Associated Mathieu Functions.

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§ 1. *Definition of the Functions.*

The periodic solutions of the linear differential equation

$$(1) \qquad \frac{d^2u}{dz^2} + \left[a + 2\theta \cos 2z - \frac{\nu(\nu-1)}{\sin^2 z}\right] u = 0,$$

which reduce to Mathieu functions when $\nu = 0$ or 1, will be known as the associated Mathieu functions. The significance of this terminology will appear in the following section.

The differential equation is not new to analysis; it appears, for example, in a paper by Max Abraham * on the problem of damped wave motion in stated modes of vibration. The present writer, in the course of other investigations,† obtained the integral equation

$$(2) \qquad u(z) = \lambda \int_0^{2\pi} e^{k \cos z \cos s} \sin^\nu z \sin^\nu s \, u(s) \, ds, \qquad (k^2 = 4\theta)$$

whose solutions appear to be solutions also of (1). A recent paper by M. P. Humbert, read before this Society,‡ gives this integral equation, which is said to hold only when ν is an integer, and is to be replaced, when ν is not an integer, by an integral relation between two distinct solutions of (1). The purpose of this present paper is to show that the integral equation holds for *all* real values of $\nu > -\frac{1}{2}$, and that the limits of integration may be taken to be 0 and π, which is a great advantage in that no branch point of $\sin^\nu s$ occurs within the range of integration.

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* *Math. Annalen,* 52 (1899) pp. 81–112.

† *Proc. R.S.E.*, 42 (1922) p. 47.

‡ *Proceedings,* 40 (1922) pp. 28–29.

§ 2. *Identification of the Equation.*

If equation (1) be reduced to its algebraic form by the substitution $\zeta = \cos^2 z$, it then has three regular singular points, viz., an elementary* singularity at $\zeta = 0$, a regular singularity with exponent-difference $\nu - \frac{1}{2}$ at $\zeta = 1$, and an irregular singularity of the first species at $\zeta = \infty$. In the ordinary Mathieu equation, the singularity at $\zeta = 1$ is elementary. If now the irregular singularity at infinity be simplified, becoming a regular singularity, equation (1) becomes the associated Gegenbauer equation, and Mathieu's eqnation becomes the ordinary Gegenbauer equation. Thus equation (1) bears the same relation to Mathieu's equation as the associated bears to the ordinary Gegenbauer equation; † for this reason it has been called the equation of the associated Mathieu functions.

§ 3. *The Four Types of Function.*

It is known, from the general theory of equations whose coefficients are periodic, that the two fundamental solutions, when a has a pre-assigned value, are

$$u = \sin^{\nu} z \, u_1(z)$$

and

$$u = \sin^{1-\nu} z \, u_2(z),$$

where $u_1(z)$ and $u_2(z)$ are *uniform* functions of z, but are not, in general, periodic. For certain *characteristic* values of a, either $u_1(z)$ or $u_2(z)$, but not necessarily both, may be periodic.‡

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* *I.e.* a regular singularity with exponent-difference $\frac{1}{2}$. It is to be remembered that the coalescence of two elementary singularities produces in general a regular singularity with arbitrary exponent-difference; the coalescence of three elementary singularities generates an irregular singularity of the first species, and so on.

† When $\nu = \frac{1}{2}$ the equation bears the same relation to Legendre's equation as its general form bears to the associated Legendre equation.

‡ It has been proved by the present writer, *Proc. Camb. Phil. Soc.*, 21 (1922) p. 117, that when $\nu = 0$ or 1 the equation cannot have two periodic solutions except for $\theta = 0$. It is shown in the present section that this is true for all values of ν.

These periodic solutions, when they exist, are of four distinct types:

$$\text{I.}\quad u = \sin^{\nu} z\, F_o(n, \nu, z),\quad \text{where } a = (2n+1+\nu)^2 + f_o(n, \theta, \nu)$$

$$\text{II.}\quad u = \sin^{1-\nu} z\, F_o(n, 1-\nu, z),\quad \text{where } a = (2n+2-\nu)^2 + f_o(n, \theta, 1-\nu).$$

$$\text{III.}\quad u = \sin^{\nu} z\, F_e(n, \nu, z),\quad \text{where } a = (2n+\nu)^2 + f_e(n, \theta, \nu)$$

$$\text{IV.}\quad u = \sin^{1-\nu} z\, F_e(n, 1-\nu, z),\quad \text{where } a = (2n+1-\nu)^2 + f_e(n, \theta, 1-\nu).$$

In each case $n = 0, 1, 2, \ldots$; F_o and F_e are respectively odd and even functions of $\cos z$, and f_o and f_e vanish when $\theta = 0$. When $\nu = \frac{1}{2}$, Types I. and II. and Types III. and IV. become identical. $F_o(n, \nu, z)$ and $F_e(n, \nu, z)$ are uniform periodic solutions of

$$\frac{d^2 u}{dz^2} + 2\nu \cot z \frac{du}{dz} + (a - \nu^2 + 2\theta \cos 2z)\, u = 0. \tag{3}$$

The a corresponding to Type I. is obtained by equating to zero the determinant

$$\begin{vmatrix} a-(\nu+1)^2, & -a+(\nu-3)^2+\theta, & -\theta, & 0, & \ldots \\ \theta, & a-(\nu+3)^2-\theta, & -a+(\nu-5)^2+\theta, & -\theta, & \ldots \\ 0, & \theta, & a-(\nu+5)^2-\theta, & -a+(\nu-7)^2+\theta, & \ldots \\ 0, & 0, & \theta, & a-(\nu+7)^2-\theta, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

and is clearly not unchanged when $1-\nu$ is written for ν except when $\nu = \frac{1}{2}$. Similarly the a corresponding to Type III. is obtained by equating to zero the determinant

$$\begin{vmatrix} a-\nu^2-\theta, & -a+(\nu-2)^2+2\theta, & -\theta, & 0, & \ldots \\ \theta, & a-(\nu+2)^2-\theta, & -a+(\nu-4)^2+\theta, & -\theta, & \ldots \\ 0, & \theta, & a-(\nu+4)^2-\theta. & -a+(\nu-6)^2+\theta, & \ldots \\ 0, & 0, & \theta, & a-(\nu+6)^2-\theta, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

and is also unchanged by the substitution of $1-\nu$ for ν except when $\nu = \frac{1}{2}$.

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If another periodic solution could coexist with one of Type I., it would be of Type IV., but the forms of the determinants show that this is impossible unless $\theta=0$ Similarly Types II. and III. might coexist, but this also is impossible unless $\theta=0$.

It is to be noted that, when $\nu=0$ or 1, and $\theta \neq 0$,

$$
\begin{aligned}
F_o(n,\ 0,\ z) &= ce_{2n+1}(z),\\
\sin z\ F_o(n,\ 1,\ z) &= se_{2n+2}(z),\\
F_e(n,\ 0,\ z) &= ce_{2n}(z),\\
\sin z\ F_e(n,\ 1,\ z) &= se_{2n+1}(z).
\end{aligned}
$$

When $\theta=0$, the functions degenerate into the Gegenbauer functions as follows:

$$
\begin{aligned}
F_o(n,\ \nu,\ z) &= C^{\nu}_{2n+1}(\cos z),\\
F_e(n,\ \nu,\ z) &= C^{\nu}_{2n}(\cos z),\\
\sin^{1-2\nu} z\ F_o(n,\ 1-\nu,\ z) &= H^{\nu}_{2n+2}(\cos z),\\
\sin^{1-2\nu} z\ F_e(n,\ 1-\nu,\ z) &= H^{\nu}_{2n+1}(\cos z).
\end{aligned}
$$

§ 4. *The Integral Equation.*

It may easily be proved, that if $u(z)$ be a solution of equation (1), of Type I. or III., say

$$u(z)=\sin^{\nu}(z)\ F(z),$$

then the integral

$$y(z)=\int_{\alpha}^{\beta} e^{k\cos z\cos s}\sin^{\nu} z\sin^{\nu} s\ u(s)\,ds \tag{4}$$

is a solution of (1) provided that

$$\sin^{2\nu} s\ \{k\sin s\cos x\ F(s)+F'(s)\}=0,$$

for all values of z, when $s=\alpha$ and $s=\beta$. This condition certainly holds if $\alpha=0$ and $\beta=\pi$, provided $\nu>-\frac{1}{2}$, which is also a necessary condition for the convergence of the integral.

If ν be not integral, let $\sin^{\nu} s$ denote the principal branch of the function, *i.e.* that branch which is real and positive in the range $0<s<\pi$.

The integral $y(z)$ is then not zero, and it clearly is a function of Type I. or III. But since the equation cannot admit at the same time two distinct solutions of those types, $y(z)$ must be identified with $u(z)$. Consequently the solutions of (1) of Types I. and III. satisfy the integral equation

$$(5) \qquad u(z) = \lambda \int_0^\pi e^{k \cos z \cos s} \sin^\nu z \sin^\nu s \, u(s) \, ds,$$

provided $\nu > -\frac{1}{2}$.

Similarly, solutions of Types II. and IV. are solutions of the integral equation

$$(5a) \qquad u(z) = \lambda \int_0^\pi e^{k \cos z \cos s} \sin^{1-\nu} z \sin^{1-\nu} s \, u(s) \, ds,$$

provided $\nu < \frac{3}{2}$.

It is to be noted that if, in (4), $u(s)$ be taken to be a solution of Type II. or IV., β must be taken to be $\alpha + 2\pi$, and then $y(z)$ becomes identically zero.

§ 5. *Notation.*

Write

$$\begin{aligned}
\text{I.} \quad & ce^\nu_{2n+1}(z) = \sin^\nu z \, F_o(z, n, \nu) \\
\text{II.} \quad & se^\nu_{2n+2}(z) = \sin^{1-\nu} z \, F_o(z, n, 1-\nu) \\
\text{III.} \quad & ce^\nu_{2n} \quad (z) = \sin^\nu z \, F_e(z, n, \nu) \\
\text{IV.} \quad & se^\nu_{2n+1}(z) = \sin^{1-\nu} z \, F_e(z, n, 1-\nu),
\end{aligned}$$

so that $ce^{1-\nu}_{2n+1}(z) = se^\nu_{2n+2}(z)$, $ce^{1-\nu}_{2n}(z) = se^\nu_{2n+1}(z)$, and in particular,

$$\begin{aligned}
ce^0_{2n+1}(z) &= ce_{2n+1}(z) = se^1_{2n+2}(z) \\
se^0_{2n+2}(z) &= se_{2n+2}(z) = ce^1_{2n+1}(z) \\
ce^0_{2n} \quad (z) &= ce_{2n} \quad (z) = se^1_{2n+1}(z) \\
se^0_{2n+1}(z) &= se_{2n+1}(z) = ce^1_{2n} \, (z)
\end{aligned}$$

By considering the development of the nuclei of integral equations (5) and (5a), it is seen that the associated Mathieu functions as above defined may be expressed in terms of ordinary Mathieu functions in the following ways:

$$\begin{aligned}
ce^\nu_{2n+1}(z) &= \sin^\nu z \sum_{r=0}^\infty a_r \, ce_{2r+1}(z) \\
&= \sin^{\nu-1} z \sum_{r=0}^\infty a'_r \, se_{2r+2}(z), \\
ce^\nu_{2n}(z) &= \sin^\nu z \sum_{r=0}^\infty b_r \, ce_{2r}(z) \\
&= \sin^{\nu-1} z \sum_{r=0}^\infty b'_r \, se_{2r+1}(z),
\end{aligned}$$

and $se_{2n+1}(z)$ and $se_{2n+2}(z)$ may be similarly developed.

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§ 6. *An Analogous Equation.*

The equation

$$(6) \qquad \frac{d^2u}{dz^2} + \left[a + 2\theta \cos 2z - \frac{\mu(\mu-1)}{\cos^2 z}\right] u = 0$$

is satisfied by the solutions of the integral equations

$$(7) \qquad u(z) = \lambda \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ik \sin z \sin s} \cos^{\mu} z \cos^{\mu} s \, u(s) \, ds,$$

provided $\mu > -\frac{1}{2}$, and

$$(7a) \qquad u(z) = \lambda \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{ik \sin z \sin s} \cos^{1-\mu} z \cos^{1-\mu} s \, u(s) \, ds,$$

provided $\mu > \frac{3}{2}$.

§ 7 *The Associated Lamé Functions.*

These functions are the solutions of the equation *

$$(8) \qquad \frac{d^2u}{dz^2} - \left[h + n(n+1) k^2 sn^2 z + \frac{m(m-1) k^2 cn_2 z}{dn^2 z}\right] u = 0$$

which are of the forms

$$dn^m z \, F_1(z) \text{ and } dn^{1-m} z \, F_2(z),$$

when $F_1(z)$ and $F_2(z)$ are uniform doubly-periodic functions of z.

The equation, in the algebraic form obtained by writing $\zeta = sn^2 z$, has two elementary and two regular singularities; it reduces, when $m(m-1) = 0$, to an equation having three elementary singularities and one regular singularity, that is, to the algebraic form of Lamé's equation.

The functions satisfy an integral equation of the form

$$(9) \qquad u(x) = \lambda \int_0^{2K} \Phi_n^{\gamma}(k \, sn \, z \, sn \, s) \, dn^{\gamma} z \, dn^{\gamma} s \, u(s) \, ds,$$

where $\gamma = m > -\frac{1}{2}$ or $\gamma = 1 - m > -\frac{1}{2}$, and $\Phi_n^{\gamma}(\mu)$ is the Legendre function $(1-\mu^2)^{-\frac{1}{2}\gamma} P_n^{\gamma}(\mu)$, or the Gegenbauer function $C_{n-\gamma}^{\gamma+\frac{1}{2}}(\mu)$.

* Hermite, *Crelle's Journal*, 89 (1881) p. 18, *Œuvres*, 4 p. 18. A still further generalisation of Lamé's equation is given by Darboux, *Comptes rendus* 1882, and de Sparre, *Acta Math.* 4 (1883) pp. 105–140, 289–321.