ON THE SECOND NATURAL REPRESENTATION OF THE SYMMETRIC GROUPS

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(Received 13 November, 1967)

1. Introduction. In **[1]**, the natural representation module of the symmetric groups, hereafter called the first natural representation module of the symmetric groups, was analysed. It is the purpose of this paper to analyse the second natural representation module of the symmetric groups.

We begin by defining these modules. Let K be a field of characteristic p, and let $x_1, \ldots, x_n$ be commuting, independent indeterminates over K. Let Φ_n denote the group algebra of the symmetric group S_n on $\{x_1, \ldots, x_n\}$ over K. The ring of polynomials $K[x_1, \ldots, x_n]$ may be turned into a Φ_n-module in the obvious manner, namely by taking

$$\tau f(x_1, \ldots, x_n) = f(\tau x_1, \ldots, \tau x_n)$$

for all $f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ and $\tau \in S_n$. We select certain Φ_n-submodules of $K[x_1, \ldots, x_n]$ which are finite-dimensional vector spaces over K. The first natural representation module $M^1(n)$ consists of all polynomials of the form $\sum_{i=1}^{n} \alpha_i x_i$ with $\alpha_i \in K$. The second natural representation module $M^2(n)$ consists of all polynomials of the form $\sum_{1 \leqq i < j \leqq n} \alpha_{ij} x_i x_j$ with $\alpha_{ij} \in K$. They have K-bases $\{x_i : 1 \leqq i \leqq n\}$ and $\{x_i x_j : 1 \leqq i < j \leqq n\}$ respectively.

In **[2]**, we are given a method of constructing a full set of irreducible inequivalent representation modules of S_n over a field of characteristic zero. These modules, which we shall call Specht modules, are constructed as submodules of $K[x_1, \ldots, x_n]$ as follows. We require a module for each partition of n. Let $n = \lambda_1 + \ldots + \lambda_r$ $(\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_r > 0)$ be a partition of n, denoted by (λ). We write down the associated tableau

$$\begin{array}{ccccccccccc}
x_1 & x_2 & x_3 & \cdot & \cdot & \cdot & \cdot & & \cdot & x_{\lambda_1 - 1} & x_{\lambda_1} \\
x_{\lambda_1+1} & x_{\lambda_1+2} & \cdot & \cdot & \cdot & \cdot & \cdot & x_{\lambda_1+\lambda_2} & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
x_{n-\lambda_r+1} & \cdot & \cdot & \cdot & \cdot & \cdot & x_n & & & &
\end{array}$$

with r rows, and λ_i entries in the i-th row.

If $\{a_i : i = 1, \ldots, s\}$ is a set of elements of any ring, the difference product $\Delta(a_1, \ldots, a_s)$ is defined by

$$\Delta(a_1, \ldots, a_s) = \prod_{1 \leqq i < j \leqq s} (a_i - a_j), \qquad \Delta(a_1) = 1.$$

https://doi.org/10.1017/S0017089500000495 Published online by Cambridge University Press

We define the polynomial $f^{(\lambda)}(x_1, \ldots, x_n)$ to be the product of the difference products of the entries in each column of the tableau, i.e.

$$f^{(\lambda)}(x_1, \ldots, x_n) = \Delta(x_1, x_{\lambda_1+1}, \ldots, x_{n-\lambda_r+1})\Delta(x_2, x_{\lambda_1+2}, \ldots, x_{n-\lambda_r+2}) \ldots \Delta(x_{\lambda_1}, \ldots, x_{\lambda_1+\ldots+\lambda_s}),$$

where s is the smallest integer such that $\lambda_s \neq \lambda_{s+1}$, or, if no such integer exists, $s = r$. The Specht module $S^{(\lambda)}$ corresponding to the partition (λ) is none other than $\Phi_n . f^{(\lambda)}(x_1, \ldots, x_n)$. Over a field of characteristic zero, these Specht modules are irreducible and no two are Φ_n-isomorphic. Over a field of non-zero characteristic they may reduce, although they are indecomposable, except for characteristic 2.

Let $M_0^1(n)$ denote the set of polynomials of the form $\sum_{i=1}^{n} \lambda_i x_i$ with $\sum_{i=1}^{n} \lambda_i = 0$, and let $M_0^2(n)$ denote the set of all polynomials of the form $\sum_{1 \leqq i < j \leqq n} \lambda_{ij} x_i x_j$ with $\sum_{1 \leqq i < j \leqq n} \lambda_{ij} = 0$. Then these are Φ_n-submodules of $M^1(n)$ and $M^2(n)$ respectively. $M_0^1(n)$ is generated over K by polynomials of the form $(x_i - x_j)$, and is clearly the Specht module $S^{(\lambda)}$ corresponding to the partition (λ) of n defined by $n = \lambda_1 + \lambda_2$, where $\lambda_1 = n-1$ and $\lambda_2 = 1$.

$M_0^1(n)$ is irreducible when p does not divide n. When p divides n, $s = \sum_{i=1}^{n} x_i$ is contained in $M_0^1(n)$, and $M_0^1(n)/Ks$ is irreducible. These are the results of Theorem (5.2) of **[1]**.

If (μ) is the partition $n = \mu_1 + \mu_2$, $\mu_1 = n-2$, $\mu_2 = 2$, then clearly $S^{(\mu)}$ is a Φ_n-submodule of $M_0^2(n)$; it is generated over K by the set of polynomials of the form $(x_i - x_j)(x_k - x_l)$ with i, j, k, l distinct integers between 1 and n. Note that $S^{(\mu)}$ is not defined if $n < 4$. Indeed, we need only consider $n \geqq 4$ since $M^2(3) \approx M^1(3)$ and $M^2(2) \approx K$. We shall write $S(n)$ for $S^{(\mu)}$ in the following.

The first result is that

$$M_0^2(n) / S(n) \approx M_0^1(n).$$

We show that $S(n)$ is a direct summand of $M_0^2(n)$ if and only if p does not divide $n-2$. We find that $S(n)$ is irreducible if p does not divide $n-1$ or $n-2$, and we find how $S(n)$ reduces if p divides $n-1$ or $n-2$. We also show that $M_0^2(n)$ is a direct summand of $M^2(n)$ if and only if p does not divide $n(n-1)/2$.

In the following, the range of any summation symbol will be 1 to n unless otherwise stated. Further, $\sum_{i \neq k} x_i$ will mean $x_1 + \ldots + x_{k-1} + x_{k+1} + \ldots + x_n$. Also, whenever we have defined a set $\{\lambda_{ij} : 1 \leqq i < j \leqq n\}$ of elements of K, we shall suppose that λ_{ij} is also defined for $i > j$ by $\lambda_{ij} = \lambda_{ji}$.

The author wishes to thank Dr H. K. Farahat for his guidance throughout the work which led to the preparation of this paper, the referee for his helpful comments, and the Science Research Council for their financial support.

2. Two exact sequences. We first construct the tools which will enable us to solve the problem. We have already denoted by s the element $\sum_{i=1}^{n} x_i$ of $M^1(n)$. Set

$$a_i = x_i(s - x_i) \qquad (i = 1, \ldots, n).$$

If $\tau \in S_n$ is such that $\tau x_i = x_k$, then $\tau a_i = a_k$. We denote by N the Φ_n-submodule of $M^2(n)$ generated over K by $\{a_1, \ldots, a_n\}$, i.e.

$$N = \Phi_n a_1.$$

Set

$$b_{ij} = a_i - a_j = (x_i - x_j)(s - x_i - x_j).$$

We denote by $\tilde{N}$ the Φ_n-submodule of N generated over K by the b_{ij}. $\tilde{N}$ is in fact generated over K by $\{b_{1j} : j = 2, \ldots, n\}$. Finally, let

$$\sigma = \sum_{1 \leqq i < j \leqq n} x_i x_j.$$

Then, by induction on n, we have

$$\sum_{i=1}^{n} a_i = 2\sigma.$$

In §4 we shall write $\tilde{N}(n)$ and $N(n)$ for $\tilde{N}$ and N as defined above, so that we may speak of $\tilde{N}(n-1)$ and $N(n-1)$ defined in terms of $x_1, \ldots, x_{n-1}$ without confusion. We shall then denote by $\sigma(n-1)$, $a_i(n-1)$ $(i = 1, \ldots, n-1)$, etc., elements of $M^2(n-1)$, and by σ, a_i, etc., elements of $M^2(n)$.

A linear transformation of vector spaces is uniquely determined by its action on a basis of the domain space. Any Φ_n-module is a vector space over K, and any Φ_n-homomorphism is a K-linear transformation. We shall define certain Φ_n-homomorphisms by giving the action of the map on a basis, leaving it to the reader to check that the resulting linear transformation is indeed a Φ_n-homomorphism. This principle is illustrated in the next paragraph.

Define the Φ_n-homomorphism $\bar{d}: M^2(n) \to M^1(n)$ by

$$\bar{d}(x_i x_j) = x_i + x_j$$

and let d denote the restriction of $\bar{d}$ to $M_0^2(n)$. Clearly, $\bar{d}$ may be written in terms of the partial differentiation operators $\partial/\partial x_i$ in the following way:

$$\bar{d}(x_i x_j) = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(x_i x_j). \tag{1}$$

$M_0^2(n)$ has K-basis $\{x_i x_j - x_1 x_2 : i < j \text{ and } (i, j) \neq (1, 2)\}$. Clearly, d maps $M_0^2(n)$ into $M_0^1(n)$. In fact, d maps $M_0^2(n)$ onto $M_0^1(n)$, since

$$x_i - x_j = d(x_i x_s - x_j x_s)$$

if $s \neq i, j$. We now obtain an expression for the image of an element of $M^2(n)$ in $M^1(n)$.

Lemma 1. *Let*

$$x = \sum_{i<j} \lambda_{ij} x_i x_j.$$

Then

$$\bar{d}(x) = \sum_k \beta_k x_k,$$

https://doi.org/10.1017/S0017089500000495 Published online by Cambridge University Press

where

$$\beta_k = \sum_{j \neq j} \lambda_{kj}.$$

Proof. By definition

$$d(x) = \sum_{i<k} \lambda_{ij}(x_i + x_j).$$

We proceed by induction on n. The proof for $n = 1$ is immediate. Suppose that

$$\sum_{1 \leqq i < j \leqq n} \lambda_{ij}(x_i + x_j) = \sum_{k=1}^{n} \left(\sum_{j \neq k} \lambda_{jk} \right) x_k.$$

Then

$$\begin{aligned} \sum_{1 \leqq i < j \leqq n+1} \lambda_{ij}(x_i + x_j) &= \sum_{k=1}^{n} \left(\sum_{j \neq k} \lambda_{jk} \right) x_k + \sum_{i=1}^{n} \lambda_{i(n+1)}(x_i + x_{n+1}) \\ &= \sum_{k=1}^{n} \left(\sum_{\substack{j=1 \\ j \neq k}}^{n+1} \lambda_{jk} \right) x_k + \sum_{i=1}^{n} \lambda_{i(n+1)} x_{n+1} \\ &= \sum_{k=1}^{n+1} \left(\sum_{\substack{j=1 \\ j \neq k}}^{n+1} \lambda_{jk} \right) x_k. \end{aligned}$$

This proves the lemma.

We have already noted that $S(n)$ is contained in $M_0^2(n)$. In fact we have

THEOREM 1. *The following sequence of Φ_n-modules is exact*:

$$0 \longrightarrow S(n) \xrightarrow{\text{incl}} M_0^2(n) \xrightarrow{d} M_0^1(n) \longrightarrow 0. \tag{2}$$

Proof. We prove exactness at $M_0^2(n)$. First, by expressing d in the form (1), and by applying the product rule for differentiation, we see that

$$d((x_i - x_j)(x_r - x_s)) = 0,$$

and hence $S(n) \subset \operatorname{Ker} d$.

Suppose that $x = \sum_{i<j} \lambda_{ij} x_i x_j$, with $\sum_{i<j} \lambda_{ij} = 0$ and $d(x) = 0$. Then, by Lemma 1,

$$\sum_{j \neq k} \lambda_{jk} = 0 \qquad (k = 1, \ldots, n).$$

Set

$$\theta_{ij}^n = x_n(s - x_n - x_i - x_j) + x_i x_j$$

for $1 \leqq i < j < n$; then

$$\sum_{1 \leqq i < j < n} \lambda_{ij} \theta_{ij}^n = \sum_{1 \leqq i < j < n} \lambda_{ij} x_i x_j + \sum_{1 \leqq i < j < n} \lambda_{ij} x_n(s - x_n) - x_n \sum_{1 \leqq i < j < n} \lambda_{ij}(x_i + x_j).$$

But

$$\sum_{1 \leqq i < j < n} \lambda_{ij} = \sum_{1 \leqq i < j \leqq n} \lambda_{ij} - \sum_{i \neq n} \lambda_{in} = 0,$$

and hence

$$\sum_{1 \leqq i<j<n} \lambda_{ij}\theta^n_{ij} = \sum_{1 \leqq i<j<n} \lambda_{ij}x_i x_j - x_n\left(d(x) - \sum_{i=1}^{n-1} \lambda_{in}(x_i+x_n)\right)$$
$$= \sum_{1 \leqq i<j<n} \lambda_{ij}x_i x_j + x_n \sum_{i=1}^{n-1} \lambda_{in}x_i$$
$$= \sum_{1 \leqq i<j \leqq n} \lambda_{ij}x_i x_j = x.$$

Since $\sum_{1 \leqq i<j<n} \lambda_{ij} = 0$ (as proved above), $x = \sum_{1 \leqq i<j<n} \lambda_{ij}(\theta^n_{ij} - \theta^n_{12})$. However,

$$\theta^n_{ij} - \theta^n_{12} = (x_i - x_1)(x_j - x_n) + (x_1 - x_n)(x_j - x_2) \in S(n),$$

and thus $x \in S(n)$.

This proves that $\operatorname{Ker} d \subset S(n)$, and hence that $\operatorname{Ker} d = S(n)$.

COROLLARY 1. *The dimension of* $S(n)$ *over* K *is* $n(n-3)/2$, *and*

$$A = \{\theta^n_{12} - \theta^n_{ij} : 1 \leqq i < j < n,\ (i,j) \neq (1,2)\}$$

is a basis for $S(n)$.

Proof.
$$\dim_K S(n) = \dim_K M^2_0(n) - \dim_K M^1_0(n)$$
$$= \frac{n(n-3)}{2}.$$

From the theorem, the set A generates $S(n)$ over K, and we have the required number of elements; hence the set is a linearly independent basis.

COROLLARY 2. (i) *If* $p \neq 2$, $\sigma \in S(n)$ *if and only if* p *divides* $n-1$.
(ii) *If* $p = 2$, $\sigma \in S(n)$ *if and only if* n *is of the form* $4a+1$ *for some integer* a.

Proof. From the theorem, we have $S(n) = \{x \in M^2_0(n) : d(x) = 0\}$. By Lemma 1,

$$\bar{d}(\sigma) = (n-1)s.$$

Also, the sum of the coefficients of σ is $n(n-1)/2$, and $\sigma \in M^2_0(n)$ if and only if this sum is zero. The corollary follows from these statements.

THEOREM 2. *The exact sequence* (2) *splits if and only if* p *does not divide* $n-2$.

Proof. Suppose that p does not divide $n-2$. Define $\phi : M^1(n) \to M^2(n)$ by

$$\phi(x_i) = a_i \qquad (i = 1, \ldots, n).$$

Then ϕ is a Φ_n-homomorphism. Let ψ be the restriction of $\{1/(n-2)\}\phi$ to $M^1_0(n)$. Then the image of ψ is contained in $M^2_0(n)$, and we have

$$d\psi(x_i - x_j) = d(a_i - a_j)/(n-2) = d((x_i - x_j)(s - x_i - x_j))/(n-2)$$
$$= (x_i - x_j)(n-2)/(n-2),$$

using (1) and the product rule for differentiation. Hence

$$d\psi(x_i - x_j) = x_i - x_j.$$

Thus $d\psi$ is the identity on $M_0^1(n)$, and the sequence splits.

In order to prove the converse, we establish the fact that, if $x = \sum_{i<j} \lambda_{ij} x_i x_j$ is any element of $M^2(n)$, and if τ is the permutation $(x_r\, x_s)$ with $r < s$, then

$$x - \tau x = \sum_{j \neq r,s} (\lambda_{rj} - \lambda_{sj})(x_r x_j - x_s x_j). \tag{3}$$

To begin with

$$x - \tau x = \sum_{i<j} \lambda_{ij}(x_i x_j - \tau x_i x_j).$$

If $i = r$ and $j = s$, then $x_i x_j - \tau x_i x_j = 0$. Again, if i, j, r, s are all different, then $x_i x_j - \tau x_i x_j = 0$. Hence

$$\begin{aligned} x - \tau x &= \sum_{j \neq r,s} \lambda_{rj}(x_r x_j - \tau x_r x_j) + \sum_{j \neq r,s} \lambda_{sj}(x_s x_j - \tau x_s x_j) \\ &= \sum_{j \neq r,s} (\lambda_{rj} - \lambda_{sj})(x_r x_j - x_s x_j). \end{aligned}$$

Let $f\colon M_0^1(n) \to M_0^2(n)$ be a Φ_n-homomorphism, and set

$$f(x_1 - x_2) = \sum_{i<j} \lambda_{ij} x_i x_j \quad \text{with} \quad \sum_{i<j} \lambda_{ij} = 0.$$

Let τ be the permutation $(x_r\, x_s)$ with $2 < r < s \leqq n$. Then $f(x_1 - x_2)$ is invariant under τ. Using (3), we see that

$$0 = \sum_{i \neq r,s} (\lambda_{ri} - \lambda_{si})(x_r x_i - x_s x_i) \qquad (2 < r < s \leqq n).$$

Equating coefficients, we obtain

$$\begin{aligned} &\lambda_{r1} = \lambda_{s1} = v \quad \text{(say)} \quad (2 < r < s \leqq n), \\ &\lambda_{r2} = \lambda_{s2} = w \quad \text{(say)} \quad (2 < r < s \leqq n), \\ &\lambda_{rk} = \lambda_{sk} \qquad\qquad\quad (k > 2,\ 2 < r < s \leqq n). \end{aligned}$$

From the last of these we see that $\lambda_{ij} = \lambda_{\alpha\beta}$ whenever α, β, i, j are all greater than 2. Write $\lambda_{ij} = \gamma\ (2 < i < j \leqq n)$.

Then we have

$$f(x_1 - x_2) = v \sum_{j=3}^{n} x_1 x_j + w \sum_{j=3}^{n} x_2 x_j + \lambda_{12} x_1 x_2 + \gamma \sum_{2 < i < j \leqq n} x_i x_j.$$

Now let χ be the permutation $(x_1\, x_2)$. Then

$$f(x_1 - x_2) = -\chi f(x_1 - x_2).$$

This immediately gives $v = -w$ and $2\lambda_{12} = 2\gamma = 0$. Hence

$$f(x_1 - x_2) = v(a_1 - a_2) + \gamma \sum_{3 \leqq i < j \leqq n} (x_i x_j - x_1 x_2) \tag{4}$$

because $\sum_{i<j} \lambda_{ij} = 0$.

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Now suppose that the sequence (2) splits, i.e., that there exists a Φ_n-homomorphism $f: M_0^1(n) \to M_0^2(n)$ such that df is the identity on $M_0^1(n)$. Then f has the form (4), and

$$df(x_1 - x_2) = x_1 - x_2,$$

i.e.,

$$x_1 - x_2 = v(n-2)(x_1 - x_2).$$

Hence $v(n-2) = 1$, and p does not divide $n-2$. This completes the proof of the theorem.

This theorem tells us that, when p does not divide $n-2$, $M_0^2(n)$ is the direct sum of $S(n)$ and the image of ψ. But the image of ψ is clearly $\tilde{N}$, and so we have the result that

$$M_0^2(n) = S(n) \oplus \tilde{N}.$$

In particular, we have $\tilde{N} \approx M_0^1(n)$ when p does not divide $n-2$.

The second exact sequence deals with the embedding of $M_0^2(n)$ in $M^2(n)$. We consider the field K to be a Φ_n-module in which the operation of $S(n)$ is the trivial one, namely $\tau\alpha = \alpha$ for all $\alpha \in K$ and $\tau \in S(n)$. We then define a Φ_n-homomorphism $a: M^2(n) \to K$ by

$$a\left(\sum_{i<j} \lambda_{ij} x_i x_j\right) = \sum_{i<j} \lambda_{ij}.$$

THEOREM 3. *The following sequence is exact*:

$$0 \longrightarrow M_0^2(n) \xrightarrow{\text{incl}} M^2(n) \xrightarrow{a} K \longrightarrow 0.$$

Further, the sequence splits if and only if p does not divide $n(n-1)/2$.

Proof. The sequence is clearly exact.

Suppose that p does not divide $n(n-1)/2$, and define $f: K \to M^2(n)$ by

$$f(1) = \frac{2\sigma}{n(n-1)}.$$

Then f is a Φ_n-homomorphism, and clearly af is the identity on K. Hence the sequence splits.

In order to prove the converse, suppose that $f: K \to M^2(n)$ is a Φ_n-homomorphism, and let

$$f(1) = \sum_{i<j} \lambda_{ij} x_i x_j.$$

Since $f(1)$ is invariant under all transpositions, the coefficients λ_{ij} are all equal, say, to γ (using (3)). Then

$$f(1) = \gamma\sigma.$$

Now suppose that af is the identity on K. Then

$$1 = af(1) = a(\gamma\sigma) = \gamma n(n-1)/2.$$

Hence p does not divide $n(n-1)/2$. This completes the proof.

Theorem 3 implies that, when p does not divide $n(n-1)/2$,

$$M^2(n) = M_0^2(n) \oplus K\sigma.$$

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We can also deduce that $M_0^2(n)$ is not a direct summand of $M^2(n)$ when p divides $n(n-1)/2$. For if $M_2(n) = M_0^2(n) \oplus B$, then B contains an element y which is invariant under all permutations. Applying formula (3), we see that $y = \lambda\sigma$ for some $\lambda \in K$. But $\sigma \in M_0^2(n)$ when p divides $n(n-1)/2$, and this is a contradiction. This result may be compared with (2.1) of [**1**]. A similar argument shows that, when $p \mid n-2$, $S(n)$ is not a direct summand of $M_0^2(n)$.

3. Analysis of $S(n)$ when p does not divide $n-2$. Recall that

$$\theta_{ij}^n = x_n(s - x_n - x_i - x_j) + x_i x_j.$$

LEMMA 2.

$$\sum_{1 \le i < j < n} \theta_{ij}^n = \begin{cases} \sigma & \text{if } p \ne 2 \text{ and } p \mid n-1, \text{ or if } p = 2 \text{ and } n = 4a+1, \\ \sigma - a_n & \text{if } p \ne 2 \text{ and } p \mid n-2, \text{ or if } p = 2 \text{ and } n = 2(2a+1). \end{cases}$$

Proof.

$$\sum_{1 \le i < j < n} \theta_{ij}^n = \sum_{1 \le i < j < n} x_i x_j + \frac{(n-1)(n-2)}{2} x_n(s - x_n) - x_n \sum_{1 \le i < j < n} (x_i + x_j).$$

Under each of the four conditions described in the statement of the lemma, $(n-1)(n-2)/2$ is zero in K, and, by applying Lemma 1 to the last term,

$$\sum_{1 \le i < j < n} \theta_{ij}^n = \sum_{1 \le i < j < n} x_i x_j - x_n(n-2) \sum_{i=1}^{n-1} x_i = \sigma - a_n - (n-2)a_n.$$

If $p \ne 2$ and p divides $n-1$, then $n-2 \equiv -1 \pmod{p}$, and so

$$\sum_{1 \le i < j < n} \theta_{ij}^n = \sigma.$$

If $p \ne 2$ and p divides $n-2$,

$$\sum_{1 \le i < j < n} \theta_{ij}^n = \sigma - a_n.$$

The results for $p = 2$ follow similarly.

We now turn to the problem of analysing $S(n)$.

THEOREM 4. (i) *Suppose that p is not equal to 2. $S(n)$ is irreducible when p divides neither $n-1$ nor $n-2$. When p divides $n-1$, a composition series for $S(n)$ is given by*

$$0 \subset K\sigma \subset S(n).$$

(ii) *Suppose that p is equal to 2. $S(n)$ is irreducible when $n = 2a+1$ with a odd. When $n = 2a+1$ with a even, a composition series for $S(n)$ is given by*

$$0 \subset K\sigma \subset S(n).$$

Proof. Let $x \in S(n)$. We may write

$$x = \sum_{1 \le i < j < n} \lambda_{ij} \theta_{ij}^n \quad \text{with} \quad \sum_{1 \le i < j < n} \lambda_{ij} = 0$$

(Corollary 1, Theorem 1).

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Supposing either that $p \neq 2$ and p does not divide $n-1$ or $n-2$ or that $p = 2$ and $n = 2a+1$ with a odd, we shall assume that $x \neq 0$ and prove that $\Phi_n x = S(n)$; supposing either that $p \neq 2$ and p divides $n-1$, or that $p = 2$ and $n = 2a+1$ with a even, we shall assume that $\bar{x}$, the coset of x in $S(n)/K\sigma$, is not zero, and we shall prove that $\Phi_n \bar{x} = S(n)/K\sigma$. The theorem will follow from this, except for the case $n = 4$.

Let η denote the permutation $(x_k\, x_l)$, where $1 \leqq k < l < n$. By the method of the proof of (3), we find that

$$y_{kl} = x - \eta x = \sum_{i \neq k,l,n} (\lambda_{il} - \lambda_{ik})(\theta^n_{il} - \theta^n_{ik}).$$

Further, let r and s be two integers less than n such that r, s, k, l are four distinct integers. If μ denotes the permutation $(x_r\, x_s)$, we have

$$\begin{aligned} z_{klrs} = y_{kl} - \mu y_{kl} &= (\lambda_{sl} - \lambda_{sk} - \lambda_{rl} + \lambda_{rk})(\theta^n_{sl} - \theta^n_{sk} - \theta^n_{rl} + \theta^n_{rk}) \\ &= (\lambda_{sl} - \lambda_{sk} - \lambda_{rl} + \lambda_{rk})(x_s - x_r)(x_l - x_k). \end{aligned}$$

If, for some set $\{r, s, k, l\}$, $(\lambda_{sl} - \lambda_{sk} - \lambda_{rl} + \lambda_{rk}) \neq 0$, then clearly $\Phi_n x$ contains $(x_s - x_r)(x_k - x_l)$, which generates $S(n)$, and hence $\Phi_n x = S(n)$. Otherwise

$$\lambda_{sl} - \lambda_{sk} = \lambda_{rl} - \lambda_{rk}$$

for all r, s, k, l. In this case,

$$\begin{aligned} y_{kl} &= (\lambda_{rl} - \lambda_{rk}) \sum_{i \neq k,l,n} (\theta^n_{il} - \theta^n_{ik}) \\ &= (\lambda_{rl} - \lambda_{rk}) \sum_{i \neq k,l,n} (x_i x_l - x_i x_k - x_l x_n + x_k x_n) \\ &= (\lambda_{rl} - \lambda_{rk}) \left\{ \left[\sum_{i \neq k,l} (x_i x_l - x_i x_k) \right] - (x_l\, x_n - x_k x_n) - (n-3)(x_l\, x_n - x_k x_n) \right\} \\ &= (\lambda_{rl} - \lambda_{rk})\big((a_l - a_k) - (n-2)x_n(x_l - x_k)\big). \end{aligned}$$

Suppose there exist l, k such that $\lambda_{rl} - \lambda_{rk} \neq 0$. Then $\Phi_n x$ contains $(a_l - a_k) - (n-2)x_n(x_l - x_k)$, and, if t is different from k, l, n, $\Phi_n x$ also contains $(a_l - a_k) - (n-2)x_t(x_l - x_k)$. $\Phi_n x$ also contains the difference of these two, namely $(n-2)(x_n - x_t)(x_l - x_k)$, and, since p does not divide $n-2$, $\Phi_n x$ contains $(x_n - x_t)(x_l - x_k)$, which generates $S(n)$. Thus $\Phi_n x = S(n)$. Otherwise, $\lambda_{rl} = \lambda_{rk}$ for all r, k, l, i.e. the coefficients are all equal, say to λ. But $\sum_{1 \leqq i < j < n} \lambda_{ij} = 0$, and hence

$$\tfrac{1}{2}(n-1)(n-2)\lambda = 0.$$

This is where we must distinguish between the different cases.

If $p \neq 2$ and p does not divide $n-1$ or $n-2$, then $\lambda = 0$ and $x = 0$, contrary to assumption. If $p \neq 2$ and p divides $n-1$, then, by Lemma 2,

$$\begin{aligned} x &= \lambda \sum_{1 \leqq i < j < n} \theta^n_{ij} \\ &= \lambda\sigma, \end{aligned}$$

and thus $\bar{x} = 0$ in $S(n)/K\sigma$, contrary to assumption. This completes the proof of part (i).

If $p = 2$ and $n = 2a+1$ with a odd, then $\frac{1}{2}(n-1)(n-2)\lambda = 0$ implies $\lambda = 0$ and hence $x = 0$, contrary to assumption. If $p = 2$ and $n = 2a+1$ with a even, we again have $x = \lambda\sigma$, and hence $\bar{x} = 0$, contrary to assumption. This completes the proof, except for the case $n = 4$, $p = 3$. But this case is trivial since the K-dimension of $S(4)$ is 2, and $S(4)$ contains the submodule $K\sigma$.

This completes the study of the cases when p does not divide $n-2$. We have $M_0^2(n) = \tilde{N} \oplus S(n)$ with $\tilde{N} \approx M_0^1(n)$, and $M^2(n)/M_0^2(n) \approx K$.

For the situations considered in this section, composition series can easily be constructed from the chain

$$0 \subset \tilde{N} \subset \tilde{N} + S(n) = M_0^2(n) \subset M^2(n),$$

using Theorem 4 and the results from [1] quoted in Section 1. For instance, in the case $p \neq 2$, $p \mid n-1$, we obtain the composition series

$$0 \subset \tilde{N} \subset \tilde{N} + K\sigma \subset M_0^2(n) \subset M^2(n).$$

4. The case p divides $n-2$. In order to calculate the composition factors of $S(n)$ when p divides $n-2$, we look at $S(n)$ considered as a Φ_{n-1}-module. Any Φ_n-module may be regarded as a Φ_{n-1}-module. The operation of S_{n-1} on the module is determined by the process of restriction from S_n to its subgroup S_{n-1}.

By Corollary 1, Theorem 1, $S(n)$ has a K-basis

$$\{\theta^n_{12} - \theta^n_{ij} : 1 \leqq i < j < n,\ (i,j) \neq (1,2)\}.$$

$M_0^2(n-1)$ is the set of polynomials of the form $\sum_{1 \leqq i < j < n} \lambda_{ij} x_i x_j$ with $\sum_{1 \leqq i < j < n} \lambda_{ij} = 0$. It has a K-basis

$$\{x_1 x_2 - x_i x_j : 1 \leqq i < j < n,\ (i,j) \neq (1,2)\}.$$

There is an obvious Φ_{n-1}-isomorphism

$$f : M_0^2(n-1) \to S(n), \tag{5}$$

namely that defined by

$$f(x_1 x_2 - x_i x_j) = \theta^n_{12} - \theta^n_{ij} \qquad (1 \leqq i < j < n,\ (i,j) \neq (1,2)).$$

This proves

LEMMA 3. *$S(n)$ and $M_0^2(n-1)$ are Φ_{n-1}-isomorphic.*

In the case $n = 4$, p divides $n-2$ implies that $p = 2$. $S(4) \approx M_0^2(3)$ over Φ_3, and $M_0^2(3) \approx M_0^1(3)$. The latter is irreducible over Φ_3, and hence $S(4)$ is irreducible over Φ_3. But any Φ_4-submodule of $S(4)$ is also a Φ_3-submodule of $S(4)$, and hence $S(4)$ is irreducible over Φ_4. We shall next use Lemma 3 and the results of the last section to find the composition factors of $M^2(n)$ when p divides $n-2$ and $n > 4$.

Set $n' = n-1$. Then p divides $n'-1$. The composition factors of $M^2(n')$ are known. We treat the cases $p = 2$ and $p \neq 2$ separately, taking first the case $p \neq 2$. A composition series for $M^2(n')$ is given by

$$0 \subset \tilde{N}(n-1) \subset \tilde{N}(n-1) + K\sigma(n-1) \subset \tilde{N}(n-1) + S(n-1) = M_0^2(n-1) \subset M^2(n-1).$$

$\tilde{N}(n-1)$ has a K-basis $\{b_{1j}(n-1):j=2,\ldots,n-1\}$, for it is generated by this set, and, being isomorphic to $M_0^1(n-1)$, it has K-dimension $n-2$. By definition,

$$b_{1j}(n-1)=\sum_{k\neq 1,j,n}(x_1x_k-x_jx_k).$$

Using the isomorphism (5), we obtain

$$\begin{aligned} f(b_{1j}(n-1)) &= \sum_{k\neq 1,j,n}(\theta_{1k}^n-\theta_{jk}^n) \\ &= \sum_{k\neq 1,j,n}(x_1x_k-x_jx_k-x_1x_n+x_jx_n) \\ &= \sum_{k\neq 1,j}(x_1x_k-x_jx_k)-(n-2)(x_1x_n-x_jx_n) \\ &= b_{1j}\in\tilde{N}(n). \end{aligned}$$

Also, by Lemma 2,

$$f(\sigma(n-1))=\sum\theta_{ij}^n=\sigma-a_n.$$

We show that $\tilde{N}(n)$ and $\tilde{N}(n-1)+K\sigma(n-1)$ correspond under the isomorphism f. To do this we show that $\tilde{N}(n)$ has a basis $\{b_{12},\ldots,b_{1(n-1)},\sigma-a_n\}$. This set is linearly independent since f is an isomorphism. Also

$$\sum_{j=2}^{n}b_{1j}=\sum_{j=2}^{n}(a_1-a_j)=(n-1)a_1-\sum_{j=2}^{n}a_j=a_1-(2\sigma-a_1)=2(a_1-\sigma).$$

Since $p\neq 2$, $\sigma-a_1\in\tilde{N}(n)$, and, since this is a Φ_n-module, $\sigma-a_n\in\tilde{N}(n)$. This shows that $\dim_K\tilde{N}(n)\geqq n-1$. But $\tilde{N}(n)$ is generated over K by $\{b_{12},\ldots,b_{1n}\}$, so that $\dim_K\tilde{N}(n)\leqq n-1$. Hence $\dim_K\tilde{N}(n)=n-1$, and it follows that $\{b_{12},\ldots,b_{1(n-1)},\sigma-a_n\}$ is a K-basis for $\tilde{N}(n)$. Thus the isomorphism (5) leads to the chain

$$0\subset\tilde{N}(n)\subset S(n)$$

in which $S(n)\approx M_0^2(n-1)$ and $\tilde{N}(n)\approx\tilde{N}(n-1)+K\sigma(n-1)$ over Φ_{n-1}. Hence f induces a Φ_{n-1}-isomorphism

$$M_0^2(n-1)/\{\tilde{N}(n-1)+K\sigma(n-1)\}\rightarrow S(n)/\tilde{N}(n).$$

The former is irreducible over Φ_{n-1}, and hence the latter is irreducible over Φ_n.

We now observe that $\tilde{N}(n)$ is irreducible when $p\neq 2$ and p divides $n-2$. In fact $\tilde{N}(n)\approx M_0^1(n)$; this is clear since $\tilde{N}(n)$ has a K-basis $\{b_{12},\ldots,b_{1n}\}$, whilst $M_0^1(n)$ has a K-basis $\{x_1-x_2,\ldots,x_1-x_n\}$ and the map g, defined by

$$g(b_{1j})=x_1-x_j\qquad(j=2,\ldots,n),$$

is a Φ_n-isomorphism. The results of [**1**] show that $\tilde{N}(n)$ is irreducible. This completes the proof of

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THEOREM 5. *When p is not equal to 2 and p divides n−2, the following is a composition series of $M^2(n)$:*

$$0 \subset \tilde{N}(n) \subset S(n) \subset M_0^2(n) \subset M^2(n).$$

Recall that, by Theorem 1,

$$M_0^2(n)/S(n) \approx M_0^1(n).$$

We use a similar method to find a composition series for $M^2(n)$ when $p = 2$ and n is even.

THEOREM 6. *Let $p = 2$.*

(i) *If $n = 2a$ with a even, $S(n)$ has a composition series*

$$0 \subset \tilde{N}(n) \subset S(n).$$

(ii) *If $n = 2a$ with a odd, then $S(n)$ has a composition series*

$$0 \subset \tilde{N}(n) \subset \Phi_n(\sigma - a_n) \subset S(n).$$

Proof. We use the Φ_{n-1}-isomorphism (5).

(i) If $n = 2a$ with a even, then $n' = n-1 = 2(a-1)+1$, with $a-1$ odd. A composition series for $M_0^2(n-1)$ is given by

$$0 \subset \tilde{N}(n-1) \subset \tilde{N}(n-1)+S(n-1) = M_0^2(n-1).$$

(ii) If $n = 2a$ with a odd, then $n' = n-1 = 2(a-1)+1$ with $a-1$ even, and a composition series for $M_0^2(n-1)$ is given by

$$0 \subset \tilde{N}(n-1) \subset \tilde{N}(n-1)+K\sigma(n-1) \subset \tilde{N}(n-1)+S(n-1) = M_0^2(n-1).$$

In both cases we have $f(b_{1j}(n-1)) = b_{1j} \in \tilde{N}(n)$ for $j = 2, \ldots, n-1$, and so we know that $\{b_{1j} : j = 2, \ldots, n-1\}$ is a linearly independent set. Also, $\{b_{1j} : j = 2, \ldots, n\}$ generates $\tilde{N}(n)$, and

$$\sum_{j=2}^{n} b_{1j} = \sum_{j=2}^{n} (a_1 + a_j) = (n-1)a_1 + \sum_{j=2}^{n} a_j = a_1 + (2\sigma + a_1) = 0.$$

Thus $\{b_{1j} : j = 2, \ldots, n-1\}$ generates $\tilde{N}(n)$. This set is therefore a K-basis for $\tilde{N}(n)$, and $\dim_K \tilde{N}(n) = n-2$.

This proves that $\tilde{N}(n-1)$ and $\tilde{N}(n)$ are Φ_{n-1}-isomorphic. Since the former is irreducible over Φ_{n-1}, the latter is irreducible over Φ_n.

In the case $n = 2a$ with a even we also have a Φ_{n-1}-isomorphism

$$M_0^2(n-1)/\tilde{N}(n-1) \to S(n)/\tilde{N}(n)$$

induced by f. Hence we have $S(n)/\tilde{N}(n)$ irreducible over Φ_n. This completes the proof of part (i).

Now consider the case $n = 2a$ with a odd. By Lemma 2, $f(\sigma(n-1)) = \sigma - a_n$. $\Phi_n(\sigma - a_n)$ is a submodule of $S(n)$ containing $\tilde{N}(n)$, and $\{b_{12}, \ldots, b_{1(n-1)}, \sigma - a_n\}$ is a linearly independent

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set. Further, $\Phi_n(\sigma-a_n)$ is generated over K by $\{\sigma-a_2,\ldots,\sigma-a_n\}$, and so both of these sets form a K-basis for $\Phi_n(\sigma-a_n)$. Hence f induces a Φ_{n-1}-isomorphism between $\tilde{N}(n-1)+K\sigma(n-1)$ and $\Phi_n(\sigma-a_n)$. Hence f induces a Φ_{n-1}-isomorphism

$$M_0^2(n-1)/\{\tilde{N}(n-1)+K\sigma(n-1)\}\rightarrow S(n)/\Phi_n(\sigma-a_n).$$

The former is irreducible over Φ_{n-1}, and so the latter is irreducible over Φ_n. This completes the proof of part (ii).

This theorem and the fact that $M_0^2(n)/S(n)$ is isomorphic to $M_0^1(n)$ tell us the composition factors of $M^2(n)$ when $p=2$ and n is even. In this case, it was found that $M_0^1(n)$ has an irreducible factor space of K-dimension $n-2$. We show that this is isomorphic to $\tilde{N}(n)$.

We have the following exact sequences of Φ_n-modules:

$$0\longrightarrow Ks\xrightarrow{\text{incl}}M^1(n)\xrightarrow{\phi}N(n)\longrightarrow 0$$

and

$$0\longrightarrow M_0^1(n)\xrightarrow{\text{incl}}M^1(n)\xrightarrow{a}Ks\longrightarrow 0.$$

The mappings ϕ and a are defined by

$$\phi(x_i)=a_i \qquad (i=1,\ldots,n)$$

and

$$a(x_i)=s \qquad (i=1,\ldots,n),$$

respectively. Hence $N\approx M^1(n)/Ks$ and $Ks\approx M^1(n)/M_0^1(n)$. This shows that N and $M_0^1(n)$ have the same composition factors. From a knowledge of the composition factors of $M_0^1(n)$, we deduce that $0\subset\tilde{N}(n)\subset N(n)$ is a composition series for $N(n)$ and that $\tilde{N}(n)$ is isomorphic to the $(n-2)$-dimensional composition factor of $M_0^1(n)$.

Thus when $p=2$ and p divides $n-2$, the irreducible Φ_n-module $\tilde{N}$ appears twice in a composition series for $M^2(n)$. We saw that the same was true when $p\neq 2$ and p divides $n-2$.

This completes the analysis of the second natural representation module of the symmetric groups. We have obtained certain irreducible representation modules, namely factor modules of $S(n)$, in addition to the irreducible representation modules obtained in (5.2) of [1].

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