BULL. AUSTRAL. MATH. SOC. 47B47, 42B20
VOL. 33 (1986), 329-333.

SIMILARITY BETWEEN KLEINECKE-SHIROKOV THEOREM AND FUGLEDE-PUTNAM THEOREM

TAKAYUKI FURUTA

Recently in this journal we have shown the similarity between the Kleinecke-Shirokov theorem for subnormal operators and the Fuglede-Putnam theorem. The purpose of this paper is to show that this similarity can be generalized to operators which belong to some classes of non-normal operators wider than the class of subnormal operators.

1. Introduction

An operator means a bounded linear operator on a complex Hilbert space. Let $B(H)$ denote the set of all bounded linear operators on a complex Hilbert space H. An operator T is called dominant if there is a real constant $M_\lambda \geqslant 1$ such that

$$\|(T-\lambda)^* x\| \leqslant M_\lambda \|(T-\lambda)x\|$$

for all x in H and for all complex numbers λ. If there is a constant M such that $M_\lambda \leqslant M$ for all λ, T is called M-hyponormal. An operator T is called subnormal if T has a normal extension. It is known that every subnormal operator is hyponormal $(T^*T \geqslant TT^*)$ and every hyponormal operator is M-hyponormal. For two arbitrary operators A and B, $[A,B]$ denotes the commutator of A and B, that is, $[A,B] = AB - BA$.

Following [1], we introduce the following definition as an extension of the usual commutator.

Received 16 July 1985.

DEFINITION 1. $[C_j, B]_* = C_1B - BC_2$ for $C_1, C_2, B \in B(H)$.

Ackermans, van Eijndhoven and Martens in [1], and Roitman in [7] investigated independently the similarity between the Kleinecke-Shirokov theorem ([4],[8]) for normal operators and the Fuglede-Putnam theorem ([2],[6]) as follows.

THEOREM A ([1],[7]). *Let* N_1 *and* N_2 *be normal. If* $[N_j, [N_j, B]_*]_* = 0$ *then* $[N_j, B]_* = 0$.

As an extension of Theorem A, recently, among others, we showed the following similarity between the Kleinecke-Shirokov theorem for subnormal operators and the Fuglede-Putnam theorem.

THEOREM B [3]. *Let* A_1 *and* A_2^* *be subnormal. If* $[A_j, [A_j, B]_*]_* = 0$, *then* $[A_j, B]_* = 0$.

In this paper we show the similarity between the Kleinecke-Shirokov theorem for M-hyponormal operators and the Fuglede-Putnam theorem, that is, we show Theorem 1 and Corollary 2 as an extension of Theorem B.

2. Statement of theorems

THEOREM 1. *The following two conditions are equivalent:*

(1) $[A_j^*, [A_j, B]_*]_* = 0$ *and* $[A_2, [A_j, B]_*^*[A_j, B]_*] = 0$,

(2) $[A_j, B]_* = 0$.

COROLLARY 1. *The following two conditions are equivalent:*

(1) $[A_j^*, [A_j, B]_*]_* = 0$ *and* $[A_j, [A_j, B]_*]_* = 0$,

(2) $[A_j, B]_* = 0$.

COROLLARY 2. *Let* A_1 *and* A_2^* *be* M*-hyponormal. Then* $[A_j, [A_j, B]_*]_* = 0$ *holds if and only if* $[A_j, B]_* = 0$ *holds.*

Corollary 1 is shown in ([7], Remark) and Corollary 2 is an extension of Theorem B since every subnormal operator is always M-hyponormal.

https://doi.org/10.1017/S0004972700003890 Published online by Cambridge University Press

3. Proofs of theorems

We prove the following lemma before we give the proof of Theorem 1.

LEMMA. *The following two conditions are equivalent:*

(1) $[A^*,[A,B]] = 0$ *and* $[A,[A,B]^*[A,B]] = 0$,

(2) $[A,B] = 0$.

Proof of Lemma. (2) easily implies (1). Assume (1), then by

$$\begin{aligned} [A,B]^*[A,B] &= B^*A^*[A,B] - A^*B^*[A,B] \\ &= B^*[A,B]A^* - A^*B^*[A,B] \\ &= [B^*[A,B],A^*] \end{aligned}$$

using $[A^*,[A,B]] = 0$.

Hence $[A,B]^*[A,B]$ is a commutator and commutes with A by the second half of assumption (1), so that $[A,B]^*[A,B]$ is quasinilpotent by the Kleinecke-Shirokov theorem ([4],[8]), consequently $[A,B] = 0$ since $[A,B]^*[A,B]$ is a positive operator.

Proff of Theorem 1. We have only to prove that (1) implies (2) since the reverse implication is obvious. Assume (1). Then we define $\hat{A}$ and $\hat{B}$ as follows:

$$\hat{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

so that we have

$$[\hat{A},\hat{B}] = \begin{pmatrix} 0 & [A_j,B]_* \\ 0 & 0 \end{pmatrix},$$

$$[\hat{A}^*,[\hat{A},\hat{B}]] = \begin{pmatrix} 0 & [A_j^*,[A_j,B]_*]_* \\ 0 & 0 \end{pmatrix},$$

$$= 0$$

by the first half of assumption (1). Also we have

$$[\hat{A},\hat{B}]^*[\hat{A},\hat{B}] = \begin{pmatrix} 0 & 0 \\ 0 & [A_j,B]_*^*[A_j,B]_* \end{pmatrix},$$

https://doi.org/10.1017/S0004972700003890 Published online by Cambridge University Press

and

$$[\hat{A}, [\hat{A},\hat{B}]^*[\hat{A},\hat{B}]] = \left[\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & [A_j,B]_*^*[A_j,B]_* \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & [A_2, [A_j,B]_*^*[A_j,B]_*] \end{pmatrix}$$

$$= 0$$

by the second half of assumption (1). By Lemma 1 we have $[\hat{A},\hat{B}] = 0$, that is, $[A_j,B]_* = 0$, so the proof of Theorem 1 is complete.

Proof of Corollary 1. We have only to show that (1) implies (2) since the reverse implication is obvious. Assume (1), then we have $A_2[A_j,B]_*^* = [A_j,B]_*^*A_1$, by taking the adjoint of the first half of assumption (1) and this relation yields

$$A_2[A_j,B]_*^*[A_j,B]_* = [A_j,B]_*^*A_1[A_j,B]_*$$

$$= [A_j,B]_*^*[A_j,B]_*A_2$$

by the second half of assumption (1), so that $[A_j,B]_* = 0$ by Theorem 1 and the proof is complete.

We state the following Theorem C in order to prove Corollary 2.

THEOREM C [5]. *If* A *and* B^* *are M-hyponormal and* $AX = XB$, *then* $A^*X = XB^*$.

Proof of Corollary 2. The proof of the "if" part is obvious and we have only to show the proof of "only if" part. If A_1 , A_2^* are M-hyponormal and $[A_j, [A_j,B]_*]_* = 0$, then $[A_j^*, [A_j,B]_*]_* = 0$ by Theorem C, so that $[A_j,B]_* = 0$ by Corollary 1.

Addendum. After we wrote this manuscript, Professor Yoshino kindly sent his preprint [9] to us. "If A is dominant and B^* is M-hyponormal and $AX = XB$, then $A^*X = XB^*$" is cited in [9]. This result and Corollary 1 yield the following result in the same way as the proof of Corollary 2.

COROLLARY 3. *Let* A *be dominant and* B^* *be* M*-hyponormal. Then* $[A_j, [A_j, B]_*]_* = 0$ *holds if and only if* $[A_j, B]_* = 0$ *holds.*

Corollary 3 is an extension of Corollary 2 since every M-hyponormal is dominant.

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Department of Mathematics,
Faculty of Science,
Hirosaki University,
3 Bunkyo-Cho,
Hirosaki, Aomori 036, Japan.