big(\mu^{-1}\big)\alpha \\ &= P_{z+\tau,\mathrm{sm}}\alpha + O\big(\mu^{-1}\big)\alpha. \end{aligned}$$

□

Corollary 4.20. *For every $\tau \in \mathbb{R}$, on $L^2(M;\Lambda)$,*

$$d_{z+\tau,\mathrm{sm}} - d_{z+\tau}P_{z,\mathrm{sm}} = O\big(\mu^{-1}\big) \quad (\mu \to +\infty).$$

Proof. Since $d_{z+\tau} = d_z + \tau\eta\wedge$, it follows from Theorem 4.10 that $d_{z+\tau}$ is bounded on $E_{z,\mathrm{sm}} + E_{z+\tau,\mathrm{sm}}$, uniformly on $\mu \gg 0$. Hence, by Proposition 4.19,

$$d_{z+\tau,\mathrm{sm}} - d_{z+\tau}P_{z,\mathrm{sm}} = d_{z+\tau}(P_{z+\tau,\mathrm{sm}} - P_{z,\mathrm{sm}}) = O\big(\mu^{-1}\big).$$

□

Proposition 4.21. *On $L^2(M;\Lambda)$,*

$$P_{z,\mathrm{sm}}\,\eta\wedge, \eta\wedge P_{z,\mathrm{sm}} = O\big(\mu^{-1}\big) \quad (\mu \to +\infty).$$

Proof. By Theorem 4.10, for all $\alpha \in \Omega(M)$,

$$\|d_z P_{z,\mathrm{sm}}\alpha\|^2 = \langle \delta_z d_z P_{z,\mathrm{sm}}\alpha, P_{z,\mathrm{sm}}\alpha\rangle \le \langle \Delta_z P_{z,\mathrm{sm}}\alpha, P_{z,\mathrm{sm}}\alpha\rangle \le O\big(e^{-c\mu}\big),$$

yielding $d_z P_{z,\mathrm{sm}} = O\big(e^{-c\mu}\big)$. This is also true with the parameter $z+1$. So, by Corollary 4.20,

$$\eta\wedge P_{z,\mathrm{sm}} = (d_{z+1} - d_z)P_{z,\mathrm{sm}} = d_{z+1}P_{z+1,\mathrm{sm}} - d_z P_{z,\mathrm{sm}} + O\big(\mu^{-1}\big) = O\big(\mu^{-1}\big).$$

□

4.5. Derivatives of the small projection

Remark 4.22. For reasons of brevity, most of the results about derivatives are stated for ∂_z, which may be simply denoted with a dot. But there are obvious versions of those results for $\partial_{\bar{z}}$ with analogous proofs.

Proposition 4.23. *We have*

$$\operatorname{rank}\partial_z P_{z,\mathrm{sm}} \le 2|\mathcal{X}| \quad (\mu \gg 0), \quad \partial_z P_{z,\mathrm{sm}} = O\big(\mu^{-1}\big) \quad (\mu \to +\infty).$$

Proof. By Equation (2.4) and Theorem 4.10, for $\mu \gg 0$ and every $\omega \in \mathbb{S}^1$, a standard computation gives

$$\partial_z\big((w - D_z)^{-1}\big) = (w - D_z)^{-1}\,\eta\wedge(w - D_z)^{-1}. \tag{4.27}$$

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Then, by Equation (4.14), $\partial_z\big((w-D_z)^{-1}\big)$ defines an operator on $L^2(M;\Lambda)$, bounded uniformly on $w\in\mathbb{S}^1$ and $z\in\mathbb{C}$. By Equation (4.14) and Proposition 4.21, we also get

$$\begin{aligned}&P_{z,\mathrm{la/sm}}\partial_z\big((w-D_z)^{-1}\big)P_{z,\mathrm{sm/la}}\\&\quad=(w-D_z)^{-1}P_{z,\mathrm{la/sm}}\,\eta\wedge P_{z,\mathrm{sm/la}}(w-D_z)^{-1}=O\big(\mu^{-1}\big)\,,\end{aligned}$$

uniformly on $w\in\mathbb{S}^1$.

On the other hand, applying again basic spectral theory, we obtain

$$P_{z,\mathrm{sm}}=\frac{1}{2\pi i}\int_{\mathbb{S}^1}(w-D_z)^{-1}\,dw$$

for $\mu\gg 0$, yielding

$$\dot P_{z,\mathrm{sm}}=\frac{1}{2\pi i}\int_{\mathbb{S}^1}\partial_z\big((w-D_z)^{-1}\big)\,dw\,,\tag{4.28}$$

which defines an operator on $L^2(M;\Lambda)$, bounded uniformly on z.

Using that $P_{z,\mathrm{sm}}$ is an orthogonal projection, the argument of the proof of [7, Proposition 9.37] shows that

$$\dot P_{z,\mathrm{sm}}=P_{z,\mathrm{la}}\dot P_{z,\mathrm{sm}}P_{z,\mathrm{sm}}+P_{z,\mathrm{sm}}\dot P_{z,\mathrm{sm}}P_{z,\mathrm{la}}.\tag{4.29}$$

So $\operatorname{rank}\dot P_{z,\mathrm{sm}}\le 2\operatorname{rank}P_{z,\mathrm{sm}}\le 2|\mathcal{X}|$ by Corollary 4.9, and

$$\begin{aligned}\dot P_{z,\mathrm{sm}}&=\frac{1}{2\pi i}\int_{\mathbb{S}^1}P_{z,\mathrm{la}}\partial_z\big((w-D_z)^{-1}\big)P_{z,\mathrm{sm}}\,dw\\&\quad+\frac{1}{2\pi i}\int_{\mathbb{S}^1}P_{z,\mathrm{sm}}\partial_z\big((w-D_z)^{-1}\big)P_{z,\mathrm{la}}\,dw=O\big(\mu^{-1}\big).\end{aligned}$$

□

Lemma 4.24. *For all $p\in\mathcal{X}$,*

$$\partial_z e_{p,z}=\left(\frac{n}{8\mu}-\frac{|x_p^+|^2}{2}+O(e^{-c\mu})\right)e_{p,z}\quad(\mu\to+\infty).$$

Proof. Using integration by parts, and since ρ is an even function and ρ' vanishes on $[-r,r]$, we obtain

$$\begin{aligned}\int_{-2r}^{2r}\rho(x)^2x^2e^{-\mu x^2}\,dx&=\frac{1}{2\mu}\int_{-2r}^{2r}(2\rho(x)\rho'(x)x+\rho(x)^2)e^{-\mu x^2}\,dx\\&=\frac{1}{2\mu}\left(\frac{\pi}{\mu}\right)^{\frac12}+O(e^{-c\mu}).\end{aligned}\tag{4.30}$$

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So

$$\begin{aligned}\partial_\mu a_\mu &= \partial_\mu\left(\left(\int_{-2r}^{2r}\rho(x)^2e^{-\mu x^2}\,dx\right)^{\frac{n}{2}}\right)\\ &= -\frac{n}{2}\left(\int_{-2r}^{2r}\rho(x)^2e^{-\mu x^2}\,dx\right)^{\frac{n}{2}-1}\int_{-2r}^{2r}\rho(x)^2x^2e^{-\mu x^2}\,dx\\ &= -\frac{n}{2}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}-\frac{1}{2}}\frac{1}{2\mu}\left(\frac{\pi}{\mu}\right)^{\frac{1}{2}}+O(e^{-c\mu}) = -\frac{n}{4\mu}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}}+O(e^{-c\mu}).\end{aligned}$$

Hence, by Equation (4.11),

$$\partial_\mu\left(\frac{1}{a_\mu}\right) = -\frac{\partial_\mu a_\mu}{a_\mu^2} = \frac{n}{4\mu}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}}\left(\frac{\mu}{\pi}\right)^{\frac{n}{2}}+O(e^{-c\mu}) = \frac{n}{4\mu}\left(\frac{\mu}{\pi}\right)^{\frac{n}{4}}+O(e^{-c\mu}). \tag{4.31}$$

It also follows from Proposition 4.1 (iii) and Equations (4.9), (4.11) and (4.31) that

$$\begin{aligned}\partial_\mu e_{p,\mu} &= \partial_\mu\left(\frac{\rho_p}{a_\mu}e^{-\mu|x_p|^2/2}\,dx_p^1\wedge\cdots\wedge dx_p^{\operatorname{ind}(p)}\right)\\ &= \left(\partial_\mu\left(\frac{1}{a_\mu}\right)a_\mu-\frac{|x_p|^2}{2}\right)e_{p,\mu} = \left(\frac{n}{4\mu}-\frac{|x_p|^2}{2}+O(e^{-c\mu})\right)e_{p,\mu}.\end{aligned} \tag{4.32}$$

So, by Equation (4.10),

$$\partial_\mu e_{p,z} = \left(\frac{n}{4\mu}-\frac{|x_p|^2}{2}+O(e^{-c\mu})\right)e_{p,z}\,,\quad \partial_\nu e_{p,z} = -ih_pe_{p,z}. \tag{4.33}$$

Then the result follows using the right-hand side of Equation (4.1). □

Proposition 4.25. *For all $p\in\mathcal{X}$,*

$$\|\partial_z(D_ze_{p,z})\|_{L^\infty} = O(e^{-c\mu})\quad(\mu\to+\infty).$$

Proof. From Equation (4.12), we get

$$\begin{aligned}\partial_z(D_ze_{p,z}) = \frac{1}{2}\Bigg(&e^{-i\nu h_p}\partial_\mu\left(\frac{1}{a_\mu}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}}\right)\hat{c}(d\rho_p)e_{p,\mu}\\ &+e^{-i\nu h_p}\frac{1}{a_\mu}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}}\hat{c}(d\rho_p)\partial_\mu e_{p,\mu}-h_pe^{-i\nu h_p}\frac{1}{a_\mu}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}}\hat{c}(d\rho_p)e_{p,\mu}\Bigg).\end{aligned} \tag{4.34}$$

By Equations (4.11) and (4.31),

$$\begin{aligned}\partial_\mu\left(\frac{1}{a_\mu}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}}\right) &= \partial_\mu\left(\frac{1}{a_\mu}\right)\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}}-\frac{n\pi}{4a_\mu\mu^2}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}-1}\\ &= \frac{n}{4\mu}\left(\frac{\mu}{\pi}\right)^{\frac{n}{4}}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}}-\frac{n\pi}{4\mu^2}\left(\frac{\pi}{\mu}\right)^{\frac{n}{4}-1}\left(\frac{\mu}{\pi}\right)^{\frac{n}{4}}+O(e^{-c\mu}) = O(e^{-c\mu}).\end{aligned} \tag{4.35}$$

The result follows applying Proposition 4.1 (iii) and Equations (4.9), (4.11), (4.32) and (4.35) to Equation (4.34), and using that $d\rho_p = 0$ around p. □

Proposition 4.26. *For every $p\in\mathcal{X}$,*

$$\|\partial_z(P_{z,\mathrm{sm}}e_{p,z}-e_{p,z})\|_{L^\infty} = O(e^{-c\mu})\quad(\mu\to+\infty).$$

Proof. By Equation (4.17),

$$\partial_z(P_{z,\mathrm{sm}}e_{p,z}-e_{p,z})=\frac{1}{2\pi i}\int_{\mathbb{S}^1}w^{-1}\partial_z\big((w-D_z)^{-1}\big)D_ze_{p,z}\,dw+\frac{1}{2\pi i}\int_{\mathbb{S}^1}w^{-1}(w-D_z)^{-1}\partial_z(D_ze_{p,z})\,dw.$$

Now, apply Equations (4.18) and (4.27) and Propositions 4.6 and 4.25. □

5. Small and large zeta invariants of Morse forms

5.1. Small and large zeta invariants

According to Sections 3.2 and 4.2, if B is an operator in $L^2(M;\Lambda)$ so that $\zeta(s,\Delta_z,B)$ is defined, we have

$$\zeta(s,\Delta_z,B)=\zeta_{\mathrm{sm}}(s,\Delta_z,B)+\zeta_{\mathrm{la}}(s,\Delta_z,B)\,,$$

where

$$\zeta_{\mathrm{sm/la}}(s,\Delta_z,B)=\zeta(s,\Delta_z,B_{z,\mathrm{sm/la}}).$$

These are the contributions from the small/large spectrum to $\zeta(s,\Delta_z,B)$, which are called the *small/large zeta functions* of (Δ_z,B). In particular, we can write

$$\zeta(s,z)=\zeta_{\mathrm{sm}}(s,z)+\zeta_{\mathrm{la}}(s,z)\,,$$

where $\zeta_{\mathrm{sm/la}}(s,z)=\zeta_{\mathrm{sm/la}}(s,z,\eta)$ is the small/large zeta function of $(\Delta_z,\eta\wedge D_z\mathsf{w})$. Since $\zeta_{\mathrm{sm}}(s,z)$ is an entire function, $\zeta_{\mathrm{la}}(s,z)$ has the same poles as $\zeta(s,z)$ (Remark 3.10), with the same residues. The value $\zeta_{\mathrm{sm/la}}(1,z)$ will be called the *small/large zeta invariant* of (M,g,η,z). The following results follow like Corollaries 3.9 and 3.11.

Corollary 5.1. *If* $\Re s>1/2$, *then*

$$\zeta_{\mathrm{la}}(s,z)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\,\mathrm{Str}\left(\eta\wedge D_ze^{-t\Delta_z}P_{z,\mathrm{la}}\right)dt\,,$$

where the integral is absolutely convergent.

Corollary 5.2. *We have*

$$\zeta_{\mathrm{sm}}(1,z)=\mathrm{Str}(\eta\wedge D_z^{-1}(\Pi_z^{\perp})_{\mathrm{sm}})\,,$$
$$\zeta_{\mathrm{la}}(1,z)=\lim_{t\downarrow 0}\mathrm{Str}\left(\eta\wedge D_z^{-1}e^{-t\Delta_z}P_{z,\mathrm{la}}\right).$$

5.2. Truncated heat invariants of perturbed operators

For $k=0,\dots,n$, let $K'_{z,k,t}(x,y)$ and $\widetilde{K}_{z,k,t}(x,y)$ denote the Schwartz kernels of $e^{-t\Delta_{z,k}}\Pi_z^{\perp}$ and $e^{-t\Delta_{z,k}}P_{z,\mathrm{la},k}$, respectively. According to Section 3.1, their restrictions to the diagonal have asymptotic expansions (as $t\downarrow 0$),

$$K'_{z,k,t}(x,x)\sim\sum_{l=0}^{\infty}e'_{k,l}(x,z)t^{(l-n)/2}\,,\quad \widetilde{K}_{z,k,t}(x,x)\sim\sum_{l=0}^{\infty}\tilde{e}_{k,l}(x,z)t^{(l-n)/2}. \tag{5.1}$$

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We have

$$e'_{k,l}(x,z) = \begin{cases} e_{k,l}(x,z) & \text{if } l < n \\ e_{k,n}(x,z) - \beta_z^k & \text{if } l = n\,, \end{cases}$$

$$\tilde{e}_{k,l}(x,z) = \begin{cases} e_{k,l}(x,z) & \text{if } l < n \\ e_{k,n}(x,z) - H_{z,k,0}(x,x) & \text{if } l = n\,, \end{cases} \tag{5.2}$$

where $H_{z,k,t}(x,y)$ is the Schwartz kernel of $e^{-t\Delta_{z,k}}P_{z,\mathrm{sm},k}$, which is defined for all $t\in\mathbb{R}$ and is smooth. We also have asymptotic expansions

$$h'_k(t,z) := \operatorname{Tr}\left(e^{-t\Delta_{z,k}}\Pi_z^{\perp}\right) \sim \sum_{l=0}^{\infty} a'_{k,l}(z)t^{(l-n)/2}, \tag{5.3}$$

$$\tilde{h}_k(t,z) := \operatorname{Tr}\left(e^{-t\Delta_{z,k}}P_{z,\mathrm{la},k}\right) \sim \sum_{l=0}^{\infty} \tilde{a}_{k,l}(z)t^{(l-n)/2}. \tag{5.4}$$

By Equations (3.4), (3.5) and (3.9),

$$a'_{k,l}(z) = \int_M \operatorname{str} e'_{k,l}(x,z)\,\operatorname{dvol}(x) = \begin{cases} a_{k,l}(z) & \text{if } l < n \\ a_{k,l}(z) - \beta_z^k & \text{if } l = n. \end{cases} \tag{5.5}$$

$$\tilde{a}_{k,l}(z) = \int_M \operatorname{str} \tilde{e}_{k,l}(x,z)\,\operatorname{dvol}(x) = \begin{cases} a_{k,l}(z) & \text{if } l < n \\ a_{k,l}(z) - \dim E_{z,\mathrm{sm}}^k & \text{if } l = n. \end{cases} \tag{5.6}$$

The operators $e^{-t\Delta_z}\Pi_z^{\perp}\mathsf{w}$ and $e^{-t\Delta_z}P_{z,\mathrm{la}}\mathsf{w}$ have Schwartz kernels

$$K'_{z,t}(x,y) = \sum_{k=0}^{n}(-1)^k K'_{z,k,t}(x,y)\,, \quad \widetilde{K}_{z,t}(x,y) = \sum_{k=0}^{n}(-1)^k \widetilde{K}_{z,k,t}(x,y)\,,$$

with induced asymptotic expansions

$$K'_{z,t}(x,x) \sim \sum_{l=0}^{\infty} e'_l(x,z)t^{(l-n)/2}\,, \quad \widetilde{K}_{z,t}(x,x) \sim \sum_{l=0}^{\infty} \tilde{e}_l(x,z)t^{(l-n)/2}\,,$$

where

$$e'_l(x,z) = \sum_{k=0}^{n}(-1)^k e'_{k,l}(x,z)\,, \quad \tilde{e}_l(x,z) = \sum_{k=0}^{n}(-1)^k \tilde{e}_{k,l}(x,z).$$

We also have induced asymptotic expansions,

$$h'(t,z) := \operatorname{Str}\left(e^{-t\Delta_z}\Pi_z^{\perp}\right) \sim \sum_{l=0}^{\infty} a'_l(z)t^{(l-n)/2}\,,$$

$$\tilde{h}(t,z) := \operatorname{Str}\left(e^{-t\Delta_z}P_{z,\mathrm{la}}\right) \sim \sum_{l=0}^{\infty} \tilde{a}_l(z)t^{(l-n)/2}\,,$$

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where

$$a'_l(z) = \sum_{k=0}^{n} (-1)^k a'_{k,l}(z)\,, \quad \tilde{a}_l(z) = \sum_{k=0}^{n} (-1)^k \tilde{a}_{k,l}(z).$$

If $\mu \gg 0$, by Equation (2.9), Corollary 4.9 and Theorem 4.10, $e'_{k,l}(x,z)$ and $\tilde{e}_{k,l}(x,z)$ depend smoothly on z (Section 3.1), and therefore so do $h'_k(t,z)$, $\tilde{h}_k(t,z)$, $a'_{k,l}(z)$, $\tilde{a}_{k,l}(z)$, $e'_l(x,z)$, $\tilde{e}_l(x,z)$, $h'(t,z)$, $\tilde{h}(t,z)$, $a'_l(z)$ and $\tilde{a}_l(z)$.

5.3. Truncated derived heat invariants of perturbed operators

For $k = 0,\dots,n$ and $j = 1,2$, let

$$h^j_k(t,z) = \operatorname{Tr}\left(e^{-t\Delta_{z,k}}\Pi^j_{z,k}\right)\,, \quad \tilde{h}^j_k(t,z) = \operatorname{Tr}\left(e^{-t\Delta_{z,k}}\Pi^j_{z,\mathrm{la},k}\right).$$

Lemma 5.3. *We have*

$$h^1_{k+1}(t,z) = h^2_k(t,z) = \sum_{p=0}^{k} (-1)^{k-p} h'_p(t,z) = \sum_{q=k+1}^{n} (-1)^{q-k-1} h'_q(t,z).$$

Proof. This follows by induction on k, using that

$$h^1_0(t,z) = h^2_n(t,z) = 0\,, \quad h'_k(t,z) = h^1_k(t,z) + h^2_k(t,z)\,, \quad h^2_k(t,z) = h^1_{k+1}(t,z).$$

The last equality holds because the diagram of Equation (2.7) is commutative. □

Let

$$h^j(t,z) = \operatorname{Str}\left(e^{-t\Delta_z}\Pi^j_z\right) = \sum_{k=0}^{n} (-1)^k h^j_k(t,z)\,,$$
$$\tilde{h}^j(t,z) = \operatorname{Str}\left(e^{-t\Delta_z}\Pi^j_{z,\mathrm{la}}\right) = \sum_{k=0}^{n} (-1)^k \tilde{h}^j_k(t,z).$$

Thus,

$$h'(t,z) = h^1(t,z) + h^2(t,z)\,, \quad \tilde{h}(t,z) = \tilde{h}^1(t,z) + \tilde{h}^2(t,z). \tag{5.7}$$

Corollary 5.4. *We have $h'(t,z) = 0$.*

Proof. This is a direct consequence of Lemma 5.3 and Equation (5.7). □

Corollary 5.5. *We have*

$$h^1(t,z) = -h^2(t,z) = \sum_{k=0}^{n} (-1)^k k h'_k(t,z) = \operatorname{Str}\left(\mathsf{N} e^{-t\Delta_z}\Pi^\perp_z\right).$$

Proof. Corollary 5.4 and Equation (5.7) give the first equality. By Lemma 5.3 and Corollary 5.4,

$$\begin{aligned} h^1(t,z) &= \sum_{k=0}^{n}(-1)^k \sum_{q=k}^{n}(-1)^{q-k} h'_q(t,z) = \sum_{q=0}^{n}(-1)^q (q+1) h'_q(t,z) \\ &= h'(t,z) + \sum_{q=0}^{n}(-1)^q q h'_q(t,z) = \sum_{q=0}^{n}(-1)^q q h'_q(t,z). \end{aligned}$$

□

Remark 5.6. Note the similarity between Corollaries 4.17 and 5.5.

Applying Equation (5.3) and Lemma 5.3, we get

$$h^j_k(t,z) \sim \sum_{l=0}^{\infty} a^j_{k,l}(z) t^{(l-n)/2}\,, \quad h^j(t,z) \sim \sum_{l=0}^{\infty} a^j_l(z) t^{(l-n)/2}\,, \tag{5.8}$$

where

$$\begin{aligned} a^1_{k+1,l}(z) = a^2_{k,l}(z) &= \sum_{p=0}^{k}(-1)^{k-p} a'_{p,l}(t,z) = \sum_{q=k+1}^{n}(-1)^{q-k-1} a'_{q,l}(t,z)\,, \\ a^1_l(z) &= -a^2_l(z) = \sum_{k=0}^{n}(-1)^k k a'_{k,l}(z). \end{aligned}$$

Lemma 5.3, Corollary 5.4 and Equation (5.8) have obvious versions for $\tilde{h}^j_k(t,z)$ and $\tilde{h}^j(t,z)$, with similar proofs. The coefficients of the corresponding asymptotic expansions are denoted by $\tilde{a}^j_{k,l}(z)$ and $\tilde{a}^j_l(z)$.

Corollary 5.7. *For all $l \le n$ and $\mu \gg 0$, $a^1_l(z)$ and $\tilde{a}^1_l(z)$ are independent of z.*

Proof. Apply Equations (2.9), (5.5) and (5.6); Corollary 4.9; and Theorems 3.4 and 4.10. □

5.4. Zeta function versus theta function

Consider also the meromorphic function

$$\theta(s,z) = \theta(s,z,\eta) = -\zeta(s,\Delta_z,\mathsf{Nw})\,, \tag{5.9}$$

called theta function of Δ_z, and write

$$\theta(s,z) = \theta_{\mathrm{sm}}(s,z) + \theta_{\mathrm{la}}(s,z)\,,$$

where

$$\theta_{\mathrm{sm/la}}(s,z) = \theta_{\mathrm{sm/la}}(s,z,\eta) = -\zeta_{\mathrm{sm/la}}(s,\Delta_z,\mathsf{Nw}). \tag{5.10}$$

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By Corollary 5.5,

$$-\zeta(s,\Delta_z,\Pi_z^1\mathsf{w}) = \zeta(s,\Delta_z,\Pi_z^2\mathsf{w}) = \theta(s,z)\,,$$
$$-\zeta_{\mathrm{sm/la}}(s,\Delta_z,\Pi_z^1\mathsf{w}) = \zeta_{\mathrm{sm/la}}(s,\Delta_z,\Pi_z^2\mathsf{w}) = \theta_{\mathrm{sm/la}}(s,z). \tag{5.11}$$

Recall that $\zeta(s,z)$ is smooth at $s=1$ (Corollary 3.9). Moreover, $\theta(s,z)$ is smooth at $s=0$ [66]. The same is true for $\zeta_{\mathrm{la}}(s,z)$ and $\theta_{\mathrm{la}}(s,z)$.

Proposition 5.8. *If $\mu \gg 0$, then*

$$\partial_z\theta_{\mathrm{la}}(s,z) = s\zeta_{\mathrm{la}}(s+1,z).$$

Proof. Recall that a dot may be used to denote ∂_z. Like in Equation (4.29),

$$\dot\Pi_z^1 = (\Pi_z^1)^\perp\dot\Pi_z^1\Pi_z^1 + \Pi_z^1\dot\Pi_z^1(\Pi_z^1)^\perp.$$

Therefore, since Π_z^1 and $(\Pi_z^1)^\perp$ commute with Δ_z^{-s} and $P_{z,\mathrm{la}}$, for $\Re s \gg 0$,

$$\zeta_{\mathrm{la}}(s,\Delta_z,\dot\Pi_z^1\mathsf{w}) = \operatorname{Str}\left(\dot\Pi_z^1\Delta_z^{-s}P_{z,\mathrm{la}}\right) = 0\,,$$

yielding $\zeta_{\mathrm{la}}(s,\Delta_z,\dot\Pi_z^1\mathsf{w}) = 0$ for all s because this is a meromorphic function. Hence, since Δ_z and $\Pi_{z,\mathrm{la}}^1\mathsf{w}$ commute, Proposition 3.1 (i),(v) gives

$$\partial_z\zeta_{\mathrm{la}}(s,\Delta_z,\Pi_z^1\mathsf{w}) = -s\zeta_{\mathrm{la}}(s+1,\Delta_z,\dot\Delta_z\Pi_z^1\mathsf{w}) = -s\operatorname{Str}\left(\dot\Delta_z\Delta_z^{-s-1}\Pi_{z,\mathrm{la}}^1\right). \tag{5.12}$$

Next, by Equation (2.4),

$$\dot\Delta_z\Pi_{z,\mathrm{la}}^1 = (\eta\wedge\delta_z + \delta_z\,\eta\wedge)\Pi_{z,\mathrm{la}}^1 = \eta\wedge\delta_z\Pi_{z,\mathrm{la}}^1 + \delta_z\,\eta\wedge\Pi_{z,\mathrm{la}}^1. \tag{5.13}$$

But, since $\Pi_z^1\delta_z = 0$,

$$\operatorname{Str}\left(\delta_z\,\eta\wedge\Delta_z^{-s-1}\Pi_{z,\mathrm{la}}^1\right) = -\operatorname{Str}\left(\eta\wedge\Delta_z^{-s-1}\Pi_{z,\mathrm{la}}^1\delta_z\right) = 0. \tag{5.14}$$

From Equations (5.11)–(5.14) and Proposition 3.1 (i), we get

$$\begin{aligned}\partial_z\theta_{\mathrm{la}}(s,z) &= -\partial_z\zeta_{\mathrm{la}}(s,\Delta_z,\Pi_z^1\mathsf{w}) = s\operatorname{Str}\left(\eta\wedge\delta_z\Delta_z^{-s-1}\Pi_{z,\mathrm{la}}^1\right)\\ &= s\operatorname{Str}\left(\eta\wedge D_z\Delta_z^{-s-1}\Pi_{z,\mathrm{la}}^1\right) = s\zeta_{\mathrm{la}}(s+1,z).\end{aligned}$$

□

Remark 5.9. In the case where η is a Morse form and $\mu \gg 0$, the regularity of $\zeta(s,z)$ indicated in Remark 3.10 also follows from Corollary 5.7 and Proposition 5.8.

Corollary 5.10. *If $\mu \gg 0$, then Equation (1.6) is true.*

Proof. Apply Proposition 5.8 and Corollary 5.1. □

5.5. The case of the differential of a Morse function

Let us consider the special case where $\eta = dh$ for a Morse function h. The following four results follow like Lemma 3.12 and Corollaries 3.13 to 3.15.

Lemma 5.11. *For $\mu \gg 0$,*

$$\operatorname{Str}\left(\eta \wedge d_z^{-1} \Pi_{z,\mathrm{sm}}^1\right) = -\operatorname{Str}\left(h\,(\Pi_z^{\perp})_{\mathrm{sm}}\right),$$
$$\operatorname{Str}\left(\eta \wedge d_z^{-1} e^{-t\Delta_z} \Pi_{z,\mathrm{la}}^1\right) = -\operatorname{Str}\left(h\,e^{-t\Delta_z} P_{z,\mathrm{la}}\right).$$

Corollary 5.12. *For $\mu \gg 0$,*

$$\zeta_{\mathrm{sm}}(1,z) = -\operatorname{Str}\left(h\,(\Pi_z^{\perp})_{\mathrm{sm}}\right),$$
$$\zeta_{\mathrm{la}}(1,z) = -\lim_{t\downarrow 0} \operatorname{Str}\left(h\,e^{-t\Delta_z} P_{z,\mathrm{la}}\right).$$

Corollary 5.13. *If $\mu \gg 0$, then $\zeta_{\mathrm{sm/la}}(1,z) \in \mathbb{R}$.*

Corollary 5.14. *If M is oriented and $|\mu| \gg 0$, then*

$$\zeta_{\mathrm{sm/la}}(1,z) = \zeta_{\mathrm{sm/la}}(1,-\bar{z}) = \zeta_{\mathrm{sm/la}}(1,-z) = \zeta_{\mathrm{sm/la}}(1,\bar{z}).$$

Corollary 5.15. *The value $\zeta_{\mathrm{sm}}(1,z)$ is uniformly bounded on z for $\mu \gg 0$.*

Proof. The operator $h\,(\Pi_z^{\perp})_{\mathrm{sm}}$ is uniformly bounded and, for $\mu \gg 0$, has uniformly bounded rank. So $\operatorname{Str}(h\,(\Pi_z^{\perp})_{\mathrm{sm}})$ is uniformly bounded on z for $\mu \gg 0$, and therefore the result follows from Corollary 5.12. □

Theorem 5.16. *The following limit holds uniformly on ν:*

$$\lim_{\mu\to+\infty} \zeta_{\mathrm{la}}(1,z) = -\int_M h\,e(M,\nabla^M)\,\mathrm{dvol} + \sum_{p\in\mathcal{X}}^{n} (-1)^{\mathrm{ind}(p)} h(p).$$

Proof. By Equations (5.1) and (5.2), Theorem 3.2 and Corollary 5.12, for $\mu \gg 0$,

$$\begin{aligned}
\zeta_{\mathrm{la}}(1,z) &= -\lim_{t\downarrow 0} \operatorname{Str}\left(h\,e^{-t\Delta_z} P_{z,\mathrm{la}}\right) = -\int_M h(x) \operatorname{str} \tilde{e}_n(x,z)\,\mathrm{dvol}(x) \\
&= -\int_M h(x) \operatorname{str} e_n(x,z)\,\mathrm{dvol}(x) + \operatorname{Str}(hP_{z,\mathrm{sm}}) \\
&= -\int_M h\,e(M,\nabla^M)\,\mathrm{dvol} + \operatorname{Str}(hP_{z,\mathrm{sm}}).
\end{aligned}$$

According to Corollary 4.9, the elements $P_{z,\mathrm{sm}} e_{p,z}$ $(p \in \mathcal{X})$ form a base of $E_{z,\mathrm{sm}}^k$ when $\mu \gg 0$. Applying the Gram–Schmidt process to this base, we get an orthonormal frame $\tilde{e}_{p,z}$ $(p \in \mathcal{X})$ of $E_{z,\mathrm{sm}}$. By Proposition 4.7 for $m = 0$ and Equations (4.8)–(4.11),

$$\lim_{\mu\to+\infty} \langle h\,\tilde{e}_{p,z}, \tilde{e}_{q,z} \rangle = \lim_{\mu\to+\infty} \langle h e_{p,z}, e_{q,z} \rangle = h(p)\delta_{pq}.$$

Hence,

$$\lim_{\mu\to+\infty} \operatorname{Str}(hP_{z,\mathrm{sm}}) = \sum_{k=0}^{n} (-1)^k \sum_{p\in\mathcal{X}_k} h(p).$$

□

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6. The small complex versus the Morse complex

6.1. Preliminaries on Morse and Smale vector fields

6.1.1. Vector fields with Morse-type zeros. Let X be a real smooth vector field on M with flow $\phi = \{\phi^t\}$. Let $\mathcal{Y} = \mathrm{Zero}(X)$ denote the set of zeros of X (or rest points ϕ). It is said that a zero p of X is of *Morse type* with (*Morse*) *index* of $\mathrm{ind}(p)$ if, using the notation of Equation (4.2),

$$X = -\sum_{j=1}^{n} \epsilon_{p,j} x_p^j \frac{\partial}{\partial x_p^j} \tag{6.1}$$

on the domain U_p of some coordinates $x_p = (x_p^1, \ldots, x_p^n)$ at p, also called *Morse coordinates*. This condition means that $X = -\operatorname{grad}_g h_{X,p}$ on U_p, where $h_{X,p}$ and g are given on U_p by the center and right-hand side of Equations (4.1) and (4.3). The coordinates x_p used in Equation (6.1) are not unique; that expression is invariant by taking positive multiples of the coordinates (contrary to the expressions of Equations (4.1), (4.3) and (4.4)). But $\mathrm{ind}(p)$ is independent of x_p. Note that the Hopf index of $-X$ at p is $(-1)^{\mathrm{ind}(p)}$.

Let us consider $\eta \in Z^1(M,\mathbb{R})$ and use the notation of Section 4.1. For $p \in \mathcal{X} \cap \mathcal{Y}$, if Equations (4.3), (4.4) and (6.1) hold with the same coordinates, then η and g are said to be in *standard form* with respect to X around p. In this case, $C\eta$ and Cg ($C > 0$) are also in standard form with respect to X around p; indeed, $C\eta$, X and Cg satisfy Equations (4.3), (4.4) and (6.1) with the coordinates $\sqrt{C} x_p$. If $\mathcal{X} = \mathcal{Y}$, and η and g are in standard form with respect to X around every $p \in \mathcal{X}$, then η and g are said to be in *standard form* with respect to X. This concept is also applied to any Morse function h on M referring to dh and g. The reference to g may be omitted in this terminology.

Unless otherwise indicated, we assume from now on that X has Morse-type zeros. Then $\mathcal{Y}$ is finite, and the sets $\mathcal{Y}_k$, $\mathcal{Y}_+$ and $\mathcal{Y}_{<k}$ are defined like in Section 4.1.

6.1.2. Stable/unstable manifolds. For $k = 0, \ldots, n$ and $p \in \mathcal{Y}_k$, the *stable/unstable manifolds* of p are smooth injective immersions, $\iota_p^{\pm} : W_p^{\pm} \to M$, where the images $\iota_p^{\pm}(W_p^{\pm})$ consist of the points satisfying $\phi^t(x) \to p$ as $t \to \pm\infty$, and the manifolds W_p^+ and W_p^- are diffeomorphic to $\mathbb{R}^{n-k}$ and $\mathbb{R}^k$, respectively [70, Theorem 9.1]. In particular, $p \in \iota_p^{\pm}(W_p^{\pm})$, and the maps ι_p^+ and ι_p^- meet transversely at p. Let $p^{\pm} = (\iota_p^{\pm})^{-1}(p)$. Assume every U_p is connected, and let $U_p^{\pm}$ be the connected component of $(\iota_p^{\pm})^{-1}(U_p)$ that contains $p^{\pm}$. The restriction $\iota_p^{\pm} : U_p^{\pm} \to (x_p^{\mp})^{-1}(0)$ is a diffeomorphism, and therefore $(U_p^{\pm}, x_p^{\pm} \iota_p^{\pm})$ is a chart of $W_p^{\pm}$ at $p^{\pm}$.

6.1.3. Gradient-like vector fields. Given a Morse function h on M in standard form with respect to X, we have $X = -\operatorname{grad}_g h$ on M for some Riemannian metric g if and only if $Xh < 0$ on $M \setminus \mathcal{Y}$ [16, Lemma 2.1], [40, Section 6.1.3]; in this case, X is said to be *gradient-like* (with respect to h). If X is gradient-like, then the maps $\iota_p^{\pm}$ are embeddings [68, Lemma 3.8], [16, Lemma 2.2], and their images cover M [69, Theorem B

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and Lemma 1.1], [16, Corollary 2.5]. Thus, in this case, the α- and ω-limits of the orbits of X are zero points, we can write $W_p^{\pm} = \iota_p^{\pm}(W_p^{\pm})$ and $p^{\pm} = p$, and $\iota_p^{\pm}$ becomes the inclusion map.

Unless otherwise indicated, we also assume in the rest of the paper that X is gradient-like.

6.1.4. Smale vector fields. X is said to be *Smale* if $W_p^+ \pitchfork W_q^-$ for all $p,q \in \mathcal{Y}$. Then $\mathcal{M}(p,q) := W_p^+ \cap W_q^-$ is a ϕ-saturated smooth submanifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q)$. If $p = q$, we have $\mathcal{M}(p,p) = \{p\}$; in this case, define $\mathcal{T}(p,p) = \emptyset$. If $p \neq q$, the induced $\mathbb{R}$-action on $\mathcal{M}(p,q)$ is free and proper; in this case, define $\mathcal{T}(p,q) = \mathcal{M}(p,q)/\mathbb{R}$, which is a smooth manifold of dimension $\operatorname{ind}(p) - \operatorname{ind}(q) - 1$. The elements of $\mathcal{T}(p,q)$ are the (unparameterized) trajectories with α-limit p and ω-limit q, which are oriented by X. If $\operatorname{ind}(p) \leq \operatorname{ind}(q)$, then $\mathcal{T}(p,q) = \emptyset$. If $\operatorname{ind}(p) - \operatorname{ind}(q) = 1$, then $\mathcal{T}(p,q)$ consists of isolated points, each of them representing a trajectory in M. Let

$$\mathcal{T} = \bigcup_{p,q \in \mathcal{X}} \mathcal{T}(p,q)\,, \quad \mathcal{T}_p^1 = \bigcup_{q \in \mathcal{X}_{\operatorname{ind}(p)-1}} \mathcal{T}(p,q)\,, \quad \mathcal{T}_k^1 = \bigcup_{p \in \mathcal{X}_k} \mathcal{T}_p^1\,, \quad \mathcal{T}^1 = \bigcup_{k=0}^{n} \mathcal{T}_k^1\,.$$

The elements of $\mathcal{T}^1$ are called *instantons*.[1]

X can be C^∞-approximated by gradient-like Smale vector fields that agree with X around $\mathcal{X}$ [20, Proposition 2.4] (this follows from [69, Theorem A]). A well-known consequence is that, for any Morse function h, there is a C^∞-dense set of Riemannian metrics g on M such that $-\operatorname{grad}_g h$ is Smale; this density is also true in the subspace of metrics that are Euclidean with respect to Morse coordinates on given neighborhoods of the critical points.

Unless otherwise indicated, besides the above conditions, we assume from now on that X is Smale; that is, we assume (b) (Section 1.1).

6.1.5. Lyapunov forms. Any $\eta \in Z^1(M,\mathbb{R})$ is said to be *Lyapunov* for X if $\eta(X) < 0$ on $M \setminus \mathcal{Y}$ [20, Definition 2.3]. Note that this condition implies that $\operatorname{Zero}(\eta) = \mathcal{Y}$. By (b), every class in $H^1(M,\mathbb{R})$ has a representative η which is Lyapunov for X and $\eta^\sharp = -X$ for some Riemannian metric g on M, with η and g in standard form with respect to X [18, Proposition 16 (i)], [20, Observations 2.5 and 2.6], [34, Lemma 3.7], [40, Section 6.1.3].

6.1.6. Completion of the unstable manifolds.

Proposition 6.1 ([10, Appendix by F. Laudenbach, Proposition 2], [39, Chapter 2], [15, Theorem 2.1], [17, Theorem 1], [16, Theorem 4.4], [40, Sections A.2 and A.8], [52, Corollary 2.3.2]). *The following holds for every* $p \in \mathcal{Y}_k$ $(k = 0, \dots, n)$*:*

[1] In [12], the elements of $\mathcal{T}$ are called *instantons*, and the elements of $\mathcal{T}^1$ *proper instantons*.

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(i) *$\overline{W_p^-}$ is a C^1 submanifold with conic singularities*[2] *and a Whitney stratified subspace*[3]*. Its strata are the submanifolds W_q^- for $q \in \mathcal{Y}_{<k}$ with $\mathcal{T}(p,q) \neq \emptyset$. As a consequence, W_p^- has finite volume, and*

$$\overline{W_q^-} \cap \overline{W_p^-} \subset \bigcup_{x \in \mathcal{Y}_{<k}} W_x^-$$

if $q \neq p$ in $\mathcal{Y}_k$; in particular, $p \notin \overline{W_q^-}$.

(ii) *There is a compact k-manifold with corners*[4] *$\widehat{W}_p^-$ whose l-corner*[5] *is*

$$\partial_l \widehat{W}_p^- = \bigsqcup_{(q_0,\dots,q_l) \in \{p\} \times \mathcal{Y}^l} \left(\prod_{j=1}^{l} \mathcal{T}(q_{j-1},q_j) \right) \times W_{q_l}^- \quad (0 \leq l \leq k).$$

In particular, the interior of $\widehat{W}_p^-$ is $\partial_0 \widehat{W}_p^- = W_p^-$, and the set $\mathcal{T}(p,q)$ is finite if $q \in \mathcal{Y}_{k-1}$.

(iii) *There is a smooth map $\hat{\iota}_p^- : \widehat{W}_p^- \to M$ whose restriction to every component of $\partial_l \widehat{W}_p^-$ is given by the factor projection to $W_{q_l}^-$, according to (ii). In particular, $\hat{\iota}_p^- = \iota_p^-$ on W_p^-, and $\hat{\iota}_p^- : \widehat{W}_p^- \to \overline{W_p^-}$ is a stratified map.*

By Proposition 6.1 (i), we can choose the open sets U_p ($p \in \mathcal{Y}_k$, $k = 0,\dots,n$) so small that $U_p \cap \overline{W_q^-} = \emptyset$ if $q \neq p$ in $\mathcal{Y}_k$.

For every $q \in \mathcal{Y}_{k-1}$ and $\gamma \in \mathcal{T}(p,q)$, the closure $\bar{\gamma}$ in M is a compact oriented submanifold with boundary of dimension one, and $\partial \bar{\gamma} = \{p,q\}$. We may also consider $\bar{\gamma}$ as the closure of γ in $\widehat{W}_p^-$.

6.2. Preliminaries on the Morse complex

6.2.1. The Morse complex when M is oriented. For reasons of clarity, assume first that M is oriented. Fix an orientation $\mathcal{O}_p^-$ of every unstable manifold W_p^- ($p \in \mathcal{Y}_k$, $k = 0,\dots,n$), which can be also considered as an orientation of $\widehat{W}_p^-$. Then $W_p^- \equiv (W_p^-, \mathcal{O}_p^-)$ defines a current of dimension k on M, also denoted by W_p^-; namely, for $\alpha \in \Omega^k(M)$,

$$\langle W_p^-, \alpha \rangle = \int_{W_p^-} \alpha = \int_{\widehat{W}_p^-} (\hat{\iota}_p^-)^* \alpha. \tag{6.2}$$

Let $\partial_1 \mathcal{O}_p^-$ be the orientation of $\partial_1 \widehat{W}_p^-$ induced by $\mathcal{O}_p^-$ like in the Stokes' theorem; precisely, it is determined by $\mathcal{O}_p^- = \nu_p^- \otimes \partial_1 \mathcal{O}_p^-$ along $\partial_1 \widehat{W}_p^-$ for any outward-pointing normal vector ν_p^-. The restriction of $\partial_1 \mathcal{O}_p^-$ to every component $\mathcal{T}(p,q) \times W_q^-$ ($q \in \mathcal{Y}_{k'}$) of $\partial_1 \widehat{W}_p^-$ is of the form $\mathcal{O}_{p,q} \otimes \mathcal{O}_q^-$ for a unique orientation $\mathcal{O}_{p,q}$ of $\mathcal{T}(p,q)$. If $k' = k-1$,

[2] In the sense of [10, Appendix by F. Laudenbach, Section a)] and [40, Appendix A.1].
[3] Introduced by H. Whitney [72, 73], and the definition was simplified by J. Mather [45].
[4] In the sense of [47, Section 1.1.8].
[5] The union of the interiors of the boundary faces of codimension l.

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then $\mathcal{O}_{p,q}$ can be represented by a unique function $\epsilon_{p,q} : \mathcal{T}(p,q) \to \{\pm 1\}$; combining these functions, we get a map $\epsilon : \mathcal{T}^1 \to \{\pm 1\}$. By the descriptions of $\partial_1 \widehat{W}_p^-$ and $\hat\iota_p^- : \partial_1 \widehat{W}_p^- \to M$, and by the Stokes' theorem for manifolds with corners, we have [10, Appendix by F. Laudenbach], [34, Remark 1.9], [16, Theorem 3.6 and Proposition 5.3], [40, Section 6.5.3]

$$\partial W_p^- = \sum_{q \in \mathcal{Y}_{k-1},\ \gamma \in \mathcal{T}(p,q)} \epsilon(\gamma)\, W_q^- . \tag{6.3}$$

Thus, the currents W_p^- ($p \in \mathcal{X}$) generate over $\mathbb{C}$ a finite-dimensional subcomplex $(C_\bullet(X,W^-),\partial)$ of the complex $(\Omega(M)',\partial)$ of currents on M, called the *Morse complex*. The simpler notation $\mathbf{C}_\bullet = \mathbf{C}_\bullet(X) = C_\bullet(X,W^-)$ may be also used. Moreover, $\mathbf{C}_\bullet \hookrightarrow \Omega(M)'$ is a quasi-isomorphism,[6] $H_\bullet(\mathbf{C}_\bullet,\partial) \cong H_\bullet(M,\mathbb{C})$ [71, 67, 51] (see also [29, 64, 65], [35, Theorem 0.1], [10, Appendix by F. Laudenbach, Proposition 7], [40, Section 6.6.5]).

Let $(C_\bullet(X,W^+),\partial) = (C_\bullet(-X,W^-),\partial)$, involving the stable Morse cells W_p^+. If M is oriented by $\mathcal{O}_M$ and the orientation $\mathcal{O}_p^+$ of every W_p^+ is chosen so that $\mathcal{O}_p^+ \otimes \mathcal{O}_p^- = \mathcal{O}_M$ at p, then the canonical pairing

$$\langle\cdot,\cdot\rangle : C_\bullet(X,W^-) \times C_{n-\bullet}(X,W^+) \to \mathbb{K}\,, \quad \langle W_p^-,W_q^+\rangle = \delta_{pq}\,, \tag{6.4}$$

satisfies [40, Section 6.6.2]

$$\langle \partial W_p^-,W_q^+\rangle = (-1)^k \langle W_p^-,\partial W_q^+\rangle \quad (p \in \mathcal{X}_k,\ q \in \mathcal{X}_{k-1}). \tag{6.5}$$

6.2.2. The Morse complex when M may not be oriented. When M is not assumed to be oriented, the concepts of Section 6.2.1 can be extended as follows. We fix an orientation $N\mathcal{O}_p^-$ of every normal bundle NW_p^-, which can be also considered as an orientation of $N\widehat{W}_p^-$ (the normal bundle of the immersion $\hat\iota_p^-$). Then we can consider $W_p^- \equiv (W_p^-,N\mathcal{O}_p^-) \in \Omega^k(M,o(M))'$, by using $N\mathcal{O}_p^- \otimes \alpha$ as integrand in Equation (6.2) for every $\alpha \in \Omega^k(M,o(M))$; note that $N\mathcal{O}_p^- \otimes \alpha \in \Omega^k(\widehat{W}_p^-,o(\widehat{W}_p^-)) = \Omega^k(\widehat{W}_p^-)$. With the notation of Section 6.2.1, $\partial_1 N\mathcal{O}_p^- := N\mathcal{O}_p^- \otimes \nu_p^-$ describes an orientation of $N\partial_1\widehat{W}_p^-$, and the Stokes theorem has the extension (see [13, Theorem 7.7] for the case without boundary)

$$\int_{\widehat{W}_p^-} N\mathcal{O}_p^- \otimes d\beta = \int_{\partial_1 \widehat{W}_p^-} \partial_1 N\mathcal{O}_p^- \otimes \beta \quad \big(\beta \in \Omega^{k-1}(M,o(M))\big). \tag{6.6}$$

If M is oriented by O_M, then $N\mathcal{O}_p^-$ and $\mathcal{O}_p^-$ determine each other by the condition $O_M = N\mathcal{O}_p^- \otimes \mathcal{O}_p^-$. Then $\partial_1 N\mathcal{O}_p^-$ and $\partial_1\mathcal{O}_p^-$ determine each other in the same way:

$$O_M = N\mathcal{O}_p^- \otimes \mathcal{O}_p^- = N\mathcal{O}_p^- \otimes \nu_p^- \otimes \partial_1\mathcal{O}_p^- = \partial_1 N\mathcal{O}_p^- \otimes \partial_1\mathcal{O}_p^- .$$

So Equation (6.6) agrees with the usual Stokes' theorem in this way.

If M is not oriented, by using local orientations of M, the above argument shows that Equation (6.6) also agrees with the usual Stokes' theorem for $o(M)$-valued forms β with

[6] Actually, $H_\bullet(M,\mathbb{Z})$ is isomorphic to the homology of the complex of free abelian groups generated by the currents W_p^-.

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small enough support. Then, like in Section 6.2.1, we get the same map $\epsilon : \mathcal{T}^1 \to \{\pm 1\}$, and therefore the same definition of $(\mathbf{C}_\bullet, \partial)$.

6.2.3. The dual Morse complex. Let $C^k(X, W^-) = (\mathbf{C}_k)^* \equiv \mathbb{C}^{\mathcal{Y}_k}$ $(k = 0, \dots, n)$ and $\mathbf{d} = \partial^*$. The simpler notation $\mathbf{C}^\bullet = \mathbf{C}^\bullet(X)$ will be preferred. It is said that $(\mathbf{C}^\bullet, \mathbf{d})$ is the *dual Morse complex*. Boldface notation is also used for elements of $\mathbf{C}^\bullet$ and other operators on $\mathbf{C}^\bullet$. Let $\mathbf{e}_p$ $(p \in \mathcal{Y})$ denote the elements of the canonical base of $\mathbf{C}^\bullet$, determined by $\mathbf{e}_p(q) = \delta_{pq}$. By Equation (6.3), for $q \in \mathcal{Y}_{k-1}$,

$$\mathbf{d}\mathbf{e}_q = \sum_{p \in \mathcal{Y}_k,\ \gamma \in \mathcal{T}(p,q)} \epsilon(\gamma)\, \mathbf{e}_p. \tag{6.7}$$

Comparing Equations (6.3) and (6.7), we see that $(C^\bullet(X, W^-), \mathbf{d}) \equiv (C_\bullet(-X, W^+), \partial)$. Thus, from now on, $(\mathbf{C}^\bullet, \mathbf{d})$ will be also called a *Morse complex*. If M is oriented, it also follows from Equations (6.4) and (6.5) that $(C^\bullet(X, W^-), \mathsf{w}\mathbf{d}) \equiv (C_{n-\bullet}(X, W^+), \partial)$.

6.2.4. The perturbed Morse complex. Take any $\eta \in Z^1(M, \mathbb{R})$ defining a class $\xi \in H^1(M, \mathbb{R})$ (there is no need of any condition on η or g in Sections 6.2.4 to 6.2.6). For reasons of brevity, write $\eta(\gamma) = \int_\gamma \eta$ for every $\gamma \in \mathcal{T}^1$. According to [17, 18, 20], $(\mathbf{C}^\bullet, \mathbf{d})$ has an analog of the Witten's perturbation, $(\mathbf{C}^\bullet, \mathbf{d}_z = \mathbf{d}_{z\eta})$ $(z \in \mathbb{C})$, where, for $q \in \mathcal{Y}_{k-1}$ $(k = 1, \dots, n)$,

$$\mathbf{d}_z \mathbf{e}_q = \sum_{p \in \mathcal{Y}_k,\ \gamma \in \mathcal{T}(p,q)} \epsilon(\gamma) e^{z\eta(\gamma)} \mathbf{e}_p. \tag{6.8}$$

If $\eta = dh$ for some $h \in C^\infty(M, \mathbb{R})$, then $\mathbf{d}_z = e^{-zh} \mathbf{d} e^{zh}$ on $\mathbf{C}^\bullet$ because $\eta(\gamma) = h(q) - h(p)$ for $p \in \mathcal{Y}_k$, $q \in \mathcal{Y}_{k-1}$ and $\gamma \in \mathcal{T}(p,q)$; here, $e^{\pm zh}$ also denotes the operator of multiplication by the restriction of this function to $\mathcal{Y}$. It will be said that $(\mathbf{C}^\bullet, \mathbf{d}_z)$ $(z \in \mathbb{C})$ is the *perturbed dual Morse complex* defined by X and η. A perturbation $(\mathbf{C}_\bullet, \partial^z)$ is similarly defined, multiplying by $e^{z\eta(\gamma)}$ the terms of the right-hand side of Equation (6.3).

Since W_p^- $(p \in \mathcal{Y}_k,\ k = 0, \dots, n)$ is diffeomorphic to $\mathbb{R}^k$, there is a unique $h_{\eta,p}^- \in C^\infty(W_p^-, \mathbb{R})$ such that $h_{\eta,p}^-(\hat{p}^-) = 0$ and $dh_{\eta,p}^- = (\iota_p^-)^* \eta$, where $\hat{p}^- \in W_p^- \subset \widehat{W}_p^-$ is determined by $\iota_p^-(\hat{p}^-) = p$. Indeed, $h_{\eta,p}^-$ has a smooth extension $\hat{h}_{\eta,p}^-$ to $\widehat{W}_p^-$ because $\widehat{W}_p^-$ is contractile. By Proposition 6.1 (ii), for all $q \in \mathcal{Y}_{k-1}$ and $\gamma \in \mathcal{T}(p,q)$, we have $\hat{h}_{\eta,p}^-(\gamma, \hat{q}^-) = \eta(\gamma)$ at $(\gamma, \hat{q}^-) \in \{\gamma\} \times \widehat{W}_q^- \subset \partial_1 \widehat{W}_p^-$. Therefore, $\hat{h}_{\eta,q}^-$ corresponds to the restriction of $\hat{h}_{\eta,p}^- - \eta(\gamma)$ via the canonical diffeomorphism $\widehat{W}_q^- \approx \{\gamma\} \times \widehat{W}_q^-$.

According to [17, Proposition 4], [18, Proposition 10], [20, Propositions 2.15 and 2.16 and Section 6.2], a surjective homomorphism of complexes,

$$\Phi_z : (\Omega(M), d_z) \to (\mathbf{C}^\bullet, \mathbf{d}_z)\,,$$

is defined by

$$\Phi_z(\omega)(p) = \int_{W_p^-} e^{z h_{\eta,p}^-} \omega = \int_{\widehat{W}_p^-} e^{z \hat{h}_{\eta,p}^-} (\iota_p^-)^* \omega.$$

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Moreover, Φ_z is a quasi-isomorphism for all $z\in\mathbb{C}$ [10, Proposition 7 in the Appendix by F. Laudenbach] (see also [10, Theorem 2.9], [11, Theorem 1.6], [20, Proposition 2.17 and Section 6.2]). If η and g satisfy (a), then, by Equation (4.13),

$$\Phi_z : (E_{z,\mathrm{sm}},d_z) \to (\mathbf{C}^\bullet,\mathbf{d}_z)$$

is also a quasi-isomorphism. Since a direct adaptation of [18, Appendix A] shows that, for $k=0,\dots,n$, $\dim H^k(\mathbf{C}^\bullet,\mathbf{d}_z)$ is independent of $z\in\mathbb{C}$ with $|\mu|\gg 0$, we get Equation (2.9) because any $\xi\in H^1(M,\mathbb{R})$ is represented by a Morse form.

6.2.5. Morse complex with coefficients in a flat vector bundle. With more generality, for a flat vector bundle F, we may consider $(C^\bullet(X,W^-,F),\mathbf{d}^F)$, where $C^k(X,W^-,F)=\bigoplus_{p\in\mathcal{Y}_k}F_p$, and $\mathbf{d}^F\mathbf{e}$ ($\mathbf{e}\in F_q$, $q\in\mathcal{Y}_{k-1}$) is defined like in the right-hand side of Equation (6.7), replacing $\mathbf{e}_p$ with the parallel transport of $\mathbf{e}$ along $\bar\gamma^{-1}$ [10, Section 1c)]. This is the dual of the complex $(C_\bullet(X,W^-,F^*),\partial^{F^*})$, where $C_k(X,W^-,F^*)=\bigoplus_{p\in\mathcal{Y}_k}F^*_p$, and $\partial^F f$ ($f\in F^*_p$, $p\in\mathcal{X}_k$) is defined like in the right-hand side of Equation (6.3), replacing W^-_q with the parallel transport of f along $\bar\gamma$. A quasi-isomorphism

$$\Phi^F=\Phi^{X,F} : (\Omega(M,F),d) \to \big(C^\bullet(X,W^-,F),\mathbf{d}^F\big)$$

can be defined like Φ_z [10, Theorem 2.9], using the isomorphism

$$\Omega^\bullet\big(\widehat{W}^-_p,(\hat\iota^-_p)^*F\big)\cong\Omega^\bullet\big(\widehat{W}^-_p\big)\otimes F_p$$

given by the parallel transport of $(\hat\iota^-_p)^*F$. If $F=\mathcal{L}^z$ (Section 2.1.4), then

$$\big(C^\bullet(X,W^-,\mathcal{L}^z),\mathbf{d}^{\mathcal{L}^z}\big)\equiv(\mathbf{C}^\bullet,\mathbf{d}_z)\,,\quad \Phi^{\mathcal{L}^z}\equiv\Phi_z.$$

6.2.6. Hodge theory of the Morse complex. Consider the Hermitian scalar product on $\mathbf{C}^\bullet$ so that the canonical base $\mathbf{e}_p$ ($p\in\mathcal{Y}$) is orthonormal. All operators induced by $\mathbf{d}_z$ and this Hermitian structure are called *perturbed Morse operators*. For instance, besides $\mathbf{d}_z$, we have the perturbed Morse operators

$$\boldsymbol{\delta}_z=\mathbf{d}_z^*\,,\quad \mathbf{D}_z=\mathbf{d}_z+\boldsymbol{\delta}_z\,,\quad \boldsymbol{\Delta}_z=\mathbf{D}_z^2=\mathbf{d}_z\boldsymbol{\delta}_z+\boldsymbol{\delta}_z\mathbf{d}_z.$$

In particular, it will be said that $\boldsymbol{\Delta}_z$ is the *perturbed Morse Laplacian*, and its eigenvalues will be called *perturbed Morse eigenvalues*. If $z=0$, we omit the subscript 'z' and the word 'perturbed'. From Equation (6.8), we easily get

$$\boldsymbol{\delta}_z\mathbf{e}_p=\sum_{q\in\mathcal{Y}_{k-1},\ \gamma\in\mathcal{T}(p,q)} e^{\bar z\eta(\gamma)}\epsilon(\gamma)\,\mathbf{e}_q\,, \tag{6.9}$$

for $p\in\mathcal{Y}_k$. We also have

$$\mathbf{C}^\bullet=\ker\boldsymbol{\Delta}_z\oplus\operatorname{im}\mathbf{d}_z\oplus\operatorname{im}\boldsymbol{\delta}_z\,,$$
$$\ker\boldsymbol{\Delta}_z=\ker\mathbf{D}_z=\ker\mathbf{d}_z\cap\ker\boldsymbol{\delta}_z\,,\quad \operatorname{im}\boldsymbol{\Delta}_z=\operatorname{im}\mathbf{D}_z=\operatorname{im}\mathbf{d}_z\oplus\operatorname{im}\boldsymbol{\delta}_z.$$

The orthogonal projections of $\mathbf{C}^\bullet$ to $\ker\boldsymbol{\Delta}_z$, $\operatorname{im}\mathbf{d}_z$ and $\operatorname{im}\boldsymbol{\delta}_z$ are denoted by $\boldsymbol{\Pi}_z=\boldsymbol{\Pi}_z^0$, $\boldsymbol{\Pi}_z^1$ and $\boldsymbol{\Pi}_z^2$, respectively. The compositions $\mathbf{d}_z^{-1}\boldsymbol{\Pi}_z^1$, $\boldsymbol{\delta}_z^{-1}\boldsymbol{\Pi}_z^2$ and $\mathbf{D}_z^{-1}\boldsymbol{\Pi}_z^\perp$ are defined like in Section 2.1.2, and there is an obvious version of the commutative diagram of Equation (2.7).

6.3. The small complex versus the Morse complex

Our main objects of interest are the form $\eta \in Z^1(M;\mathbb{R})$ and the Riemannian metric g; X plays an auxiliary role. As indicated in Section 6.1.5, by (b), we can choose some $\eta \in \xi$ and g satisfying (a) and (c) (Section 1.1). Thus, unless otherwise indicated, assume from now on that X, η and g satisfy (c), besides (a) and (b). In particular, $\mathcal{Y} = \mathrm{Zero}(\eta)$.

For every $p \in \mathcal{Y}$, consider the functions $h_{\eta,p}$, $h_{X,p}$, $h^-_{\eta,p}$ and $\hat{h}^-_{\eta,p}$ defined in Sections 4.1, 6.1.1 and 6.2.4. By (c), we have

$$\begin{aligned} h_{\eta,p} &= h_{X,p} && \text{on} \quad U_p\,, \\ h^-_{\eta,p} &= h_{\eta,p} = -\frac{1}{2}|x^-_p|^2 && \text{on} \quad U^-_p\,, \end{aligned} \tag{6.10}$$

$$h^-_{\eta,p} < 0 \qquad \text{on} \quad W^-_p \setminus \{p\}. \tag{6.11}$$

From now on, the subscripts X and η will be dropped from the notation of these functions.

Continuing with the notation of Section 6.2.4, let $J_z : \mathbf{C}^\bullet \to E_z$ be the $\mathbb{C}$-linear isometry given by $J_z(\mathbf{e}_p) = e_{p,z}$, and let $\Psi_z = P_{z,\mathrm{sm}} J_z : \mathbf{C}^\bullet \to E_{z,\mathrm{sm}}$, which is an isomorphism for $\mu \gg 0$ (Corollary 4.9). By Proposition 4.7,

$$\|\Psi_z \mathbf{e}\| = \big(1 + O\big(e^{-c\mu}\big)\big)\|\mathbf{e}\| \quad (\mu \to +\infty)$$

for all $\mathbf{e} \in \mathbf{C}^\bullet$. Using polarization (see, e.g., [37, Section I.6.2]) and conjugation, this means that, as $\mu \to +\infty$,

$$\Psi^*_z \Psi_z = 1 + O\big(e^{-c\mu}\big)\,, \quad \Psi_z \Psi^*_z = 1 + O\big(e^{-c\mu}\big). \tag{6.12}$$

Notation 6.2. Consider functions $u(z)$ and $v(z)$ ($z \in \mathbb{C}$) with values in Banach spaces. The notation $u(z) \asymp_0 v(z)$ ($\mu \to \pm\infty$) means

$$u(z) = v(z) + O\big(e^{-c|\mu|}\big) \quad (\mu \to \pm\infty).$$

This notation may be used even when the Banach spaces depend on z.

Theorem 6.3 (Cf. [11, Theorem 6.11], [75, Theorem 6.9], [17, Theorem 4]). *For every $\tau \in \mathbb{R}$, as $\mu \to +\infty$,*

$$\Phi_{z+\tau} \Psi_z \asymp_0 \left(\frac{\pi}{\mu + \tau/2}\right)^{\mathsf{N}/2} \left(\frac{\mu}{\pi}\right)^{n/4}.$$

Proof. We adapt the proof of [75, Theorem 6.9] to the case of complex parameter. For every $p \in \mathcal{Y}_k$,

$$\Phi_{z+\tau} \Psi_z \mathbf{e}_p = \sum_{q \in \mathcal{Y}_k} \mathbf{e}_q \int_{\widehat{W}^-_q} e^{(z+\tau)\hat{h}^-_q} (\hat{\iota}^-_q)^* P_{z,\mathrm{sm}} e_{p,z}. \tag{6.13}$$

Then the result follows by checking the asymptotics of these integrals using the compactness of $\widehat{W}^-_q$.

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In the case $q=p$, by Equation (6.11) and Corollary 4.8,

$$\int_{\widehat{W}_p^-} e^{(z+\tau)\hat{h}_p^-}(\hat{\iota}_p^-)^*(P_{z,\mathrm{sm}}-1)e_{p,z} \asymp_0 0.$$

But, by Proposition 4.1 (iii) and Equations (4.8)–(4.11) and (6.10),

$$\begin{aligned}\int_{\widehat{W}_p^-} e^{(z+\tau)\hat{h}_p^-}(\hat{\iota}_p^-)^* e_{p,z} &= \int_{\widehat{W}_p^-} e^{(z+\tau)\hat{h}_p^-}(\hat{\iota}_p^-)^*\left(e^{-i\nu h_p}e_{p,\mu}\right)\\ &= \int_{\widehat{W}_p^-} e^{(\mu+\tau)\hat{h}_p^-}(\hat{\iota}_p^-)^* e_{p,\mu} = \frac{1}{a_\mu}\left(\int_{-2r}^{2r}\rho(x)e^{-(2\mu+\tau)x^2/2}\,dx\right)^k\\ &= \left(\frac{\pi}{\mu+\tau/2}\right)^{k/2}\left(\frac{\mu}{\pi}\right)^{n/4}\left(1+O\left(e^{-c\mu}\right)\right). \end{aligned} \tag{6.14}$$

(When $\tau=0$, the last equality is the same as [75, Eq. (6.30)].)

For $q\neq p$ in $\mathcal{Y}_k$, since $e_{p,z}=0$ on $\overline{W_q^-}$ because $U_p\cap\overline{W_q^-}=\emptyset$ (Section 6.1.6), like in the previous case, we get

$$\int_{\widehat{W}_q^-} e^{(z+\tau)\hat{h}_q^-}(\hat{\iota}_q^-)^* P_{z,\mathrm{sm}} e_{p,z} \asymp_0 0.$$

□

Corollary 6.4. *For every $\tau\in\mathbb{R}$, if $\mu\gg 0$, then $\Phi_{z+\tau}: E_{z,\mathrm{sm}}\to\mathbf{C}^\bullet$ is a linear*[7] *isomorphism.*

Proof. Apply Theorem 6.3 and Corollary 4.9. □

Remark 6.5. The argument of the proof of Theorem 6.3 shows that

$$\Phi_z J_z = \left(\frac{\pi}{\mu}\right)^{\mathsf{N}/2-n/4} + O\left(e^{-c\mu}\right) \quad (\mu\to+\infty).$$

So $\Phi_z : E_z\to\mathbf{C}^\bullet$ is an isomorphism for $\mu\gg 0$ (see also [20, Lemma 5.2]).

Let

$$\widetilde{\Psi}_z = \left(\frac{\mu}{\pi}\right)^{\mathsf{N}/2-n/4}\Psi_z : \mathbf{C}^\bullet\to E_{z,\mathrm{sm}}.$$

Corollary 6.6. *Consider $\widetilde{\Psi}_z^* : E_{z,\mathrm{sm}}\to\mathbf{C}^\bullet$. As $\mu\to+\infty$,*

$$\widetilde{\Psi}_z^*\widetilde{\Psi}_z = \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2} + O\left(e^{-c\mu}\right), \quad \widetilde{\Psi}_z\widetilde{\Psi}_z^* = \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2} + O\left(e^{-c\mu}\right).$$

Proof. This is a direct consequence of Equation (6.12). □

Corollary 6.7. *For any $\tau\in\mathbb{R}$, consider $\Phi_{z+\tau}: E_{z,\mathrm{sm}}\to\mathbf{C}^\bullet$. As $\mu\to+\infty$,*

$$\Phi_{z+\tau}\widetilde{\Psi}_z \asymp_0 \left(\frac{\mu}{\mu+\tau/2}\right)^{\mathsf{N}/2}, \quad \widetilde{\Psi}_z\Phi_{z+\tau} \asymp_0 \left(\frac{\mu}{\mu+\tau/2}\right)^{\mathsf{N}/2}.$$

[7] It is an isomorphism of complexes if $\tau=0$.

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Proof. The first relation is a restatement of Theorem 6.3. The second relation follows by conjugating the first one by $\widetilde{\Psi}_z$ and using Corollary 6.6. □

Corollary 6.8. *As* $\mu \to +\infty$, $\widetilde{\Psi}_z^{-1} \asymp_0 \Phi_z$ *on* $E_{z,\mathrm{sm}}$.

Proof. By Corollaries 6.6 and 6.7, on $E_{z,\mathrm{sm}}$,

$$\widetilde{\Psi}_z^{-1} \asymp_0 \widetilde{\Psi}_z^{-1}\widetilde{\Psi}_z\Phi_z = \Phi_z.$$

□

In the rest of this section, consider $\Phi_z : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$ unless otherwise indicated.

Corollary 6.9. *As* $\mu \to +\infty$,

$$\Phi_z^*\Phi_z \asymp_0 \left(\frac{\pi}{\mu}\right)^{\mathsf{N}-n/2}, \quad \Phi_z\Phi_z^* \asymp_0 \left(\frac{\pi}{\mu}\right)^{\mathsf{N}-n/2}.$$

Proof. We show the first relation, the other one being similar. By Corollaries 6.6 and 6.8, on $E_{z,\mathrm{sm}}$,

$$\Phi_z^*\Phi_z \asymp_0 \left(\widetilde{\Psi}_z^{-1}\right)^*\widetilde{\Psi}_z^{-1} = \left(\widetilde{\Psi}_z^*\right)^{-1}\widetilde{\Psi}_z^{-1} = \left(\widetilde{\Psi}_z\widetilde{\Psi}_z^*\right)^{-1} \asymp_0 \left(\frac{\pi}{\mu}\right)^{\mathsf{N}-n/2}.$$

□

Corollary 6.10. *As* $\mu \to +\infty$,

$$\widetilde{\Psi}_z \asymp_0 \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2}\Phi_z^*.$$

Proof. By Corollaries 6.7 and 6.9,

$$\widetilde{\Psi}_z \asymp_0 \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2}\widetilde{\Psi}_z\Phi_z\Phi_z^* \asymp_0 \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2}\Phi_z^*.$$

□

Corollary 6.11. *For every* $\tau \in \mathbb{R}$, *as* $\mu \to +\infty$,

$$\Phi_{z+\tau}P_{z+\tau,\mathrm{sm}}\widetilde{\Psi}_z \asymp_0 \left(\frac{\mu}{\mu+\tau/2}\right)^{\mathsf{N}/2} + O\left(\mu^{-1}\right).$$

Proof. By Corollaries 6.7, 6.6 and 6.9 and Proposition 4.19,

$$\begin{aligned}\Phi_{z+\tau}P_{z+\tau,\mathrm{sm}}\widetilde{\Psi}_z &= \Phi_{z+\tau}(P_{z+\tau,\mathrm{sm}} - P_{z,\mathrm{sm}})\widetilde{\Psi}_z + \Phi_{z+\tau}\widetilde{\Psi}_z \\ &\asymp_0 O\left(\mu^{-1}\right) + \left(\frac{\mu}{\mu+\tau/2}\right)^{\mathsf{N}/2}.\end{aligned}$$

□

Corollary 6.12. *As* $\mu \to +\infty$,

$$d_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z\mathbf{d}_z\Phi_z, \quad \delta_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z\boldsymbol{\delta}_z\Phi_z.$$

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Proof. By Theorem 4.10 and Corollary 6.7,

$$d_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z \Phi_z d_{z,\mathrm{sm}} = \widetilde{\Psi}_z \mathbf{d}_z \Phi_z .$$

Now, taking adjoints and using Corollaries 6.6, 6.9 and 6.10, we obtain

$$\delta_{z,\mathrm{sm}} = \Phi^*_{z,\mathrm{sm}} \boldsymbol{\delta}_z \widetilde{\Psi}^*_z \asymp_0 \widetilde{\Psi}_z \boldsymbol{\delta}_z \Phi_z .$$

□

Let $\widetilde{\mathbf{\Pi}}_z = \widetilde{\mathbf{\Pi}}^0_z$, $\widetilde{\mathbf{\Pi}}^1_z$ and $\widetilde{\mathbf{\Pi}}^2_z$ be the orthogonal projections of $\mathbf{C}^\bullet$ to $\Phi_z(\ker \Delta_{z,\mathrm{sm}})$, $\Phi_z(\operatorname{im} d_{z,\mathrm{sm}})$ and $\Phi_z(\operatorname{im} \delta_{z,\mathrm{sm}})$, respectively. Note that $\widetilde{\mathbf{\Pi}}^1_z = \widetilde{\mathbf{\Pi}}^1_z \mathbf{\Pi}^1_z$.

Corollary 6.13. *For $j = 0,1,2$, as $\mu \to +\infty$,*

$$\Phi_z \Pi^j_{z,\mathrm{sm}} \asymp_0 \widetilde{\mathbf{\Pi}}^j_z \Phi_z , \quad \Pi^j_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z \widetilde{\mathbf{\Pi}}^j_z \Phi_z , \quad \Pi^j_{z,\mathrm{sm}} \widetilde{\Psi}_z \asymp_0 \widetilde{\Psi}_z \widetilde{\mathbf{\Pi}}^j_{z,\mathrm{sm}} .$$

Proof. We only prove the case of $\widetilde{\mathbf{\Pi}}^2_z$, the other cases being similar. Let $\alpha_{z,1},\dots,\alpha_{z,p_z}$ be an orthonormal frame of $\delta_z(E^{k+1}_{z,\mathrm{sm}})$. So $\Phi_z\alpha_{z,1},\dots,\Phi_z\alpha_{z,p_z}$ is a base of $\Phi_z\delta_z(E^{k+1}_{z,\mathrm{sm}})$ for $\mu \gg 0$ by Corollary 6.4. Applying the Gram–Schmidt process to this base, we get an orthonormal base $\mathbf{f}_{z,1},\dots,\mathbf{f}_{z,p_z}$ of $\Phi_z\delta_z(E^{k+1}_{z,\mathrm{sm}})$. By Corollary 6.9,

$$\langle \Phi_z\alpha_{z,a}, \Phi_z\alpha_{z,b} \rangle \asymp_0 \left(\frac{\pi}{\mu}\right)^{k-n/2} \delta_{ab} ,$$

for $1 \le a,b \le p_z$. So

$$\mathbf{f}_{z,a} \asymp_0 \left(\frac{\mu}{\pi}\right)^{k/2-n/4} \Phi_z\alpha_{z,a} .$$

Hence, by Corollary 6.9, for any $\beta \in E^k_{z,\mathrm{sm}}$,

$$\begin{aligned}\widetilde{\mathbf{\Pi}}^2_z \Phi_z \beta = \sum_{a=1}^{p_z} \langle \Phi_z\beta, \mathbf{f}_{z,a} \rangle \mathbf{f}_{z,a} &\asymp_0 \left(\frac{\mu}{\pi}\right)^{k-n/2} \sum_{a=1}^{p_z} \langle \Phi_z\beta, \Phi_z\alpha_{z,a} \rangle \Phi_z\alpha_{z,a} \\ &\asymp_0 \sum_{a=1}^{m} \langle \beta, \alpha_{z,a} \rangle \Phi_z\alpha_{z,a} = \Phi_z \Pi^2_{z,\mathrm{sm}} \beta .\end{aligned}$$

This shows the first relation of the statement because $\dim E^k_{z,\mathrm{sm}} < \infty$. Then the other stated relations follow using Corollaries 6.6, 6.7 and 6.9. □

According to Corollary 6.4, in the following corollaries, we take $\mu \gg 0$ so that $\Phi_z : E_{z,\mathrm{sm}} \to \mathbf{C}^\bullet$ is an isomorphism.

Corollary 6.14. *As $\mu \to +\infty$,*

$$(\Phi_z^{-1})^* \Phi_z^{-1} \asymp_0 \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2} , \quad \Phi_z^{-1} (\Phi_z^{-1})^* \asymp_0 \left(\frac{\mu}{\pi}\right)^{\mathsf{N}-n/2} .$$

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Proof. By Corollary 6.9, for $\mathbf{e} \in \mathbf{C}^k$ with $\|\mathbf{e}\| = 1$,

$$\left\|\Phi_z^{-1}\mathbf{e}\right\| \asymp_0 \left(\frac{\mu}{\pi}\right)^{k/2-n/4} \left\|\Phi_z\Phi_z^{-1}\mathbf{e}\right\| = \left(\frac{\mu}{\pi}\right)^{k/2-n/4},$$

yielding the first stated relation. The second one has a similar proof. □

Corollary 6.15. *As $\mu \to +\infty$,*

$$\Phi_z^* \asymp_0 \left(\frac{\pi}{\mu}\right)^{\mathsf{N}-n/2} \Phi_z^{-1}, \quad \widetilde{\Psi}_z \asymp_0 \Phi_z^{-1}.$$

Proof. By Corollaries 6.9 and 6.14,

$$\Phi_z^* = \Phi_z^*\Phi_z\Phi_z^{-1} \asymp_0 \left(\frac{\pi}{\mu}\right)^{\mathsf{N}-n/2} \Phi_z^{-1}, \quad \widetilde{\Psi}_z = \widetilde{\Psi}_z\Phi_z\Phi_z^{-1} \asymp_0 \Phi_z^{-1}.$$

□

Corollary 6.16. *We have $\widetilde{\mathbf{\Pi}}_z^1 = \mathbf{\Pi}_z^1$ for $\mu \gg 0$, and $\widetilde{\mathbf{\Pi}}_z^2 \asymp_0 \mathbf{\Pi}_z^2$ as $\mu \to +\infty$.*

Proof. Since $\Phi_z(\operatorname{im} d_{z,\mathrm{sm}}) = \operatorname{im} \mathbf{d}_z$ for $\mu \gg 0$, we get $\widetilde{\mathbf{\Pi}}_z^1 = \mathbf{\Pi}_z^1$.

To prove $\widetilde{\mathbf{\Pi}}_z^2 \asymp_0 \mathbf{\Pi}_z^2$ as $\mu \to +\infty$, consider the notation of the proof of Corollary 6.13. We have $\alpha_{z,a} = \delta_z\beta_{z,a}$ $(a = 1,\dots,p_z)$ for some base $\beta_{z,1},\dots,\beta_{z,p_z}$ of $\operatorname{im} d_{z,\mathrm{sm},k}$. Hence, by Corollaries 6.7, 6.9 and 6.12,

$$\Phi_z\alpha_{z,a} = \Phi_z\delta_z\beta_{z,a} \asymp_0 \Phi_z\widetilde{\Psi}_z\boldsymbol{\delta}_z\Phi_z\beta_{z,a} \asymp_0 \boldsymbol{\delta}_z\Phi_z\beta_{z,a}, \tag{6.15}$$

and $\boldsymbol{\delta}_z\Phi_z\beta_{z,1},\dots,\boldsymbol{\delta}_z\Phi_z\beta_{z,p_z}$ is a base of $\operatorname{im}\boldsymbol{\delta}_{z,k+1}$. Applying the Gram–Schmidt process to this base, we get an orthonormal base $\mathbf{g}_{z,1},\dots,\mathbf{g}_{z,p_z}$ of $\operatorname{im}\boldsymbol{\delta}_{z,k+1}$ satisfying $\mathbf{g}_{z,a} \asymp_0 \mathbf{f}_{z,a}$ by Equation (6.15). Then, for any $\mathbf{e} \in \mathbf{C}^k$ with $\|\mathbf{e}\| = 1$,

$$\widetilde{\mathbf{\Pi}}_z^2\mathbf{e} = \sum_{a=1}^{p_z} \langle \mathbf{e}, \mathbf{g}_{z,a}\rangle \mathbf{g}_{z,a} \asymp_0 \sum_{a=1}^{p_z} \langle \mathbf{e}, \mathbf{f}_{z,a}\rangle \mathbf{f}_{z,a} = \mathbf{\Pi}_z^2\mathbf{e}.$$

□

Corollary 6.17. *We have*

$$d_{z,\mathrm{sm}} = \Phi_z^{-1}\mathbf{d}_z\Phi_z, \quad d_{z,\mathrm{sm}}^{-1}\Pi_{z,\mathrm{sm}}^1 = \Pi_{z,\mathrm{sm}}^2\Phi_z^{-1}\mathbf{d}_z^{-1}\Phi_z\Pi_{z,\mathrm{sm}}^1.$$

Proof. The first equality follows like the first relation of Corollary 6.12, using Φ_z^{-1} instead of $\widetilde{\Psi}_z$. To prove the second one, take any $\alpha \in \operatorname{im} d_{z,\mathrm{sm}}$. Since

$$d_z\Pi_{z,\mathrm{sm}}^2\Phi_z^{-1}\mathbf{d}_z^{-1}\Phi_z\alpha = d_z\Phi_z^{-1}\mathbf{d}_z^{-1}\Phi_z\alpha = \Phi_z^{-1}\mathbf{d}_z\mathbf{d}_z^{-1}\Phi_z\alpha = \alpha$$

with $\Pi_{z,\mathrm{sm}}^2\Phi_z^{-1}\mathbf{d}_z^{-1}\Phi_z\alpha \in \operatorname{im}\delta_{z,\mathrm{sm}}$, we obtain

$$\Pi_{z,\mathrm{sm}}^2\Phi_z^{-1}\mathbf{d}_z^{-1}\Phi_z\alpha = d_{z,\mathrm{sm}}^{-1}\alpha.$$

□

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6.4. Derivatives of some homomorphisms

Theorem 6.18. *As* $\mu \to +\infty$,

$$\partial_z(\Phi_z\Psi_z), \partial_{\bar z}(\Phi_z\Psi_z) \asymp_0 \left(\frac{n}{8\mu} - \frac{\mathsf{N}}{4\mu}\right)\left(\frac{\pi}{\mu}\right)^{\mathsf{N}/2-n/4}.$$

Proof. By Equation (6.13),

$$\partial_z(\Phi_z\Psi_z\mathbf{e}_p) = \sum_{q\in\mathcal{Y}_k} \mathbf{e}_q \left(\int_{\widehat{W}_q^-} \hat h_q^- e^{z\hat h_q^-} (\hat\iota_q^-)^* P_{z,\mathrm{sm}} e_{p,z} + \int_{\widehat{W}_q^-} e^{z\hat h_q^-} (\hat\iota_q^-)^* \partial_z(P_{z,\mathrm{sm}} e_{p,z}) \right), \tag{6.16}$$

for every $p \in \mathcal{Y}_k$ $(k = 0,\dots,n)$. We estimate each of these integrals.

Like in the proof of Theorem 6.3, we get, for any $q \neq p$ in $\mathcal{Y}_k$,

$$\int_{\widehat{W}_p^-} \hat h_p^- e^{z\hat h_p^-} (\hat\iota_p^-)^* (P_{z,\mathrm{sm}} - 1) e_{p,z} \asymp_0 0\,, \tag{6.17}$$

$$\int_{\widehat{W}_q^-} \hat h_q^- e^{z\hat h_q^-} (\hat\iota_q^-)^* P_{z,\mathrm{sm}} e_{p,z} \asymp_0 0. \tag{6.18}$$

Moreover, by Proposition 4.1 (iii) and Equations (4.8)–(4.11) and (4.30),

$$\begin{aligned}\int_{\widehat{W}_p^-} \hat h_p^- e^{z\hat h_p^-} (\hat\iota_p^-)^* e_{p,z} &= -\frac{k}{2a_\mu}\left(\int_{-2r}^{2r} \rho(x) e^{-\mu x^2/2}\,dx\right)^{k-1} \int_{-2r}^{2r} \rho(x) x^2 e^{-\mu x^2/2}\,dx \\ &= -\frac{k}{4\mu}\left(\frac{\pi}{\mu}\right)^{\frac{k}{2}-\frac{n}{4}} + O(e^{-c\mu}).\end{aligned} \tag{6.19}$$

On the other hand, by Equation (6.11) and Proposition 4.26,

$$\int_{\widehat{W}_q^-} e^{z\hat h_q^-} (\hat\iota_q^-)^* \partial_z(P_{z,\mathrm{sm}} e_{p,z} - e_{p,z}) \asymp_0 0\,,$$

for all $q \in \mathcal{Y}_k$. In the case $q = p$, by Equation (6.14) and Lemma 4.24,

$$\begin{aligned}\int_{\widehat{W}_p^-} e^{z\hat h_p^-} (\hat\iota_p^-)^* \partial_z e_{p,z} &= \left(\frac{n}{8\mu} + O(e^{-c\mu})\right) \int_{\widehat{W}_p^-} e^{z\hat h_p^-} (\hat\iota_p^-)^* e_{p,z} \\ &= \left(\frac{n}{8\mu} + O(e^{-c\mu})\right)\left(\left(\frac{\pi}{\mu}\right)^{\frac{k}{2}-\frac{n}{4}} + O(e^{-c\mu})\right) \\ &= \frac{n}{8\mu}\left(\frac{\pi}{\mu}\right)^{\frac{k}{2}-\frac{n}{4}} + O(e^{-c\mu}).\end{aligned} \tag{6.20}$$

In the case $q \neq p$, using Lemma 4.24 and arguing again like in the proof of Theorem 6.3, we get

$$\int_{\widehat{W}_q^-} e^{z\hat h_q^-} (\hat\iota_q^-)^* \partial_z e_{p,z} \asymp_0 0 \quad (\mu \to +\infty). \tag{6.21}$$

Now, the result for ∂_z follows from Equations (6.16)–(6.19), (6.20) and (6.21).

If we consider $\partial_{\bar z}$, the proof has to be modified as follows. In the analogue of Equation (6.16), the first term of the right-hand side must be removed. In the analogue of Lemma 4.24, we get $|x_p^-|^2$ instead of $|x_p^+|^2$ by the right-hand side of Equations (4.1) and (4.33). So $\partial_{\bar z}(\Phi_z\Psi_z)$ has the same final expression as $\partial_z(\Phi_z\Psi_z)$ by Equation (6.19). □

Theorem 6.19. *As* $\mu\to+\infty$,

$$\partial_z\big((\Psi_z^*\Psi_z)^{\pm1}\big),\partial_{\bar z}\big((\Psi_z^*\Psi_z)^{\pm1}\big)=O\big(\mu^{-1}\big).$$

Proof. We only show the case of ∂_z. Consider $P_{z,\mathrm{sm}}:E_z\to E_{z,\mathrm{sm}}$, whose adjoint is $P_z:E_{z,\mathrm{sm}}\to E_z$. Then, since $J_z:\mathbf{C}^\bullet\to E_z$ is an isometry,

$$\Psi_z^*\Psi_z=(P_{z,\mathrm{sm}}J_z)^*P_{z,\mathrm{sm}}J_z=J_z^{-1}P_zP_{z,\mathrm{sm}}J_z.$$

It follows that, for every $p\in\mathcal{Y}_k$ $(k=0,\dots,n)$,

$$\Psi_z^*\Psi_z\mathbf{e}_p=\sum_{q\in\mathcal{Y}_k}\langle P_{z,\mathrm{sm}}e_{p,z},e_{q,z}\rangle\mathbf{e}_q.$$

Therefore,

$$\begin{aligned}&\partial_z(\Psi_z^*\Psi_z)\mathbf{e}_p\\&\quad=\sum_{q\in\mathcal{Y}_k}\big(\langle\partial_z(P_{z,\mathrm{sm}})e_{p,z},e_{q,z}\rangle+\langle P_{z,\mathrm{sm}}\partial_z(e_{p,z}),e_{q,z}\rangle+\langle P_{z,\mathrm{sm}}e_{p,z},\partial_{\bar z}(e_{q,z})\rangle\big)\mathbf{e}_q.\end{aligned}$$

Then, by Propositions 4.19 and 4.23, Lemma 4.24 and its analogue for $\partial_{\bar z}$,

$$\begin{aligned}\partial_z(\Psi_z^*\Psi_z)\mathbf{e}_p&=O\big(\mu^{-1}\big)+\Big(\frac{n}{8\mu}-\frac{1}{2}\big\langle|x_p^+|^2e_{p,z},e_{p,z}\big\rangle\Big)\mathbf{e}_p+O\big(e^{-c\mu}\big)\\&=\Big(\frac{n}{8\mu}-\frac{1}{2}\big\langle|x_p^+|^2e_{p,z},e_{p,z}\big\rangle\Big)\mathbf{e}_p+O\big(\mu^{-1}\big).\end{aligned}$$

But, by Equations (4.11) and (4.30),

$$\begin{aligned}\big\langle|x_p|^2e_{p,z},e_{p,z}\big\rangle&=\Big(\int_{-2r}^{2r}\rho(x)^2e^{-\mu x^2}\,dx\Big)^{n-1}(n-k)\int_{-2r}^{2r}y^2\rho(y)^2e^{-\mu y^2}\,dy\\&=\frac{n-k}{2\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{2}}+O\big(e^{-c\mu}\big).\end{aligned}$$

Hence,

$$\partial_z(\Psi_z^*\Psi_z)\mathbf{e}_p=\Big(\frac{n}{8\mu}-\frac{n-k}{4\mu}\Big(\frac{\pi}{\mu}\Big)^{\frac{n}{2}}\Big)\mathbf{e}_p+O\big(\mu^{-1}\big)=O\big(\mu^{-1}\big)\,,$$

yielding the stated expression for $\partial_z\big(\Psi_z^*\Psi_z\big)$.

Now, arguing like in the proof of Equation (4.27) and using Equation (6.12), we get

$$\begin{aligned}\partial_z\big((\Psi_z^*\Psi_z)^{-1}\big)&=-(\Psi_z^*\Psi_z)^{-1}\partial_z(\Psi_z^*\Psi_z)(\Psi_z^*\Psi_z)^{-1}\\&=-\big(1+O\big(e^{-c\mu}\big)\big)O\big(\mu^{-1}\big)\big(1+O\big(e^{-c\mu}\big)\big)=O\big(\mu^{-1}\big).\end{aligned}$$

□

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7. Asymptotics of the large zeta invariant

7.1. Preliminaries on Quillen metrics

7.1.1. Case of a finite-dimensional complex. All vector spaces considered here are over $\mathbb{C}$. For a line λ, its dual λ^* is also denoted by λ^{-1}. For a vector space V of finite dimension, recall that $\det V = \bigwedge^{\dim V} V$. For a graded vector space $V^\bullet$ of finite dimension, let $\det V^\bullet = \bigotimes_k (\det V^k)^{(-1)^k}$.

Now, consider a finite-dimensional cochain complex $(V^\bullet, \partial)$, whose cohomology is denoted by $H^\bullet(V)$. Then there is a canonical isomorphism [38], [8, Section 1 a)]

$$\det V^\bullet \cong \det H^\bullet(V). \tag{7.1}$$

Given a Hermitian metric on $V^\bullet$ so that the homogeneous components V^k are orthogonal one another, the corresponding norm $\|\ \|_{V^\bullet}$ on $V^\bullet$ induces a metric $\|\ \|_{\det V^\bullet}$ on $\det V^\bullet$, which corresponds to a metric $\|\ \|_{\det H^\bullet(V)}$ on $\det H^\bullet(V)$ via Equation (7.1).

On the other hand, consider the induced Laplacian, $\square = (\partial + \partial^*)^2 = \partial\partial^* + \partial^*\partial$, whose kernel is a graded vector subspace $\mathcal{H}^\bullet$. Then finite-dimensional Hodge theory gives an isomorphism $H^\bullet(V) \cong \mathcal{H}^\bullet$, which induces an isomorphism

$$\det H^\bullet(V) \cong \det \mathcal{H}^\bullet. \tag{7.2}$$

The restriction of $\|\ \|_{V^\bullet}$ to $\mathcal{H}^\bullet$ induces a metric $\|\ \|_{\det \mathcal{H}^\bullet}$ on $\det \mathcal{H}^\bullet$, which corresponds to another metric $|\ |_{\det H^\bullet(V)}$ on $\det H^\bullet(V)$ via Equation (7.2).

Let $\square'$ denote the restriction $\square : \operatorname{im}\square \to \operatorname{im}\square$. For $s \in \mathbb{C}$, let

$$\theta(s) = \theta(s, \square) = -\operatorname{Str}(\mathsf{N}(\square')^{-s}). \tag{7.3}$$

This defines a holomorphic function on $\mathbb{C}$. Then the above metrics on $\det H^\bullet(V)$ satisfy [8, Proposition 1.5], [10, Theorem 1.1], [11, Theorem 1.4]

$$\|\ \|_{\det H^\bullet(V)} = |\ |_{\det H^\bullet(V)} e^{\theta'(0)/2}. \tag{7.4}$$

If $H^\bullet(V) = 0$, then $\det H^\bullet(V) \equiv \mathbb{C}$ is canonically generated by 1, and we have $\|1\|_{\det H^\bullet(V)} = e^{\theta'(0)/2}$. Using the orthogonal projection $\Pi^1 : V \to \operatorname{im}\partial$, we can write Equation (7.3) as

$$\theta(s) = -\operatorname{Str}\left((\square')^{-s}\Pi^1\right). \tag{7.5}$$

Let $(\widetilde{V}^\bullet, \tilde{\partial})$ be another finite-dimensional cochain complex, endowed with a Hermitian metric so that the homogeneous components are orthogonal to each other, and let $\phi : (V, \partial) \to (\widetilde{V}^\bullet, \tilde{\partial})$ be an isomorphism of cochain complexes, which may not be unitary. Then (see the proof of [11, Theorem 6.17])

$$\log\left(\frac{\|\ \|_{\det H^\bullet(\widetilde{V})}}{\|\ \|_{\det H^\bullet(V)}}\right)^2 = \operatorname{Str}(\log(\phi^*\phi)). \tag{7.6}$$

7.1.2. Case of an elliptic complex. Some of the concepts of Section 7.1.1 extend to the case where $V^\bullet = C^\infty(M; E^\bullet)$, for some graded Hermitian vector bundle $E^\bullet$ over M, and ∂ is an elliptic differential complex of order one. Then $\det H^\bullet(V)$ is defined because

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$\dim H^\bullet(V) < \infty$. Moreover, Hodge theory for the Laplacian $\square$ gives the isomorphism (7.2). Thus, at least the norm $|\ |_{\det H^\bullet(V)}$ is defined in this setting. Now, the expression (7.3) only defines $\theta(s) = \theta(s, \square)$ when $\Re s > n/2$, but it has a meromorphic extension to $\mathbb{C}$, denoted in the same way; indeed, Equation (7.3) becomes

$$\theta(s) = \theta(s, \square) = -\zeta(s, \square, \mathsf{Nw}) ,$$

for $\Re s > n/2$, and therefore this equality also holds for the meromorphic extensions. Furthermore, $\theta(s)$ is smooth at $s = 0$ [66], and $\theta'(0)$ can be considered as a renormalized version of the supertrace of the operator $\mathsf{N}\log(\square')$, which is not of trace class. Thus, the right-hand side of Equation (7.4) is defined in this way and plays the role of an analytic version of the metric $\|\ \|_{\det H^\bullet(V)}$, which is not directly defined. This kind of metrics were introduced by D. Quillen [60] for the case of the Dolbeault complex. The expression (7.5) also holds in this case for $\Re s \gg 0$; in fact, it becomes

$$\theta(s) = -\zeta\big(s, \square, \Pi^1 \mathsf{w}\big) ,$$

where this zeta function can be shown to define a meromorphic function on $\mathbb{C}$, even though Π^1 is not a differential operator, and this equality holds as meromorphic functions.

7.1.3. Reidemeister, Milnor and Ray–Singer metrics. Let F be a flat vector bundle over M, defined by a representation ρ of $\pi_1 M$, and let ∇^F denote its covariant derivative. Consider a smooth triangulation K of M and the corresponding cochain complex $C^\bullet(K, F)$ with coefficients in F, whose cohomology is isomorphic to $H^\bullet(M, F)$ via the quasi-isomorphism

$$\Omega(M; F) \to C^\bullet(K, F) = C_\bullet(K, F^*)^*$$

defined by integration of differential forms on smooth simplices. Given a Hermitian structure g^F on F, its restriction to the fibers over the barycenters of the simplices induces a metric on $C^\bullet(K, F)$, and the concepts of Section 7.1.1 can be applied. In this case, the left-hand side of Equation (7.4) is called the *Reidemeister metric*, denoted by $\|\ \|^{\mathrm{R}}_{\det H^\bullet(M,F)}$.

If $\nabla^F g^F = 0$ (ρ is unitary) and $H^\bullet(M, F) = 0$, then the Reidemeister torsion $\tau_M(\rho)$ is defined using K, and it is a topological invariant of M [30, 62, 23]. Moreover, $\tau_M(\rho) = \|1\|^{\mathrm{R}}_{\det H^\bullet(M,F)}$ is the exponential factor of the right-hand side of Equation (7.4) [61, Proposition 1.7]. If we only assume $\nabla^F g^F = 0$, then $\|\ \|^{\mathrm{R}}_{\det H^\bullet(M,F)}$ is still a topological invariant of M.

Next, given a vector field X on M satisfying (b), $H^\bullet(M, F)$ is also isomorphic to the cohomology of $(C^\bullet(-X, W^-, F), \mathbf{d}^F)$ via the quasi-isomorphism

$$\Phi^{-X,F} : \Omega(M, F) \to C^\bullet(-X, W^-, F) = C_\bullet(-X, W^-, F^*)^* .$$

This complex has a metric induced by g^F, like in Section 6.2.4, and the concepts of Section 7.1.1 can be also applied. In this case, the left-hand side of Equation (7.4) is called the *Milnor metric*, denoted by $\|\ \|^{\mathrm{M},-X}_{\det H^\bullet(M,F)}$, and the metric factor of the right-hand

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side of Equation (7.4) is denoted by $|\ |^{\mathrm{M},-X}_{\det H^\bullet(M,F)}$. If $\nabla^F g^F=0$, then $\|\ \|^{\mathrm{M},-X}_{\det H^\bullet(M,F)} = \|\ \|^{\mathrm{R}}_{\det H^\bullet(M,F)}$ [50, Theorem 9.3].

Finally, the concepts of Section 7.1.2 can be applied to $(\Omega(M,F),d^F)$, whose cohomology is again $H^\bullet(M,F)$. In this case, the right-hand side of Equation (7.4) is called the *Ray–Singer metric*, denoted by $\|\ \|^{\mathrm{RS}}_{\det H^\bullet(M,F)}$, and the metric factor of the right-hand side of Equation (7.4) is denoted by $|\ |^{\mathrm{RS}}_{\det H^\bullet(M,F)}$. If $H^\bullet(M,F)=0$, then the exponential factor of the right-hand side of Equation (7.4) is called the *analytic torsion* or *Ray–Singer torsion*, denoted by $T_M(\rho)$. These concepts were introduced by Ray and Singer [61], who conjectured that $T_M(\rho)=\tau_M(\rho)$ if $\nabla^F g^F=0$ and $H^\bullet(M,F)=0$. Independent proofs of this conjecture were given by Cheeger [21] and Müller [54]. This conjecture still holds true if the induced Hermitian structure $g^{\det F}$ on $\det F$ is flat, as shown at the same time by Bismut and Zhang [10] and Müller [54]. Actually, in [10], Bismut and Zhang reformulated the conjecture in the form $\|\ \|^{\mathrm{RS}}_{\det H^\bullet(M,F)} = \|\ \|^{\mathrm{R}}_{\det H^\bullet(M,F)}$. Moreover, they also considered the case where $g^{\det F}$ is not assumed to be flat [10, 11], extending the above results by introducing an additional term. The first ingredient of this extra term is the 1-form

$$\theta(F,g^F)=\operatorname{tr}\left((g^F)^{-1}\nabla^F g^F\right), \tag{7.7}$$

which vanishes if and only if $g^{\det F}$ is flat. Moreover, $\theta(F,g^F)$ is closed and its cohomology class of $\theta(F,g^F)$ is independent of the choice of g^F [10, Proposition 4.6]; this class measures the obstruction to the existence of a flat Hermitian structure on $\det F$.

Let $e(M,\nabla^M)$ be the representative of the Euler class of M given by the Chern–Weil theory using g^M; it belongs to $\Omega^n(M,o(M))$ because M may not be oriented. Let $\psi(M,\nabla^M)$ be the current of degree $n-1$ on TM constructed in [44] (see also [9, Section 3], [10, Section 3], [19, Section 2], [20, Section 4]). Identify the image of the zero section of TM with M, and identify the conormal bundle of M in TM with T^*M. Let δ_M be the current on TM defined by integration on M, and let $\pi:TM\to M$ be the vector bundle projection.

Proposition 7.1 (Bismut–Zhang [10, Theorem 3.7]). *The following holds:*

(i) *For any smooth function $\lambda:TM\to\mathbb{R}^\pm$, under the mapping $v\mapsto\lambda v$, $\psi(M,\nabla^M)$ is changed into $(\pm1)^n\psi(M,\nabla^M)$.*

(ii) *The current $\psi(M,\nabla^M)$ is locally integrable, and its wave front set is contained in T^*M. Thus, $\psi(M,\nabla^M)$ is smooth on $TM\setminus M$.*

(iii) *The restriction of $-\psi(M,\nabla^M)$ to the fibers of $TM\setminus M$ coincides with the solid angle defined by g^M.*

(iv) *We have*

$$d\psi(M,\nabla^M)=\pi^*e(M,\nabla^M)-\delta_M.$$

Remark 7.2. In Proposition 7.1, observe that (i) and (iv) are compatible because $e(M,\nabla^M)=0$ if n is odd. By (ii)–(iv), the restriction of $\psi(M,\nabla^M)$ to $TM\setminus M$ is induced by a smooth differential form on the sphere bundle which transgresses $e(M,\nabla^M)$ (such a differential form was already defined and used in [22]).

Theorem 7.3 (Bismut–Zhang [10, Theorem 0.2], [11, Theorem 0.2]). *We have*

$$\log\left(\frac{\|\ \|^{\mathrm{RS}}_{\det H^\bullet(M,F)}}{\|\ \|^{\mathrm{M},-X}_{\det H^\bullet(M,F)}}\right)^2 = -\int_M \theta(F,g^F)\wedge(-X)^*\psi(M,\nabla^M).$$

Remark 7.4. By (b), $X=-\operatorname{grad}_{g'}h$ for some Morse function h and some Riemannian metric g' on M, which may not be the given metric g^M. If we fix h, the right-hand side of the equality in Theorem 7.3 is independent of the choice of X satisfying $X=-\operatorname{grad}_{g'}h$ for some g' [10, Proposition 6.1].

Theorem 7.3 will be applied to the case of the flat complex line bundle $\mathcal{L}^z$ with a Hermitian structure $g^{\mathcal{L}^z}$ (Section 2.1.2). By Equations (2.13) and (7.7),

$$\theta(\mathcal{L}^z,g^{\mathcal{L}^z}) = -2\mu\eta. \tag{7.8}$$

7.2. Asymptotics of the large zeta invariant

We prove Theorem 1.2 (i) here. With the notation of Section 7.1.2, consider the meromorphic function $\theta(s,z)=\theta(s,\Delta_z)$, also defined in Equation (5.9), as well as its components $\theta_{\mathrm{sm/la}}(s,z)$ defined in Equation (5.10). Consider also the current $\psi(M,\nabla^M)$ of degree $n-1$ on TM (Section 7.1.3). By Proposition 7.1 (i),

$$-\mathbf{z}_{\mathrm{la}}(-\eta) = (-1)^n\mathbf{z}_{\mathrm{la}}(\eta). \tag{7.9}$$

Notation 7.5. Let $\asymp_1$ be defined like $\asymp_0$ in Notation 6.2, using $O(|\mu|^{-1})$ instead of $O(e^{-c|\mu|})$.

Take some Morse function h on M such that $Xh<0$ on $M\setminus\mathcal{Y}$, and h is in standard form with respect to X. Then $X=-\operatorname{grad}_{g'}h$ for some Riemannian metric g' (Section 6.1.3), which may not be the given metric g. Consider the flat complex line bundle $\mathcal{L}_{z\eta-dh}$ with the Hermitian structure $g^{\mathcal{L}_{z\eta-dh}}$ (Section 2.1.2). Note that $\mathbf{d}_{dh}^{\mathcal{L}_{z\eta-dh}}\equiv\mathbf{d}_{z\eta}$ on $C^\bullet(X,W^-,\mathcal{L}_{z\eta-dh})\equiv\mathbf{C}^\bullet(X)$. So, by Equation (7.8), Theorem 7.3 and Remark 7.4,

$$\log\frac{\|\ \|^{\mathrm{RS}}_{\det H_z^\bullet(M)}}{\|\ \|^{\mathrm{M},-X}_{\det H_z^\bullet(M)}} = \int_M(\mu\eta-dh)\wedge(-X)^*\psi(M,\nabla^M)\,, \tag{7.10}$$

where $H_z^\bullet(M)=H_{z\eta}^\bullet(M)$. With the notation of Section 7.1.3, let

$$\|\ \|^{\mathrm{RS,sm}}_{\det H_z^\bullet(M)} = |\ |^{\mathrm{RS}}_{\det H_z^\bullet(M)}e^{\theta'_{\mathrm{sm}}(0,z)/2}.$$

By Equation (7.4),

$$\log\frac{\|\ \|^{\mathrm{RS}}_{\det H_z^\bullet(M)}}{\|\ \|^{\mathrm{M},-X}_{\det H_z^\bullet(M)}} = \log\frac{\|\ \|^{\mathrm{RS,sm}}_{\det H_z^\bullet(M)}}{\|\ \|^{\mathrm{M},-X}_{\det H_z^\bullet(M)}} + \frac{\theta'_{\mathrm{la}}(0,z)}{2}. \tag{7.11}$$

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By Equation (7.6) and Corollary 6.4, for $\mu \gg 0$,

$$\log\left(\frac{\|\ \|^{\mathrm{RS,sm}}_{\det H^\bullet_z(M)}}{\|\ \|^{\mathrm{M},-X}_{\det H^\bullet_z(M)}}\right)^2 = -\operatorname{Str}(\log(\Phi_z^*\Phi_z)) = -\operatorname{Str}\left(\log\left(\Psi_z^{-1}\Phi_z^*\Phi_z\Psi_z\right)\right)$$
$$= -\operatorname{Str}\left(\log\left((\Psi_z^*\Psi_z)^{-1}(\Phi_z\Psi_z)^*\Phi_z\Psi_z\right)\right). \tag{7.12}$$

From Equation (6.12) and Theorems 6.3, 6.18 and 6.19, we obtain

$$\left((\Psi_z^*\Psi_z)^{-1}(\Phi_z\Psi_z)^*\Phi_z\Psi_z\right)^{-1} = \left(\frac{\pi}{\mu}\right)^{\frac{n}{2}-\mathsf{N}} + O\left(e^{-c\mu}\right),$$
$$\partial_z\left((\Psi_z^*\Psi_z)^{-1}(\Phi_z\Psi_z)^*\Phi_z\Psi_z\right) = \partial_z\left((\Psi_z^*\Psi_z)^{-1}\right)(\Phi_z\Psi_z)^*\Phi_z\Psi_z + (\Psi_z^*\Psi_z)^{-1}(\partial_{\bar z}(\Phi_z\Psi_z))^*\Phi_z\Psi_z$$
$$+ (\Psi_z^*\Psi_z)^{-1}(\Phi_z\Psi_z)^*\partial_z(\Phi_z\Psi_z)$$
$$\asymp_0 \left(O\left(\mu^{-1}\right) + \left(\frac{n}{4\mu} - \frac{\mathsf{N}}{2\mu}\right)\right)\left(\frac{\pi}{\mu}\right)^{\mathsf{N}-\frac{n}{2}}.$$

So

$$\partial_z \operatorname{Str}\left(\log\left((\Psi_z^*\Psi_z)^{-1}(\Phi_z\Psi_z)^*\Phi_z\Psi_z\right)\right)$$
$$= \operatorname{Str}\left(\left((\Psi_z^*\Psi_z)^{-1}(\Phi_z\Psi_z)^*\Phi_z\Psi_z\right)^{-1}\partial_z\left((\Psi_z^*\Psi_z)^{-1}(\Phi_z\Psi_z)^*\Phi_z\Psi_z\right)\right)$$
$$= O\left(\mu^{-1}\right) + \operatorname{Str}\left(\frac{n}{4\mu} - \frac{\mathsf{N}}{2\mu}\right) + O\left(e^{-c\mu}\right) = O\left(\mu^{-1}\right).$$

Then, by Equation (7.12),

$$\partial_z \log \frac{\|\ \|^{\mathrm{RS,sm}}_{\det H^\bullet_z(M)}}{\|\ \|^{\mathrm{M},-X}_{\det H^\bullet_z(M)}} = O\left(\mu^{-1}\right). \tag{7.13}$$

By taking the derivative with respect to z of both sides of Equation (7.10) and using Equations (7.11) and (7.13) and Corollary 5.10, we get $\zeta_{\mathrm{la}}(1,z) \asymp_1 \mathbf{z}_{\mathrm{la}}$, as stated in Theorem 1.2 (i).

Remark 7.6. In the case where $\eta = dh$, Theorem 1.2 (i) agrees with Theorem 5.16. In fact, by Proposition 7.1 (iv), Theorem 1.2 (i) and the Stokes formula,

$$\zeta_{\mathrm{la}}(1,z) \asymp_1 -\int_M h(-X)^* d\psi(M,\nabla^M) = -\int_M h(-X)^*(\pi^* e(M,\nabla^M) - \delta_M)$$
$$= -\int_M h\, e(M,\nabla^M) + \sum_{p\in\mathcal{Y}} (-1)^{\mathrm{ind}(p)} h(p).$$

8. Asymptotics of the small zeta invariant

8.1. Condition on the integrals along instantons

Let

$$\mathcal{M}_p = \mathcal{M}_p(\eta, X) = -\max\{\eta(\gamma) \mid \gamma \in \mathcal{T}_p^1\} \quad (p \in \mathcal{Y}_+),$$
$$\mathcal{M}_k = \mathcal{M}_k(\eta, X) = \min_{p\in\mathcal{Y}_k} \mathcal{M}_p \quad (k = 1, \dots, n).$$

Thus, (d) means that $\mathcal{M}_p = \mathcal{M}_k$ for all $k = 1,\dots,n$ and $p \in \mathcal{Y}_k$. The following result will be proved in Appendix A.

Theorem 8.1. *For every $\xi \in H^1(M,\mathbb{R})$ and numbers $a_n \geq \cdots \geq a_1 \gg 0$ or $a_1 \geq \cdots \geq a_n \gg 0$, there is some $\eta \in \xi$, satisfying (a) and (c) with the given X and some metric g, such that $\mathcal{M}_p(\eta,X) = a_k$ for all $k = 1,\dots,n$ and $p \in \mathcal{Y}_k$.*

Remark 8.2. If $\xi \neq 0$, for $p \in \mathcal{Y}_k$, $q \in \mathcal{Y}_{k-1}$ and $\gamma,\delta \in \mathcal{T}(p,q) \subset \mathcal{T}_p^1$, the period $\langle \xi, \bar{\gamma}\bar{\delta}^{-1}\rangle = \eta(\gamma) - \eta(\delta)$ may not be zero. Hence, it may not be possible to get $\eta(\gamma) = -a_k$ for all $\gamma \in \mathcal{T}_p^1$, contrary to the case where $\xi = 0$.

From now on, we assume η satisfies (d), besides (a) and (c). By Theorem 8.1, this is possible for any prescription of the class $\xi = [\eta] \in H^1(M,\mathbb{R})$. Let $a_k = \mathcal{M}_k(\eta,X)$ ($k = 1,\dots,n$). Then $-\eta$ also satisfies (a), (c) and (d) with $-X$ and g, and $\mathcal{M}_k(-\eta,-X) = a_{n-k+1}$. So, if M is oriented, by Corollaries 4.15 and 4.16,

$$-\mathbf{z}_{\mathrm{sm}}(-\eta) = -\sum_{k=1}^{n} (-1)^k \left(1 - e^{a_{n-k+1}}\right) m^1_{n-k+1}. \tag{8.1}$$

8.2. Asymptotics of the perturbed Morse operators

Consider the notation of Section 6.2.4. By Equation (6.8),

$$\mathbf{d}_{z,k-1} = e^{-a_k z}(\mathbf{d}'_{k-1} + \mathbf{d}''_{z,k-1}), \tag{8.2}$$

for $k = 1,\dots,n$, where

$$\mathbf{d}'_{k-1}\mathbf{e}_q = \sum_{p\in\mathcal{Y}_k,\ \gamma\in\mathcal{T}(p,q),\ \eta(\gamma)=-a_k} \epsilon(\gamma)\mathbf{e}_p\,, \tag{8.3}$$

$$\mathbf{d}''_{z,k-1}\mathbf{e}_q = \sum_{p\in\mathcal{Y}_k,\ \gamma\in\mathcal{T}(p,q),\ \eta(\gamma)<-a_k} e^{z(a_k+\eta(\gamma))}\epsilon(\gamma)\mathbf{e}_p\,, \tag{8.4}$$

for $q \in \mathcal{Y}_{k-1}$. Observe that

$$e^{a_k z}\mathbf{d}_{z,k-1} = \mathbf{d}'_{k-1} + O(e^{-c\mu}) \quad (\mu \to +\infty). \tag{8.5}$$

So

$$\mathbf{d}'_k\mathbf{d}'_{k-1} = \lim_{\mu\to+\infty} e^{(a_{k+1}+a_k)z}\mathbf{d}_{z,k}\mathbf{d}_{z,k-1} = 0.$$

Hence, the operator $\mathbf{d}' = \sum_k \mathbf{d}'_k$ on $\mathbf{C}^\bullet$ satisfies $(\mathbf{d}')^2 = 0$. Taking adjoints in Equations (8.2)–(8.4) or using Equation (6.9), we also get

$$\boldsymbol{\delta}_{z,k} = e^{-a_k\bar{z}}(\boldsymbol{\delta}'_k + \boldsymbol{\delta}''_{z,k})\,, \tag{8.6}$$

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for $k = 1, \dots, n$, where

$$\boldsymbol{\delta}'_k \mathbf{e}_p = \sum_{q \in \mathcal{Y}_{k-1},\ \gamma \in \mathcal{T}(p,q),\ \eta(\gamma) = -a_k} \epsilon(\gamma)\, \mathbf{e}_q, \tag{8.7}$$

$$\boldsymbol{\delta}''_{z,k} \mathbf{e}_p = \sum_{q \in \mathcal{Y}_{k-1},\ \gamma \in \mathcal{T}(p,q),\ \eta(\gamma) = -a_k} e^{\bar{z}(a_k + \eta(\gamma))} \epsilon(\gamma) \mathbf{e}_q\,, \tag{8.8}$$

for $p \in \mathcal{Y}_k$. Moreover, Equation (8.5) yields

$$e^{a_k \bar{z}} \boldsymbol{\delta}_{z,k} = \boldsymbol{\delta}'_k + O(e^{-c\mu}) \quad (\mu \to +\infty). \tag{8.9}$$

Let $\boldsymbol{\delta}' = \sum_k \boldsymbol{\delta}'_k = (\mathbf{d}')^*$, which satisfies $(\boldsymbol{\delta}')^2 = 0$, and let

$$\mathbf{D}' = \mathbf{d}' + \boldsymbol{\delta}'\,, \quad \boldsymbol{\Delta}' = (\mathbf{D}')^2 = \mathbf{d}'\boldsymbol{\delta}' + \boldsymbol{\delta}'\mathbf{d}'.$$

We have

$$\mathbf{C}^\bullet = \ker \boldsymbol{\Delta}' \oplus \operatorname{im} \mathbf{d}' \oplus \operatorname{im} \boldsymbol{\delta}'\,,$$
$$\operatorname{im} \boldsymbol{\Delta}' = \operatorname{im} \mathbf{D}' = \operatorname{im} \mathbf{d}' \oplus \operatorname{im} \boldsymbol{\delta}'\,, \quad \ker \boldsymbol{\Delta}' = \ker \mathbf{D}' = \ker \mathbf{d}' \cap \ker \boldsymbol{\delta}'.$$

The orthogonal projections of $\mathbf{C}^\bullet$ to $\ker \boldsymbol{\Delta}'$, $\operatorname{im} \mathbf{d}'$ and $\operatorname{im} \boldsymbol{\delta}'$ are denoted by $\boldsymbol{\Pi}' = \boldsymbol{\Pi}'^0$, $\boldsymbol{\Pi}'^1$ and $\boldsymbol{\Pi}'^2$, respectively. Like in Sections 2.1.2 and 6.2.6, the composition $(\mathbf{d}')^{-1}\boldsymbol{\Pi}'^1$ is defined on $\mathbf{C}^\bullet$. From Equations (8.5) and (8.9), we easily get that, as $\mu \to +\infty$,

$$\boldsymbol{\Pi}^j_{z,k} = \boldsymbol{\Pi}'^j_k + O(e^{-c\mu}) \quad (j = 0,1,2)\,, \tag{8.10}$$

$$e^{-a_k z} (\mathbf{d}_{z,k-1})^{-1} \boldsymbol{\Pi}^1_{z,k} = (\mathbf{d}'_{k-1})^{-1} \boldsymbol{\Pi}'^1_k + O(e^{-c\mu}). \tag{8.11}$$

By Equations (8.5) and (8.9), on $\operatorname{im} \boldsymbol{\delta}_{z,k} \oplus \operatorname{im} \mathbf{d}_{z,k-1}$,

$$\boldsymbol{\Delta}_z = e^{-2a_k\mu} \boldsymbol{\Delta}' + O(e^{-(2a_k + c)\mu}) \quad (\mu \to +\infty). \tag{8.12}$$

Proposition 8.3. *For $k = 0, \dots, n$ and $\mu \gg 0$, the spectrum of $\boldsymbol{\Delta}_z$ on $\operatorname{im} \boldsymbol{\delta}_{z,k} \oplus \operatorname{im} \mathbf{d}_{z,k-1}$ is contained in an interval of the form*

$$\left[C e^{-2a_k\mu}, C' e^{-2a_k\mu}\right] \quad (C' \geq C).$$

Proof. The positive eigenvalues of $\boldsymbol{\Delta}'$ are contained in an interval $[C_0, C'_0]$ ($C'_0 \geq C_0 > 0$). By Equation (8.12), for $\mu \gg 0$ and $\mathbf{e} \in \operatorname{im} \boldsymbol{\delta}_{z,k} \oplus \operatorname{im} \mathbf{d}_{z,k-1}$,

$$\langle \boldsymbol{\Delta}_z \mathbf{e}, \mathbf{e} \rangle \geq e^{2a_k\mu} \langle \boldsymbol{\Delta}' \mathbf{e}, \mathbf{e} \rangle - C_1 e^{-(2a_k + c)\mu} \|\mathbf{e}\|^2 \geq \left(C_0 e^{-2a_k\mu} - C_1 e^{-(2a_k + c)\mu}\right) \|\mathbf{e}\|^2\,,$$
$$\langle \boldsymbol{\Delta}_z \mathbf{e}, \mathbf{e} \rangle \leq e^{2a_k\mu} \langle \boldsymbol{\Delta}' \mathbf{e}, \mathbf{e} \rangle + C_1 e^{-(2a_k + c)\mu} \|\mathbf{e}\|^2 \leq \left(C'_0 e^{-2a_k\mu} + C_1 e^{-(2a_k + c)\mu}\right) \|\mathbf{e}\|^2.$$

Then result follows taking $0 < C < C_0$ and $C' > C'_0$. □

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8.3. Estimates of the nonzero small spectrum

Theorem 8.4. *If $\mu \gg 0$, the spectrum of $\Delta_{z,\mathrm{sm}}$ on $\operatorname{im}\delta_{z,\mathrm{sm},k} \oplus \operatorname{im} d_{z,\mathrm{sm},k-1}$ is contained in an interval of the form*

$$[C\mu e^{-2a_k\mu}, C'\mu e^{-2a_k\mu}] \quad (C' \geq C).$$

Proof. By the commutativity of the diagram of Equation (2.7), for every eigenvalue λ of $\Delta_{z,\mathrm{sm}}$ on $\operatorname{im}\delta_{z,\mathrm{sm},k} \oplus \operatorname{im} d_{z,\mathrm{sm},k-1}$, there are normalized λ-eigenforms, $e \in \operatorname{im}\delta_{z,\mathrm{sm},k}$ and $e' \in \operatorname{im} d_{z,\mathrm{sm},k-1}$ so that $d_z e = \lambda^{1/2} e'$ and $\delta_z e' = \lambda^{1/2} e$. So the maximum and minimum of the spectrum of $\Delta_{z,\mathrm{sm}}$ on $\operatorname{im}\delta_{z,\mathrm{sm},k} \oplus \operatorname{im} d_{z,\mathrm{sm},k-1}$ is $\|d_{z,\mathrm{sm},k-1}\|^2$ and $\|d_{z,\mathrm{sm},k-1}^{-1}\Pi^1_{z,\mathrm{sm},k}\|^{-2}$, respectively. Similarly, the maximum and minimum of the spectrum of $\boldsymbol{\Delta}_z$ on $\operatorname{im}\boldsymbol{\delta}_{z,k} \oplus \operatorname{im}\mathbf{d}_{z,k-1}$ is $\|\mathbf{d}_{z,k-1}\|^2$ and $\|\mathbf{d}_{z,k-1}^{-1}\boldsymbol{\Pi}^1_{z,k}\|^{-2}$, respectively. Then the result follows from Corollaries 6.9, 6.14 and 6.17 and Proposition 8.3:

$$\begin{aligned}
\|d_{z,\mathrm{sm},k-1}\|^2 &\leq \|\Phi_{z,k}^{-1}\|^2 \|\mathbf{d}_{z,k-1}\|^2 \|\Phi_{z,k-1}\|^2 \\
&\leq \left(\left(\frac{\mu}{\pi}\right)^{k-\frac{n}{2}} + O(e^{-c\mu})\right) C_0' e^{-2a_k\mu} \left(\left(\frac{\pi}{\mu}\right)^{k-1-\frac{n}{2}} + O(e^{-c\mu})\right) \\
&\leq C'\mu e^{-2a_k\mu}, \\
\|d_{z,\mathrm{sm},k-1}^{-1}\Pi^1_{z,\mathrm{sm},k-1}\|^{-2} &\geq \|\Phi_{z,k-1}^{-1}\|^{-2} \|\mathbf{d}_{z,k-1}^{-1}\boldsymbol{\Pi}^1_{z,k}\|^{-2} \|\Phi_{z,k}\|^{-2} \\
&\geq \left(\left(\frac{\pi}{\mu}\right)^{k-1-\frac{n}{2}} + O(e^{-c\mu})\right) C_0 e^{-2a_k\mu} \left(\left(\frac{\mu}{\pi}\right)^{k-\frac{n}{2}} + O(e^{-c\mu})\right) \\
&\geq C\mu e^{-2a_k\mu}.
\end{aligned}$$

□

8.4. Asymptotics of the small zeta invariant

Theorem 1.2 (ii) is proved here.

Theorem 8.5. *As $\mu \to +\infty$,*

$$\eta \wedge d_z^{-1}\Pi^1_{z,\mathrm{sm},k} \asymp_1 (1 - e^{a_k})\Pi^1_{z,\mathrm{sm},k}.$$

Proof. Consider the notation of Sections 6.3 and 8.2. By Corollaries 6.13 and 6.16, for $\mu \gg 0$,

$$\Pi^1_{z,\mathrm{sm}} \asymp_0 \widetilde{\Psi}_z \widetilde{\boldsymbol{\Pi}}^1_z \Phi_{z,\mathrm{sm}} = \widetilde{\Psi}_z \boldsymbol{\Pi}^1_z \Phi_{z,\mathrm{sm}}. \tag{8.13}$$

For brevity, let $S_z = \Phi_z \widetilde{\Psi}_{z-1}$ and $T_z = \Phi_{z-1} P_{z-1,\mathrm{sm}} \widetilde{\Psi}_z$ on $\mathbf{C}^\bullet$. By Corollaries 6.9 and 6.13,

$$S_z, T_z \asymp_1 1. \tag{8.14}$$

Moreover, by Proposition 4.19 and Corollary 6.7, and the definitions of Ψ_z and $\widetilde{\Psi}_z$, considered as maps $\mathbf{C}^\bullet \to L^2(M;\Lambda)$, we get

$$\begin{aligned}
\widetilde{\Psi}_z S_z = \widetilde{\Psi}_z \Phi_z P_{z-1,\mathrm{sm}} \widetilde{\Psi}_{z-1} &\asymp_1 \widetilde{\Psi}_z \Phi_z P_{z,\mathrm{sm}} \widetilde{\Psi}_{z-1} \\
&\asymp_1 P_{z,\mathrm{sm}} \widetilde{\Psi}_{z-1} \asymp_1 P_{z-1,\mathrm{sm}} \widetilde{\Psi}_{z-1} = \widetilde{\Psi}_{z-1}.
\end{aligned} \tag{8.15}$$

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By Equations (8.5), (8.10), (8.11), and (8.13)–(8.15); Proposition 8.3; and Corollaries 4.20, 6.6, 6.7, 6.9, 6.11, 6.13 and 6.15 to 6.17; and Theorem 8.4,

$$\begin{aligned}
e^{a_k}\Pi^1_{z,\mathrm{sm},k} &\asymp_0 e^{a_k}\widetilde{\Psi}_z\mathbf{\Pi}^1_{z,k}\Phi_{z,\mathrm{sm}} \asymp_1 e^{a_k}\widetilde{\Psi}_z\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}}\\
&= e^{a_k}\widetilde{\Psi}_z\mathbf{d}'_{k-1}(\mathbf{d}'_{k-1})^{-1}\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}}\\
&\asymp_1 e^{a_k}\widetilde{\Psi}_zS_z\mathbf{d}'_{k-1}T_z(\mathbf{d}'_{k-1})^{-1}\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}}\\
&\asymp_1 e^{a_k}\widetilde{\Psi}_{z-1}\mathbf{d}'_{k-1}T_z(\mathbf{d}'_{k-1})^{-1}\mathbf{\Pi}'^1_k\Phi_{z,\mathrm{sm}}\\
&\asymp_1 e^{a_k}\widetilde{\Psi}_{z-1}e^{a_k(z-1)}\mathbf{d}_{z-1,k-1}T_ze^{-a_kz}\mathbf{d}^{-1}_{z,k-1}\mathbf{\Pi}^1_{z,k}\Phi_{z,\mathrm{sm}}\\
&= \widetilde{\Psi}_{z-1}\mathbf{d}_{z-1,k-1}T_z\mathbf{d}^{-1}_{z,k-1}\mathbf{\Pi}^1_{z,k}\Phi_{z,\mathrm{sm}}\\
&\asymp_0 \widetilde{\Psi}_{z-1}\mathbf{d}_{z-1,k-1}T_z\widetilde{\mathbf{\Pi}}^2_{z,k-1}\mathbf{d}^{-1}_{z,k-1}\widetilde{\mathbf{\Pi}}^1_{z,k}\Phi_{z,\mathrm{sm}}\\
&\asymp_0 \widetilde{\Psi}_{z-1}\mathbf{d}_{z-1,k-1}\Phi_{z-1}P_{z-1,\mathrm{sm}}\Pi^2_{z,k-1}\widetilde{\Psi}_z\mathbf{d}^{-1}_{z,k-1}\Phi_z\Pi^1_{z,\mathrm{sm}}\\
&\asymp_0 \widetilde{\Psi}_{z-1}\mathbf{d}_{z-1,k-1}\Phi_{z-1}P_{z-1,\mathrm{sm}}\Pi^2_{z,k-1}\Phi_z^{-1}\mathbf{d}^{-1}_{z,k-1}\Phi_z\Pi^1_{z,\mathrm{sm}}\\
&= \widetilde{\Psi}_{z-1}\Phi_{z-1}d_{z-1,\mathrm{sm},k-1}d^{-1}_{z,\mathrm{sm},k-1}\Pi^1_{z,\mathrm{sm},k} \asymp_0 d_{z-1}d_z^{-1}\Pi^1_{z,\mathrm{sm},k}.
\end{aligned}$$

Therefore,

$$\eta\wedge d_z^{-1}\Pi^1_{z,\mathrm{sm},k} = (d_z-d_{z-1})d_z^{-1}\Pi^1_{z,\mathrm{sm},k} \asymp_1 (1-e^{a_k})\Pi^1_{z,\mathrm{sm},k}.$$

□

Theorem 1.2 (ii) follows from Corollaries 4.9 and 5.2 and Theorem 8.5.

Remark 8.6. Theorem 1.2 (ii) agrees with Corollaries 5.13 to 5.15 by Equation (8.1).

9. Prescription of the asymptotics of the zeta invariant

We prove Theorem 1.3 here. By Theorem 8.1, given $a\gg 0$, there is some $\eta_0\in\xi$ and some metric g satisfying (a) and (d) with the given X and so that $\mathcal{M}_k(\eta_0,X)=a$ for all $k=1,\dots,n$. Using the notation of Section 4.1, we are going to modify η_0 only in every U_p for $p\in\mathcal{Y}_0\cup\mathcal{Y}_n$.

Fix any $\epsilon>0$ such that, for every $p\in\mathcal{Y}_0\cup\mathcal{Y}_n$, the open ball $B(p,3\epsilon)$ is contained in U_p. Let

$$V=\bigcup_{p\in\mathcal{Y}_0\cup\mathcal{Y}_n}B(p,\epsilon)\,,\quad V'=\bigcup_{p\in\mathcal{Y}_0\cup\mathcal{Y}_n}B(p,2\epsilon).$$

Take a smooth function $\sigma:[0,3\epsilon]\to[0,1]$ so that

$$\sigma'\le 0\,,\quad \sigma([0,\epsilon])=1\,,\quad \sigma([2\epsilon,3\epsilon])=0.$$

Let $f_j\in C^\infty(M,\mathbb{R})$ $(j=0,n)$ be the extension by zero of the combination of the functions $\sigma(|x_p|)\in C^\infty_{\mathrm{c}}(B(p,3\epsilon),\mathbb{R})$ $(p\in\mathcal{Y}_j)$. We have

$$\operatorname{supp}df_j\subset V'_j\setminus V_j\,,\quad f_j(V_j)=1\,,\quad f_j(M\setminus V'_j)=0\,,\quad Xf_0\ge 0\,,\quad Xf_n\le 0.$$

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For any $c_0,c_n \geq 0$, let $\eta = \eta(c_0,c_n) = \eta_0 - c_0\,df_0 + c_n\,df_n$. This closed 1-form satisfies (a) and (d) with X and g, and we have

$$\mathcal{M}_1(\eta,X) = a + c_0\,, \quad \mathcal{M}_n(\eta,X) = a + c_n\,, \quad \mathcal{M}_k(\eta_1,X) = a \quad (1 < k < n).$$

Hence, by Corollary 4.15,

$$\mathbf{z}_{\mathrm{sm}}(\eta) - \mathbf{z}_{\mathrm{sm}}(\eta_0) = e^a(e^{c_0}-1)m_1^1 + (-1)^n e^a(1-e^{c_n})m_n^1. \tag{9.1}$$

By (a), $e(M,\nabla^M) = 0$ on every U_p ($p \in \mathcal{Y}$). So, using the Stokes formula,

$$\begin{aligned}
\mathbf{z}_{\mathrm{la}}(\eta) - \mathbf{z}_{\mathrm{la}}(\eta_0) &= \int_M (c_n\,df_n - c_0\,df_0) \wedge (-X)^*\psi(M,\nabla^M)\\
&= \int_M (c_0 f_0 - c_n f_n)(-X)^* d\psi(M,\nabla^M)\\
&= \int_M (c_0 f_0 - c_n f_n)\, e(M,\nabla^M) - \sum_{p\in\mathcal{Y}} (-1)^{\mathrm{ind}(p)}(c_0 f_0 - c_n f_n)(p)\\
&= (-1)^n c_n|\mathcal{Y}_n| - c_0|\mathcal{Y}_0|. \qquad (9.2)
\end{aligned}$$

Combining Equations (9.1) and (9.2), we obtain

$$\mathbf{z}(\eta) - \mathbf{z}(\eta_0) = e^a(e^{c_0}-1)m_1^1 + (-1)^n e^a(1-e^{c_n})m_n^1 + (-1)^n c_n|\mathcal{Y}_n| - c_0|\mathcal{Y}_0|.$$

Using local changes of X and applying [69, Lemmas 1.1 and 1.2], we can increase $|\mathcal{Y}_0|$ or $|\mathcal{Y}_n|$ as much as desired. By Lemma 4.12 and Equation (4.21), we have

$$m_1^1 = |\mathcal{Y}_0| - \beta^0_{\mathrm{No}}\,, \quad m_n^1 = |\mathcal{Y}_n| - \beta^n_{\mathrm{No}}\,, \tag{9.3}$$

which can be increased as much as desired. So, if n is even (resp., odd), given any $\tau \in \mathbb{R}$ (resp., $\tau \gg 0$), we get $\mathbf{z}(\eta(c_0,c_n)) = \tau$ for some $c_0,c_n \geq 0$.

Now, assume n is even. To prove that $\pm\mathbf{z}(\pm\eta) = \tau$, by Equations (7.9), (9.1) and (9.2), it is enough to prove that we can choose $|\mathcal{Y}_0|$, $|\mathcal{Y}_n|$, c_0 and c_n so that

$$\begin{aligned}
\mathbf{z}_{\mathrm{sm}}(\eta) &= \mathbf{z}_{\mathrm{sm}}(\eta_0) + e^a(e^{c_0}-1)m_1^1 + e^a(1-e^{c_n})m_n^1 = 0,\\
\mathbf{z}_{\mathrm{la}}(\eta) &= \mathbf{z}_{\mathrm{la}}(\eta_0) + c_n|\mathcal{Y}_n| - c_0|\mathcal{Y}_0| = \tau.
\end{aligned}$$

Using Equation (9.3) and writing $u = -e^{-a}\mathbf{z}_{\mathrm{sm}}(\eta_0)$ and $v = \tau - \mathbf{z}_{\mathrm{la}}(\eta_0)$, the above system becomes

$$\begin{aligned}
(e^{c_0}-1)(|\mathcal{Y}_0| - \beta^0_{\mathrm{No}}) + (1-e^{c_n})(|\mathcal{Y}_n| - \beta^n_{\mathrm{No}}) &= u,\\
c_n|\mathcal{Y}_n| - c_0|\mathcal{Y}_0| &= v.
\end{aligned}$$

The following result states that these equalities are satisfied by some $c_0,c_n \geq 0$ and $|\mathcal{Y}_0|,|\mathcal{Y}_n| \gg 0$.

Lemma 9.1. *Given $u,v \in \mathbb{R}$ and $\beta,\gamma \geq 0$, there are $c,d \geq 0$ and integers $p,q \gg 0$ such that*

$$\begin{aligned}
(e^c-1)(p-\beta) + (1-e^d)(q-\gamma) &= u,\\
dq - cp &= v.
\end{aligned}$$

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Proof. Taking $q>0$, we get

$$d=(cp+v)/q.$$

Thus, $cp+v\geq 0$; that is, $c\geq -v/p$. Let

$$F_{p,q}(c)=(e^c-1)(p-\beta)+\left(1-e^{(cp+v)/q}\right)(q-\gamma).$$

We have to find integers $p,q\gg 0$ and $c\geq 0,-v/p$ such that $F_{p,q}(c)=u$.

Observe that

$$\beta<p<q\Rightarrow \lim_{c\to+\infty}F_{p,q}(c)=+\infty\,, \tag{9.4}$$

$$\gamma<q<p\Rightarrow \lim_{c\to+\infty}F_{p,q}(c)=-\infty. \tag{9.5}$$

Note also that, if (c,d,p,q) is a solution for some (u,v,β,γ), then (d,c,q,p) is a solution for $(-u,-v,\gamma,\beta)$. So it is sufficient to consider the case $v\geq 0$. In this case, c can reach 0 and

$$F_{p,q}(0)=\left(1-e^{v/q}\right)(q-\gamma)\,,$$

which is independent of p. Choose $q\gg\beta,\gamma$; thus, $F_{p,q}(0)\leq 0$. If $u\geq F_{p,q}(0)$, take p so that $\beta\ll p<q$, yielding $u\in\operatorname{im}F_{p,q}$ by Equation (9.4). If $u<F_{p,q}(0)$, take $p>q$, yielding $u\in\operatorname{im}F_{p,q}$ by Equation (9.5). □

10. The switch of the order of integration

The proof of Theorem 1.4 is given in this section. Let $\mathcal{S}$ be the Schwartz space on $\mathbb{R}$. Recall that the space of tempered distributions is the continuous dual space $\mathcal{S}'$, with the strong topology. Suppose first that Equation (1.7) is used as definition of Z_μ. By Theorems 1.1 and 1.2, the expression (1.7) defines a tempered distribution Z_μ for $\mu\gg 0$. Moreover, using also the formula of the inverse Fourier transform, we get, for $f\in\mathcal{S}$,

$$\langle Z_\mu,f\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}\zeta(1,z)\,\hat f(\nu)\,d\nu\to\frac{\mathbf{z}}{2\pi}\int_{-\infty}^{\infty}\hat f(\nu)\,d\nu=\mathbf{z}f(0)\,,$$

as $\mu\to+\infty$, uniformly on ν. For every $C>0$, this convergence is also uniform on $f\in\mathcal{S}$ with $|\hat f(\nu)|,|\nu^2\hat f(\nu)|\leq C$. So $Z_\mu\to\mathbf{z}\delta_0$ in $\mathcal{S}'$ as $\mu\to+\infty$. To get Theorem 1.4, it only remains to prove the following.

Theorem 10.1. *Both Equations (1.4) and (1.7) define the same tempered distribution Z_μ for $\mu\gg 0$.*

Proposition 10.2. *For $\mu\gg 0$, $t>0$ and $f\in\mathcal{S}$,*

$$\int_{-\infty}^{\infty}\int_t^{\infty}\left|\operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\right||\hat f(\nu)|\,du\,d\nu<\infty.$$

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Proof. By [26, Corollary XI.9.8 and Lemma XI.9.9 (d)],

$$\begin{aligned}\left|\operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\right| &\le \left|\eta\wedge\delta_z e^{-u\Delta_z}\right|_1 \le \|\eta\wedge\|\left|\delta_z e^{-u\Delta_z}\right|_1\\ &= \|\eta\|_{L^\infty}\operatorname{Tr}\left((d_z\delta_z)^{1/2}e^{-u\Delta_z}\right) \le \|\eta\|_{L^\infty}\operatorname{Tr}\left(\Delta_z^{1/2}e^{-u\Delta_z}\right),\end{aligned}$$

where $|\ |_1$ denotes the trace norm. Hence,

$$\begin{aligned}\int_t^\infty \left|\operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\right| du &\le \|\eta\|_{L^\infty}\int_t^\infty \operatorname{Tr}\left(\Delta_z^{1/2}e^{-u\Delta_z}\right) du\\ &= \|\eta\|_{L^\infty}\operatorname{Tr}\left(\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\right).\end{aligned}$$

The operator $(1+\Delta)^{-N}$ is of trace class for any $N>n$. Therefore,

$$\operatorname{Tr}\left(\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\right) \le \left|(1+\Delta)^{-N}\right|_1\left\|(1+\Delta)^N\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\right\|.$$

By Corollary 2.3 and Theorem 8.4, for $\mu\gg 0$ and $\alpha\in L^2(M;\Lambda)$,

$$\begin{aligned}&\left\|(1+\Delta)^N\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\alpha\right\|\\ &\quad\le C_0\left\|\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\alpha\right\|_{2N} \le C_1|z|^{2N}\left\|\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\alpha\right\|_{2N,z}\\ &\quad= C_2|z|^{2N}\sum_{k=0}^{2N}\left\|D_z^k\Delta_z^{-1/2}e^{-t\Delta_z}\Pi_z^\perp\alpha\right\| \le C_3|z|^{2N}\sum_{k=0}^{2N}\frac{1}{t^{k/2}}\left\|\Delta_z^{-1/2}\Pi_z^\perp\alpha\right\|\\ &\quad\le C|z|^{2N}\left(1+t^{-N}\right)e^{c\mu}\|\alpha\|.\end{aligned}$$

Thus, since $f\in\mathcal{S}$,

$$\begin{aligned}&\int_{-\infty}^\infty\int_t^\infty \left|\operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\right|\left|\hat f(\nu)\right| du\,d\nu\\ &\quad\le C\|\eta\|_{L^\infty}\left|(1+\Delta)^{-N}\right|_1\left(1+t^{-N}\right)e^{c\mu}\int_{-\infty}^\infty |z|^{2N}\left|\hat f(\nu)\right| d\nu<\infty.\end{aligned}$$

□

Proof of Theorem 10.1. We compute

$$\begin{aligned}&-\frac{1}{2\pi}\int_{-\infty}^\infty \lim_{t\downarrow 0}\operatorname{Str}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\hat f(\nu)\,d\nu\\ &\quad= -\frac{1}{2\pi}\lim_{t\downarrow 0}\int_{-\infty}^\infty \operatorname{Str}\left(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1\right)\hat f(\nu)\,d\nu\\ &\quad= \frac{1}{2\pi}\lim_{t\downarrow 0}\int_{-\infty}^\infty\int_t^\infty \operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\hat f(\nu)\,du\,d\nu\\ &\quad= \frac{1}{2\pi}\lim_{t\downarrow 0}\int_t^\infty\int_{-\infty}^\infty \operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\hat f(\nu)\,d\nu\,du\\ &\quad= \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty \operatorname{Str}\left(\eta\wedge\delta_z e^{-u\Delta_z}\right)\hat f(\nu)\,d\nu\,du.\end{aligned}$$

Here, the first equality is given by the Lebesgue's dominated convergence theorem, whose hypothesis is satisfied because $\hat f\in\mathcal{S}$ and $|\operatorname{Str}(\eta\wedge d_z^{-1}e^{-t\Delta_z}\Pi_z^1)|\le C$ for all $t>0$, $|\mu|\gg 0$

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and $\nu \in \mathbb{R}$ by Theorems 1.1 and 1.2. The third equality is given by Fubini's theorem, whose hypothesis is satisfied by Proposition 10.2. □

A. Integrals along instantons

Theorem 8.1 is proved here. We show the case where $a_n \geq \cdots \geq a_1 \gg 0$. Then the case where $a_1 \geq \cdots \geq a_n \gg 0$ follows by using $-X$ and $-\xi$.

By [69, Theorem B], there is some Morse function h on M such that $h(\mathcal{Y}_k) = \{k\}$ $(k = 0,\dots,n)$, $Xh < 0$ on $M \setminus \mathcal{Y}$, and h is in standard form with respect to X; in particular, $\mathrm{Crit}_k(h) = \mathcal{Y}_k$. Now, we proceed like in the proof of [18, Proposition 16 (i)]. Since $\mathcal{Y}$ is finite, there is some $\eta' \in \xi$ such that $\eta' = 0$ on some open neighborhood U_p of every $p \in \mathcal{Y}$. Let $U_k = \bigcup_{p\in\mathcal{Y}_k} U_p$ and $U = \bigcup_k U_k$. We can assume $h(U_k) \subset (k-1/4, k+1/4)$ for all $k = 0,\dots,n$. If $C \gg 0$, then the representative $\eta'' := \eta' + C\,dh$ of ξ satisfies $\eta''(X) < 0$ on $M \setminus \mathcal{Y}$.

For $k = 0,\dots,n$, let $I_k^\pm \subset \mathbb{R}$ be the closed interval with boundary points $k \pm 1/4$ and $k \pm 1/2$. Since there are no critical values of h in $I_k^\pm$, every $T_k^\pm := h^{-1}(I_k^\pm)$ is compact submanifold with boundary of dimension n, every $\Sigma_k^\pm := h^{-1}(k \pm 1/2)$ is a closed submanifold of codimension 1, and there are identities $T_k^\pm \equiv \Sigma_k^\pm \times I_k^\pm$ given by $x \equiv (\pi_k^\pm(x), h(x))$ $(x \in T_k^\pm)$, where $\pi_k^\pm(x)$ is the unique point of $\Sigma_k^\pm$ that meets the ϕ-orbit of x. Of course, $\Sigma_k^- = \Sigma_{k-1}^+$ $(k = 1,\dots,n)$ and $T_0^- = \Sigma_0^- = T_n^+ = \Sigma_n^+ = \emptyset$. (See Figure 1.)

We have $\Sigma_k^\pm \pitchfork \iota_p^\pm(W_p^\pm)$ for $p \in \mathcal{Y}_k$. Let $K_p^\pm = \Sigma_k^\pm \cap \iota_p^\pm(W_p^\pm)$ and $K_k^\pm = \bigcup_{p\in\mathcal{Y}_k} K_p^\pm$, which are closed submanifolds of $\Sigma_k^\pm$; K_k^- is of codimension k in Σ_k^-, and K_k^+ of codimension $n-k$ in Σ_k^+. Since the α- and ω-limits of the orbits of X are zero points, the orbit of ϕ through every point $x \in \Sigma_k^+ \setminus K_k^+$ meets $\Sigma_k^- \setminus K_k^-$ at a unique point $\psi_k(x) := \phi^{\tau_k(x)}(x)$ $(\tau_k(x) > 0)$. This defines a diffeomorphism $\psi_k : \Sigma_k^+ \setminus K_k^+ \to \Sigma_k^- \setminus K_k^-$ and a smooth function $\tau_k : \Sigma_k^+ \setminus$

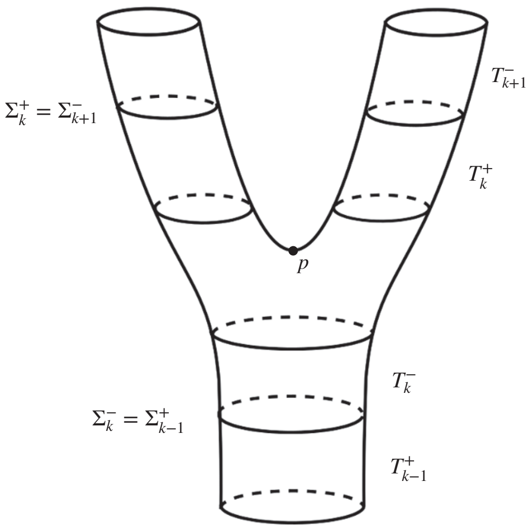

Figure 1. A representation of the sets $T_k^\pm$, $\Sigma_k^\pm$, T_{k-1}^+ and T_{k+1}^-, taking $\mathcal{Y}_k = \{p\}$.

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$K_k^+ \to \mathbb{R}^+$. Moreover, the sets $K_p^\pm$ $(p \in \mathcal{Y}_k)$ have corresponding open neighborhoods $V_p^\pm$ in $\Sigma_k^\pm$, with disjoint closures, such that $\psi_k(V_p^+ \setminus K_p^+) = V_p^- \setminus K_p^-$. Take smooth functions $\lambda_p^\pm$ $(p \in \mathcal{Y}_k)$ on $\Sigma_k^\pm$ so that $0 \le \lambda_p^\pm \le 1$, $\operatorname{supp}\lambda_p^\pm \subset V_p^\pm$, $\lambda_p^\pm = 1$ on $K_p^\pm$ and $\lambda_p^+ = \psi_k^*\lambda_p^-$ on $\Sigma_k^+ \setminus K_k^+$. Moreover, let

$$\begin{aligned} \widetilde{T}_k &= h^{-1}([k-1/2,k+1/2])\,, \quad \widetilde{K}_p = \widetilde{T}_k \cap \left(\iota_p^+(W_p^+) \cup \iota_p^-(W_p^-)\right), \\ \widetilde{V}_p &= \{\,\phi^t(x) \mid x \in V_p^+ \setminus K_p^+,\ 0 \le t \le \tau_k(x)\,\} \cup \widetilde{K}_p\,, \\ \widetilde{K}_k &= \bigcup_{p \in \mathcal{Y}_k} \widetilde{K}_p\,, \quad \widetilde{V}_k = \bigcup_{p \in \mathcal{Y}_k} \widetilde{V}_p\,, \quad M_k = h^{-1}((-\infty,k+1/2]). \end{aligned}$$

Thus, $M_k = \widetilde{T}_0 \cup \cdots \cup \widetilde{T}_k$. Note that $\widetilde{T}_k$ and M_k are compact submanifolds with boundary of dimension n, and every $\widetilde{V}_p$ (resp., $\widetilde{K}_p$) is open (resp., closed) in $\widetilde{T}_k$. We also get smooth functions $\tilde{\lambda}_p$ $(p \in \mathcal{Y}_k)$ on $\widetilde{T}_k$ determined by the condition $\tilde{\lambda}_p(\phi^t(x)) = \lambda_p^+(x)$ for all $x \in \Sigma_k^+ \setminus K_k^+$ and $0 \le t \le \tau_k(x)$. They satisfy $0 \le \tilde{\lambda}_p \le 1$, $\operatorname{supp}\tilde{\lambda}_p \subset \widetilde{V}_p$ and $\tilde{\lambda}_p = 1$ on $\widetilde{K}_p$.

Let

$$\begin{aligned} A_p &= \max\{\,|\eta'(\gamma)| \mid \gamma \in \mathcal{T}_p^1\,\} \quad \left(p \in \mathcal{Y}_+\right), \\ A_k &= \max_{p \in \mathcal{Y}_k} A_p \quad (k = 1,\dots,n)\,, \quad A = \max\{A_1,\dots,A_n\}. \end{aligned}$$

We can suppose $C > A$ and $a_1 > C + A > 0$. For $p \in \mathcal{Y}_k$, $q \in \mathcal{Y}_{k-1}$ and $\gamma \in \mathcal{T}(p,q)$,

$$dh(\gamma) = h(q) - h(p) = -1.$$

Therefore,

$$0 > \eta''(\gamma) = \eta'(\gamma) + C\,dh(\gamma) \ge -A - C > -a_1 \quad (\gamma \in \mathcal{T}^1). \tag{A.1}$$

Claim 1. *For $k = 0,\dots,n$, there is a smooth function f_k on M such that*

$$df_k(X) \le 0\,, \tag{A.2}$$

$$\operatorname{supp} df_k \subset \mathring{M}_k\,, \tag{A.3}$$

$$\max\{\,(\eta'' + df_k)(\gamma) \mid \gamma \in \mathcal{T}_p^1\,\} = -a_l \quad (p \in \mathcal{Y}_l,\ 1 \le l \le k)\,, \tag{A.4}$$

$$(\eta'' + df_k)(\delta) > -a_k \quad (\delta \in \mathcal{T}_{k+1}^1). \tag{A.5}$$

The statement follows directly from Claim 1 taking $\eta = \eta'' + df_n$. So we only have to prove this assertion.

We proceed by induction on k. For $k = 0$, we choose $f_0 = 0$. Then Equation (A.4) is vacuous, Equations (A.2) and (A.3) are trivial and Equation (A.5) is given by Equation (A.1).

Now, take any $k \ge 1$ and assume f_{k-1} is defined and satisfies Equations (A.2)–(A.5). Let

$$\begin{aligned} b_p &= -\max\{\,(\eta'' + df_{k-1})(\gamma) \mid \gamma \in \mathcal{T}_p^1\,\} \quad (p \in \mathcal{Y}_k)\,, \\ b_k &= \min\{\,b_p \mid p \in \mathcal{Y}_k\,\}. \end{aligned} \tag{A.6}$$

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For every $p \in \mathcal{Y}_k$, we have $b_p < a_{k-1} \le a_k$ because f_{k-1} satisfies Equation (A.5). So there is a smooth function h_p^- on I_k^- such that $(h_p^-)' \ge 0$, $h_p^- = 0$ around $k-1/2$, and $h_p^- = a_k - b_p$ around $k-1/4$. Let $\tilde{h}_p^-$ be the function on $V_p^- \times I_k^- \subset \Sigma_k^- \times I_k^- \equiv T_k^-$ given by $\tilde{h}_p^-(x,s) = h_p^-(s)$. We have $\tilde{h}_p^- = 0$ around $V_p^- \times \{k-1/2\}$ and $\tilde{h}_p^- = a_k - b_p$ around $V_p^- \times \{k-1/4\}$. Thus, $\tilde{h}_p^-$ has a smooth extension to $\widetilde{V}_p$, also denoted by $\tilde{h}_p^-$, which is equal to $a_k - b_p$ on $\widetilde{V}_p \setminus T_k^-$. The function $\tilde{\lambda}_p \tilde{h}_p^-$ on $\widetilde{V}_p$ can be extended by zero to get a smooth function on $\widetilde{T}_k$, also denoted by $\tilde{\lambda}_p \tilde{h}_p^-$. Let $\tilde{h}_k^- = \sum_{p\in\mathcal{Y}_k} \tilde{\lambda}_p \tilde{h}_p^-$ on $\widetilde{T}_k$.

On the other hand, let ρ_k be a smooth function on I_k^+ such that $\rho_k' \ge 0$, $\rho_k = 0$ around $k+1/4$ and $\rho_k = 1$ around $k+1/2$. Let $\tilde{\rho}_k$ be the smooth function on $T_k^+ \equiv \Sigma_k^+ \times I_k^+$ given by $\tilde{\rho}_k(x,s) = \rho_k(s)$, and let

$$\tilde{h}_k^+ = \tilde{h}_k^-(1-\tilde{\rho}_k) + (a_k - b_k)\tilde{\rho}_k$$

on T_k^+. This smooth function is equal to $\tilde{h}_k^-$ around $\Sigma_k^+ \times \{k+1/4\}$, and is equal to $a_k - b_k$ around $\Sigma_k^+ \times \{k+1/2\} \equiv \Sigma_k^+$. So the functions, $\tilde{h}_k^-$ on $\widetilde{T}_k \setminus T_k^+$ and $\tilde{h}_k^+$ on T_k^+, can be combined to produce a smooth function $\tilde{h}_k$ on $\widetilde{T}_k$. Since $\tilde{h}_k = 0$ around Σ_k^- and $\tilde{h}_k = a_k - b_k$ around Σ_k^+, there is a smooth extension of $\tilde{h}_k$ to M, also denoted by $\tilde{h}_k$, which is constant on $M \setminus \widetilde{T}_k$.

Let $f_k = f_{k-1} + \tilde{h}_k$ on M. This smooth function satisfies Equation (A.2) because f_{k-1} satisfies Equation (A.2), and X induces the opposite of the standard orientation on every fiber $\{x\} \times I_k^\pm \equiv I_k^\pm$ of $T_k^\pm$ ($x \in \Sigma_k^\pm$). It also satisfies Equations (A.3) and (A.4) for $p \in \mathcal{Y}_l$ with $1 \le l < k$ because f_{k-1} satisfies these properties and $d\tilde{h}_k$ is supported in the interior of $\widetilde{T}_k$.

Next, take any $p \in \mathcal{Y}_k$, $q \in \mathcal{Y}_{k-1}$ and $\gamma \in \mathcal{T}(p,q) \subset \mathcal{T}_p^1$. We have $\gamma \cap T_k^- \equiv \{x\} \times I_k^-$ for some $x \in K_p^- \cap K_q^+ \subset \Sigma_k^- = \Sigma_{k-1}^+$, and the orientation of $\gamma \cap T_k^-$ agrees with the opposite of the standard orientation of $\{x\} \times I_k^- \equiv I_k^-$. Then

$$\begin{aligned}(\eta'' + df_k)(\gamma) &= (\eta'' + df_{k-1} + d\tilde{h}_k)(\gamma) \le -b_p + \lambda_p^-(x) d\tilde{h}_p^-(\gamma)\\ &= -b_p - \int_{I_k^-} dh_p^- = -b_p - (a_k - b_p) = -a_k.\end{aligned}$$

Here, the equality holds when the maximum of Equation (A.6) is achieved at γ. Hence, f_k also satisfies Equation (A.4) for $p \in \mathcal{Y}_k$.

Finally, take any $p \in \mathcal{Y}_k$, $u \in \mathcal{Y}_{k+1}$ and $\delta \in \mathcal{T}(u,p) \subset \mathcal{T}_u^1 \subset \mathcal{T}_{k+1}^1$. Thus, $\delta \cap T_k^+ \equiv \{y\} \times I_k^+$ for some $y \in K_p^+ \cap K_u^- \subset \Sigma_k^+ = \Sigma_{k+1}^-$, and the orientation of $\delta \cap T_k^+$ agrees with the opposite of the standard orientation of $\{y\} \times I_k^+ \equiv I_k^+$. Then

$$\begin{aligned}(\eta'' + df_k)(\delta) &= (\eta'' + df_{k-1} + d\tilde{h}_k)(\delta) = \eta''(\delta) + d\tilde{h}_k^+(\delta)\\ &= \eta''(\delta) + \big(\tilde{h}_k^-(y) - (a_k - b_k)\big)\int_{I_k^-} d\rho_k\\ &= \eta''(\delta) + \tilde{\lambda}_p(y)\tilde{h}_p^-(y) + b_k - a_k\\ &= \eta''(\delta) + a_k - b_p + b_k - a_k = \eta''(\delta) + b_k - b_p \ge \eta''(\delta) > -a_k\,,\end{aligned}$$

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where the second equality is true because f_{k-1} satisfies Equation (A.3), and the last inequality holds by Equation (A.1). So f_k satisfies Equation (A.5).

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