Nagoya Math. J. **205** (2012), 119–148
DOI 10.1215/00277630-1543796

SCHATTEN p-CLASS PROPERTY OF PSEUDODIFFERENTIAL OPERATORS WITH SYMBOLS IN MODULATION SPACES

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Abstract. It is proved that the pseudodifferential operators $\sigma_t(X,D)$ belong to the Schatten p-class C_p, $0<p\le 2$, if the symbol $\sigma(x,\omega)$ is in certain modulation spaces on $\mathbf{R}^d_x\times\mathbf{R}^d_\omega$.

§1. Introduction

The Schatten p-class C_p is the class of compact operators introduced by von Neumann and Schatten, which is defined as follows (see [2], [20]). Let $0<p<\infty$, and let A be a compact operator on $L^2(\mathbf{R}^d)$. If the singular values $s_j(A)$ of A, that is, the eigenvalues of the positive compact operator $|A|=(A^*A)^{1/2}$, satisfy $\|A\|_{C_p}=\big(\sum_{j=1}^\infty s_j(A)^p\big)^{1/p}<\infty$, then A is said to be in the *Schatten p-class* C_p, and we denote $A\in C_p$.

If $\sigma(x,\omega)$ is a function on $\mathbf{R}^d_x\times\mathbf{R}^d_\omega$ and $t\in\mathbf{R}$, then the pseudodifferential operator $\sigma_t(X,D)$ is defined by

$$\begin{aligned}\sigma_t(X,D)f(x)&=\iint_{\mathbf{R}^{2d}}\sigma\big((1-t)x+ty,\omega\big)e^{2\pi i(x-y)\omega}f(y)\,dy\,d\omega\\&=\iint_{\mathbf{R}^{2d}}\widehat{\sigma}(\xi,\eta)e^{2\pi i(x+t\eta)\xi}f(\eta+x)\,d\xi\,d\eta.\end{aligned}$$

The operator $\sigma_t(X,D)$ is a generalization of the Kohn-Nirenberg correspondence $(t=0)$

$$\sigma(X,D)f(x)=\int_{\mathbf{R}^d}\sigma(x,\omega)\widehat{f}(\omega)e^{2\pi ix\omega}\,d\omega,$$

and the Weyl correspondence $(t=1/2)$

$$\sigma^W(X,D)f(x)=\iint_{\mathbf{R}^{2d}}\sigma\Big(\frac{x+y}{2},\omega\Big)e^{2\pi i(x-y)\omega}f(y)\,dy\,d\omega.$$

Received November 8, 2010. Accepted August 23, 2011.
2010 Mathematics Subject Classification. 42B35, 47B10.

We will consider sufficient conditions on symbols $\sigma(x,\omega)$ to ensure that the operator $\sigma_t(X,D)$ belongs to the Schatten p-class C_p. These types of issues have been studied by a number of authors using a large variety of methods (see [19], [18], [1], and [9], [10], [12] cited below). In this article, we consider the pseudodifferential operators with symbols in the modulation spaces, which are defined as follows.

DEFINITION 1.1 [3, Definition 6.1]. Fix a nonzero $\varphi \in \mathcal{S}(\mathbf{R}^n)$, $0 < p, q \leq \infty$, and a positive function m on $\mathbf{R}^n \times \mathbf{R}^n$ which satisfies

$$m(z+z', \zeta+\zeta') \leq C m(z,\zeta)(1+|z'|+|\zeta'|)^s, \quad z, \zeta, z', \zeta' \in \mathbf{R}^n, \tag{1.1}$$

for some constants $C > 0$ and $s \geq 0$. Then the modulation space $M_m^{p,q}(\mathbf{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the quasi-norm

$$\|f\|_{M_m^{p,q}} = \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |f * (M_\zeta \varphi)(z)|^p m(z,\zeta)^p \, dz \right)^{q/p} d\zeta \right)^{1/q}$$

is finite, with obvious modifications if p or $q = \infty$.

It is known that $M_m^{p,q}(\mathbf{R}^n)$ and the equivalence class of the quasi-norm do not depend on the choice of the function φ (see [3], [5], [7]). We simply write $M^{p,q}(\mathbf{R}^n)$ instead of $M_m^{p,q}(\mathbf{R}^n)$ when $m \equiv 1$.

In the present paper, we will use the modulation space on $\mathbf{R}^{2d}$. We will frequently write the elements of $\mathbf{R}^{2d}$ as

$$z = (z_1, z_2), \quad z_1, z_2 \in \mathbf{R}^d.$$

Concerning the modulation spaces and the Schatten class, the following theorem is known.

THEOREM A (see [9], [12]). *Let $0 < p < 2$. Then the pseudodifferential operators $\sigma_0(X,D) = \sigma(X,D)$ and $\sigma_{1/2}(X,D) = \sigma^W(X,D)$ are in the class C_p if $\sigma \in M_m^{2,2}(\mathbf{R}^{2d})$ with*

$$m(z,\zeta) = (1+|z_1|+|z_2|+|\zeta_1|+|\zeta_2|)^s \tag{1.2}$$

and $s > (2d/p) - d$.

As for the modulation space $M_m^{2,2}(\mathbf{R}^{2d})$ of this theorem, the following identification is known (see [7, Proposition 11.3.1]). If m is defined by (1.2),

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with $s \geq 0$ in general, then

$$\|\sigma\|_{M_m^{2,2}} \approx \left\|(1+|z_1|+|z_2|)^s \sigma(z_1,z_2)\right\|_{L^2_{z_1,z_2}} + \left\|(1+|\zeta_1|+|\zeta_2|)^s \widehat{\sigma}(\zeta_1,\zeta_2)\right\|_{L^2_{\zeta_1,\zeta_2}}, \tag{1.3}$$

where $\widehat{\sigma}$ denotes the Fourier transform defined by

$$\widehat{\sigma}(\zeta_1,\zeta_2) = \int_{\mathbf{R}^d \times \mathbf{R}^d} \sigma(z_1,z_2) e^{-2\pi i(z_1\zeta_1 + z_2\zeta_2)}\, dz_1\, dz_2.$$

One of the main purposes of the present paper is to give a refinement of Theorem A, which reads as follows.

THEOREM 1.2. *Let $0 < p < 2$, let $s > (2d/p) - d$, and let $t \in \mathbf{R}$. If $\sigma \in M_m^{2,2}(\mathbf{R}^{2d})$ with*

$$m(z,\zeta) = \left(1 + |z_1 + (1-t)\zeta_2| + |z_2 - t\zeta_1|\right)^s \tag{1.4}$$

or with

$$m(z,\zeta) = \left(1 + |z_1 - t\zeta_2| + |z_2 + (1-t)\zeta_1|\right)^s, \tag{1.5}$$

then $\sigma_t(X,D) \in C_p$.

We will also prove identifications similar to (1.3) for the modulation spaces $M_m^{2,2}(\mathbf{R}^{2d})$ of Theorem 1.2. In order to state the results, we use the following notation. We write the partial Fourier transforms of functions τ on $\mathbf{R}^{2d}$ as

$$F_1\tau(\zeta_1,z_2) = \int_{\mathbf{R}^d} \tau(z_1,z_2) e^{-2\pi i z_1\zeta_1}\, dz_1,$$
$$F_2\tau(z_1,\zeta_2) = \int_{\mathbf{R}^d} \tau(z_1,z_2) e^{-2\pi i z_2\zeta_2}\, dz_2.$$

For functions σ on $\mathbf{R}^{2d}$ and for $t \in \mathbf{R}$, we define

$$\tau_t(z_1,z_2) = e^{2\pi i t z_1 z_2} \sigma(z_1,z_2).$$

Then we have the following theorem.

THEOREM 1.3. *Let $\sigma \in L^2(\mathbf{R}^{2d})$, and let s be a nonnegative real number.*

(a) *If* $m(z,\zeta)=(1+|z_1+t\zeta_2|)^s$ *with* $t\neq 0$*, then*

$$\|\sigma\|_{M^{2,2}_m}\approx\big\|(1+|\zeta_2|)^s F_2\tau_{1/t}(z_1,\zeta_2)\big\|_{L^2_{z_1,\zeta_2}}.$$

(b) *If* $m(z,\zeta)=(1+|z_1|)^s$*, then*

$$\|\sigma\|_{M^{2,2}_m}\approx\big\|(1+|z_1|)^s\sigma(z_1,z_2)\big\|_{L^2_{z_1,z_2}}.$$

(c) *If* $m(z,\zeta)=(1+|z_2+t\zeta_1|)^s$ *with* $t\neq 0$*, then*

$$\|\sigma\|_{M^{2,2}_m}\approx\big\|(1+|\zeta_1|)^s F_1\tau_{1/t}(\zeta_1,z_2)\big\|_{L^2_{\zeta_1,z_2}}.$$

(d) *If* $m(z,\zeta)=(1+|z_2|)^s$*, then*

$$\|\sigma\|_{M^{2,2}_m}\approx\big\|(1+|z_2|)^s\sigma(z_1,z_2)\big\|_{L^2_{z_1,z_2}}.$$

If, for example, m is the weight of (1.4) and if we define

$$m_1(z,\zeta)=\big(1+|z_1+(1-t)\zeta_2|\big)^s,$$
$$m_2(z,\zeta)=\big(1+|z_2-t\zeta_1|\big)^s,$$

then obviously $m(z,\zeta)\approx m_1(z,\zeta)+m_2(z,\zeta)$, and hence,

$$\|\sigma\|_{M^{2,2}_m}\approx\|\sigma\|_{M^{2,2}_{m_1}}+\|\sigma\|_{M^{2,2}_{m_2}}.$$

This simple fact combined with Theorem 1.3 will give full identifications of the modulation spaces of Theorem 1.2. For example, for the weight $m(z,\zeta)=(1+|z_1+\zeta_2|+|z_2|)^s$, which is the m of (1.4) with $t=0$, we have

$$\|\sigma\|_{M^{2,2}_m}\approx\big\|(1+|\zeta_2|)^s F_2\tau_1(z_1,\zeta_2)\big\|_{L^2_{z_1,\zeta_2}}+\big\|(1+|z_2|)^s\sigma(z_1,z_2)\big\|_{L^2_{z_1,z_2}}.$$

Gröchenig and Heil [9] and Heil, Ramanathan, and Topiwala [12] proved Theorem A by constructing finite-rank operators that approximate the pseudodifferential operators. In fact, the argument of [9] and [12] can be modified to give a proof of Theorem 1.2 (see Section 5 below). In the present paper, we will give a different method to prove Theorem 1.2, which is another main purpose of this paper. Our method is based on a modified form of McCarthy's lemma, which characterizes the quasi-norm of C_p (see Lemma 2.3), and our argument consists of a direct estimate of the L^2-norms of functions.

Using the same method, we also prove the following theorem.

THEOREM 1.4. *Let* $0 < p \le 2$*, and let* $t \in \mathbf{R}$*. If* $\sigma \in M^{p,p}(\mathbf{R}^{2d})$*, then* $\sigma_t(X, D) \in C_p$*.*

The case $1 \le p \le 2$ of this theorem has already been proved by Gröchenig and Heil [10]. The case $0 < p < 1$ does not seem to have appeared in the literature.

Finally, we note that Theorems 1.2 and 1.4 are mutually independent; that is, Theorem 1.2 does not cover Theorem 1.4, and Theorem 1.4 does not cover Theorem 1.2. This can be seen from the following two facts. First, if $0 < p < 2$ and $s > 0$, then $M^{p,p}(\mathbf{R}^{2d}) \not\hookrightarrow M^{2,2}_m(\mathbf{R}^{2d})$, where m is one of the weights of Theorem 1.2. Second, if $0 < p \le 1$, $s < (4d/p) - 2d$, and m is one of the weights of Theorem 1.2, then $M^{2,2}_m(\mathbf{R}^{2d}) \not\hookrightarrow M^{p,p}(\mathbf{R}^{2d})$. The first fact is elementary, and the proof is left to reader. The second fact is proved by Gröchenig [6, Proposition 3] for the case $p = 1$. That proof can be generalized to the case $0 < p \le 1$ without essential change.

Notation

We write $\mathcal{S}(\mathbf{R}^n)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$, and we write $\mathcal{S}'(\mathbf{R}^n)$ to denote the space of tempered distributions on $\mathbf{R}^n$, that is, the topological dual of $\mathcal{S}(\mathbf{R}^n)$. We define

$$\|f\|_{L^p} = \Big(\int_{\mathbf{R}^n} |f(t)|^p \, dt\Big)^{1/p}$$

for $0 < p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{t \in \mathbf{R}^n} |f(t)|$. We use the pairing $\langle f, g\rangle$ between $f \in \mathcal{S}'(\mathbf{R}^n)$ and $g \in \mathcal{S}(\mathbf{R}^n)$ in a manner consistent with the inner product

$$\langle f, g\rangle = \int_{\mathbf{R}^n} f(t)\overline{g(t)} \, dt$$

on $L^2(\mathbf{R}^n)$. For a function f on $\mathbf{R}^n$, the translation and the modulation operators are defined by

$$T_x f(t) = f(t - x) \qquad \text{and} \qquad M_\omega f(t) = e^{2\pi i \omega t} f(t), \quad x, \omega \in \mathbf{R}^n,$$

respectively. We note the following:

$$(T_x f)^\wedge = M_{-x}\widehat{f}, \qquad (M_\omega f)^\wedge = T_\omega \widehat{f}, \qquad T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x.$$

We define $\widetilde{f}$ by $\widetilde{f}(t) = \overline{f(-t)}$. For $x \in \mathbf{R}^n$, we write $\langle x\rangle = (1 + |x|^2)^{1/2}$.

For nonnegative functions F and G on a set X, we write $F(x) \lesssim G(x)$ $(x \in X)$ if there exists a positive constant C independent of $x \in X$ such that $F(x) \le CG(x)$ for all $x \in X$. We write $F(x) \approx G(x)$ $(x \in X)$ if $F(x) \lesssim G(x)$ $(x \in X)$ and $G(x) \lesssim F(x)$ $(x \in X)$. We omit to write $(x \in X)$ if the variable x and the domain X are obviously recognized from the context. We also use the notation $G(x) \gtrsim F(x)$ in the same meaning as $F(x) \lesssim G(x)$.

§2. A modification of McCarthy's lemma

We first recall the definition of frames for $L^2(\mathbf{R}^d)$.

Definition 2.1. A sequence $\{f_\nu\}_{\nu=1}^{\infty}$ in $L^2(\mathbf{R}^n)$ is called a *frame* for $L^2(\mathbf{R}^n)$ if there exist constants $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \le \sum_{\nu=1}^{\infty} |\langle f, f_\nu\rangle|^2 \le B\|f\|_{L^2}^2, \quad f \in L^2(\mathbf{R}^n). \tag{2.1}$$

An orthonormal basis is a special case of the frame. In the proofs of the main results, we will use the Gabor frame $\{g_{mn}\}_{m,n\in\mathbf{Z}^d}$ given in the following example.

Example 2.2. Let g be a function in $C_0^\infty(\mathbf{R}^d)$ such that $\sup g \subset [-1,1]^d$ and $\sum_{m\in\mathbf{Z}^d} |g(x-m)|^2 = 1$ for all $x \in \mathbf{R}^d$. If we define

$$g_{mn}(x) = M_{\frac{n}{2}} T_m g(x) = g(x-m)e^{\pi i n x}, \quad m, n \in \mathbf{Z}^d,$$

then the family $\{g_{mn}\}_{m,n\in\mathbf{Z}^d}$ is a frame for $L^2(\mathbf{R}^d)$, which satisfies (2.1) with $A = B = 1$ (see, e.g., [7, Theorem 6.4.1]).

To estimate the quasi-norm on C_p, we use the following lemma, which is a modification of McCarthy's lemma (see [15, Lemma 2.2]).

Lemma 2.3. *Let T be a compact operator on $L^2(\mathbf{R}^d)$.*

(i) *If $0 < p \le 2$, then*

$$\|T\|_{C_p} \approx \inf\Big(\sum_{\nu=1}^{\infty} \|Tf_\nu\|_{L^2}^p\Big)^{1/p}.$$

(ii) *If $2 \le p < \infty$, then*

$$\|T\|_{C_p} \approx \sup\Big(\sum_{\nu=1}^{\infty} \|Tf_\nu\|_{L^2}^p\Big)^{1/p}.$$

In (i) and (ii), we take the infimum or supremum over all frames $\{f_\nu\}_{\nu=1}^{\infty}$ for $L^2(\mathbf{R}^d)$ satisfying (2.1) for fixed A and B.

McCarthy [15, Lemma 2.2] gave this lemma using orthonormal bases instead of frames, in which case the relations hold with equality in place of $\approx$. The following proof is a modification of the argument given in the book of Komatsu [14, Proposition 8.5].

Proof of Lemma 2.3.

(i) By the Schmidt representation, T can be written as

$$Tf = \sum_{j=1}^{\infty} \lambda_j \langle f, e_j \rangle e_j', \quad f \in L^2(\mathbf{R}^d),$$

where $\{e_j\}_{j=1}^{\infty}$ and $\{e_j'\}_{j=1}^{\infty}$ are orthonormal systems and $\lambda_j = s_j(T)$ (see, e.g., [16, Proposition 16.3]).

Let $\{f_\nu\}$ be an arbitrary frame for $L^2(\mathbf{R}^d)$ satisfying (2.1). Since $0 < p \leq 2$, the function $t^{p/2}$ is concave. Therefore, for any sequence $\{\alpha_j\}_{j=1}^{\infty}$ with $\alpha_j \geq 0$ and $\sum_{j=1}^{\infty} \alpha_j \leq B$, we have

$$\Big(\sum_{j=1}^{\infty} \frac{\alpha_j}{B} \lambda_j^2\Big)^{p/2} \geq \sum_{j=1}^{\infty} \frac{\alpha_j}{B} \lambda_j^p. \tag{2.2}$$

Taking $\alpha_j = |\langle f_\nu, e_j \rangle|^2$, we have $\sum_j \alpha_j = \sum_j |\langle f_\nu, e_j \rangle|^2 \leq \|f_\nu\|_{L^2}^2 \leq B$ and

$$\begin{aligned}\sum_\nu \|Tf_\nu\|_{L^2}^p &= \sum_\nu \Big(\sum_{j=1}^{\infty} |\langle f_\nu, e_j \rangle|^2 \lambda_j^2\Big)^{p/2} \geq B^{(p/2)-1} \sum_\nu \sum_{j=1}^{\infty} |\langle f_\nu, e_j \rangle|^2 \lambda_j^p \\ &\geq AB^{(p/2)-1} \sum_{j=1}^{\infty} \lambda_j^p.\end{aligned}$$

Thus,

$$\begin{aligned}\inf \Big(\sum_\nu \|Tf_\nu\|_{L^2}^p\Big)^{1/p} &\geq A^{1/p} B^{(1/2)-(1/p)} \Big(\sum_{j=1}^{\infty} \lambda_j^p\Big)^{1/p} \\ &= A^{1/p} B^{(1/2)-(1/p)} \|T\|_{C_p}.\end{aligned}$$

On the other hand, let $\{\widetilde{e}_\nu\}$ be an orthonormal basis for $L^2(\mathbf{R}^d)$ containing $\{e_j\}$, and put $f_\nu = A^{1/2} \widetilde{e}_\nu$. Then $\{f_\nu\}$ is a frame for $L^2(\mathbf{R}^d)$ satisfying (2.1). Since $Tf_\nu = \sum_j \lambda_j \langle A^{1/2} \widetilde{e}_\nu, e_j \rangle e_j' = \lambda_{j(\nu)} A^{1/2} e_{j(\nu)}'$ if $\widetilde{e}_\nu = e_{j(\nu)}$ is in $\{e_j\}$, and $= 0$ if $\widetilde{e}_\nu \notin \{e_j\}$, we have

$$\sum_\nu \|Tf_\nu\|_{L^2}^p = A^{p/2} \sum_{j=1}^{\infty} \lambda_j^p.$$

Therefore,

$$\inf\Big(\sum_{\nu}\|Tf_\nu\|_{L^2}^p\Big)^{1/p}\le A^{1/2}\Big(\sum_{j=1}^{\infty}\lambda_j^p\Big)^{1/p}=A^{1/2}\|T\|_{C_p}.$$

(ii) If $2\le p<\infty$, then we can actually prove the equality

$$\sup\Big(\sum_{\nu}\|Tf_\nu\|_{L^2}^p\Big)^{1/p}=B^{1/2}\|T\|_{C^p}.$$

In fact, for $2\le p<\infty$, the function $t^{p/2}$ is convex, and the inequality reverse to (2.2) holds. Hence, we can prove the inequality

$$\sup\Big(\sum_{\nu}\|Tf_\nu\|_{L^2}^p\Big)^{1/p}\le B^{1/2}\|T\|_{C^p}$$

by the same argument as in the first half of (i). The converse inequality can be seen by the use of the frame $f_\nu=B^{1/2}\widetilde{e}_\nu$ with $\widetilde{e}_\nu$ being the same as in (i).

$\square$

§3. Proof of Theorems 1.2 and 1.4

Before we prove Theorems 1.2 and 1.4, we recall some basic properties of modulation spaces $M_m^{p,q}(\mathbf{R}^n)$. (For the proofs, we refer to [3], [5], [7], [13], or [17].) The space $M_m^{p,q}(\mathbf{R}^n)$ is a quasi-Banach space. If $0<p,q<\infty$, then $\mathcal{S}(\mathbf{R}^n)$ is dense in $M_m^{p,q}(\mathbf{R}^n)$. If $0<p_1\le p_2\le\infty, 0<q_1\le q_2\le\infty$, and $m_2\lesssim m_1$, then $M_{m_1}^{p_1,q_1}(\mathbf{R}^n)\hookrightarrow M_{m_2}^{p_2,q_2}(\mathbf{R}^n)$. The weight functions m of Theorem A and Theorem 1.2 satisfy condition (1.1) with $n=2d$.

The following lemma will be used to represent $\sigma(x,D)$ as a superposition of “elementary operators.”

Lemma 3.1 ([7, Corollary 11.2.7]). *If $\varphi,\psi,f\in\mathcal{S}(\mathbf{R}^n)$ and if $\langle\varphi,\widetilde{\psi}\rangle\ne 0$, then*

$$f(t)=\frac{1}{\langle\varphi,\widetilde{\psi}\rangle}\int_{\mathbf{R}^{2n}} f*M_\zeta\varphi(z)T_zM_\zeta\psi(t)\,dz\,d\zeta,\quad t\in\mathbf{R}^n,$$

where the function

$$(z,\zeta,t)\mapsto f*M_\zeta\varphi(z)T_zM_\zeta\psi(t)$$

is in the class $\mathcal{S}$ on $\mathbf{R}^{3n}=\mathbf{R}_z^n\times\mathbf{R}_\zeta^n\times\mathbf{R}_t^n$.

Proof of Theorem 1.2. PART I. We prove the result for the weight (1.4) with $t=0$. Notice that $M_m^{2,2}(\mathbf{R}^{2d}) \subset L^2(\mathbf{R}^{2d})$ and that the pseudodifferential operators $\sigma(X,D)$ with symbols in $L^2(\mathbf{R}^{2d})$ are of Hilbert-Schmidt class. Hence, $\sigma(X,D)$ with $\sigma \in M_m^{2,2}(\mathbf{R}^{2d})$ are in the Hilbert-Schmidt class and, in particular, compact.

Let $\{g_{mn}\}$ be the frame given in Example 2.2. By virtue of Lemma 2.3(i), the claim of Theorem 1.2 for $\sigma(X,D)$ follows from the inequality

$$\Big(\sum_{m,n\in\mathbf{Z}^d} \|\sigma(X,D)g_{mn}\|_{L^2}^p\Big)^{1/p} \le C\|\sigma\|_{M_m^{2,2}}.$$

Since $\mathcal{S}(\mathbf{R}^{2d})$ is dense in $M_m^{2,2}(\mathbf{R}^{2d})$, it is sufficient to prove this estimate for $\sigma \in \mathcal{S}(\mathbf{R}^{2d})$. Thus, in the rest of the argument, we assume that $\sigma \in \mathcal{S}(\mathbf{R}^{2d})$.

Take $\Phi, \Psi \in \mathcal{S}(\mathbf{R}^{2d})$ such that $\langle \Phi, \widetilde{\Psi}\rangle \neq 0$. By Lemma 3.1 with $n=2d$, we have

$$\sigma(x,\omega) = \frac{1}{\langle \Phi, \widetilde{\Psi}\rangle} \int_{\mathbf{R}^{4d}} \sigma * M_\zeta \Phi(z) T_z M_\zeta \Psi(x,\omega)\, dz\, d\zeta, \quad x,\omega \in \mathbf{R}^d.$$

Let $f \in \mathcal{S}(\mathbf{R}^d)$. The above formula yields

$$\sigma(X,D)f(x) = \frac{1}{\langle \Phi, \widetilde{\Psi}\rangle} \int_{\mathbf{R}^{5d}} \sigma * M_\zeta \Phi(z) T_z M_\zeta \Psi(x,\omega) \widehat{f}(\omega) e^{2\pi i x\omega}\, dz\, d\zeta\, d\omega,$$
$$x \in \mathbf{R}^d.$$

Notice that the above integrand is a function in the class $\mathcal{S}$ on $\mathbf{R}^{6d} = \mathbf{R}_z^{2d} \times \mathbf{R}_\zeta^{2d} \times \mathbf{R}_\omega^d \times \mathbf{R}_x^d$. Thus, we obtain

$$\begin{aligned} &\|\sigma(X,D)f\|_{L^2}^2 \\ (3.1)\quad &= \frac{1}{|\langle \Phi, \widetilde{\Psi}\rangle|^2} \int_{\mathbf{R}^{11d}} \sigma * M_\zeta \Phi(z) T_z M_\zeta \Psi(x,\omega) \widehat{f}(\omega) e^{2\pi i x\omega} \\ &\quad \times \overline{\sigma * M_{\zeta'} \Phi(z')}\, \overline{T_{z'} M_{\zeta'} \Psi(x,\omega') \widehat{f}(\omega')} e^{-2\pi i x\omega'}\, dz\, d\zeta\, d\omega\, dz'\, d\zeta'\, d\omega'\, dx. \end{aligned}$$

The integrand of (3.1) is a function in the class $\mathcal{S}$ on $\mathbf{R}^{11d} = \mathbf{R}_z^{2d} \times \mathbf{R}_\zeta^{2d} \times \mathbf{R}_\omega^d \times \mathbf{R}_{z'}^{2d} \times \mathbf{R}_{\zeta'}^{2d} \times \mathbf{R}_{\omega'}^d \times \mathbf{R}_x^d$, and hence we can freely change the order of integration.

We simply write

$$\alpha(z,\zeta) = \sigma * M_\zeta \Phi(z)$$

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and apply (3.1) to $f = g_{mn}$. We have

$$\widehat{g_{mn}}(\omega) = \widehat{g}(\omega - n/2)e^{-2\pi i m(\omega - n/2)}$$

and

$$T_z M_\zeta \Psi(x,\omega) e^{2\pi i x\omega} \\ = \Theta(x - z_1, \omega - z_2) e^{2\pi i x(z_2+\zeta_1)} e^{2\pi i \omega(z_1+\zeta_2)} e^{2\pi i(-z_1 z_2 - z_1\zeta_1 - z_2\zeta_2)},$$

where $\Theta(x,\omega) = \Psi(x,\omega)e^{2\pi i x\omega}$. With these expressions, (3.1) can be written as

$$\begin{aligned} &\|\sigma(X,D)g_{mn}\|_{L^2}^2 \\ &\quad = \frac{1}{|\langle \Phi, \widetilde{\Psi}\rangle|^2} \int_{\mathbf{R}^{11d}} e^{2\pi i(-z_1 z_2 - z_1\zeta_1 - z_2\zeta_2)} \\ &\qquad \times e^{2\pi i(z_1' z_2' + z_1'\zeta_1' + z_2'\zeta_2')} \alpha(z,\zeta)\overline{\alpha(z',\zeta')} e^{2\pi i x(z_2 - z_2' + \zeta_1 - \zeta_1')} \\ &\qquad \times e^{2\pi i\omega(z_1+\zeta_2-m)} e^{2\pi i\omega'(-z_1'-\zeta_2'+m)} \Theta(x - z_1, \omega - z_2)\overline{\Theta(x - z_1', \omega' - z_2')} \\ &\qquad \times \widehat{g}(\omega - n/2)\overline{\widehat{g}(\omega' - n/2)}\, dz\, d\zeta\, d\omega\, dz'\, d\zeta'\, d\omega'\, dx. \end{aligned} \tag{3.2}$$

In the integral of (3.2), we first take the integration with respect to (x, ω, ω'). Consider the integral

$$\begin{aligned} &\int_{\mathbf{R}^{3d}} e^{2\pi i x(z_2 - z_2' + \zeta_1 - \zeta_1')} e^{2\pi i\omega(z_1+\zeta_2-m)} e^{2\pi i\omega'(-z_1'-\zeta_2'+m)} \\ &\qquad \times \Theta(x - z_1, \omega - z_2)\overline{\Theta(x - z_1', \omega' - z_2')} \\ &\qquad \times \widehat{g}(\omega - n/2)\overline{\widehat{g}(\omega' - n/2)}\, dx\, d\omega\, d\omega'. \end{aligned} \tag{3.3}$$

This is the inverse Fourier transform of the function

$$\begin{aligned} (x,\omega,\omega') \mapsto \Theta(x - z_1, \omega - z_2)&\overline{\Theta(x - z_1', \omega' - z_2')} \\ \times \widehat{g}(\omega - n/2)&\overline{\widehat{g}(\omega' - n/2)} \end{aligned} \tag{3.4}$$

evaluated at $(z_2 - z_2' + \zeta_1 - \zeta_1', z_1 + \zeta_2 - m, -z_1' - \zeta_2' + m)$. Since Θ and $\widehat{g}$ are in the class $\mathcal{S}$, we see that the absolute value of the function (3.4) is majorized by a constant times

$$\langle x - z_1\rangle^{-N}\langle \omega - z_2\rangle^{-N}\langle \omega' - z_2'\rangle^{-N}\langle z_1 - z_1'\rangle^{-N}\langle z_2 - n/2\rangle^{-N}\langle z_2' - n/2\rangle^{-N}$$

where N can be taken as arbitrarily large. The same estimate also holds for every derivative of (3.4). Hence, by integration by parts, we see that the absolute value of integral (3.3) is majorized by a constant times

$$\langle z_1 - z_1'\rangle^{-N}\langle z_2 - n/2\rangle^{-N}\langle z_2' - n/2\rangle^{-N} \\ \times \langle z_2 - z_2' + \zeta_1 - \zeta_1'\rangle^{-N}\langle z_1 + \zeta_2 - m\rangle^{-N}\langle -z_1' - \zeta_2' + m\rangle^{-N}.$$

Combining the last estimate with (3.2), we obtain

$$\begin{aligned}
&\|\sigma(X,D)g_{mn}\|_{L^2}^2 \\
&\quad \le c\int_{\mathbf{R}^{8d}} |\alpha(z,\zeta)||\alpha(z',\zeta')|\langle z_1 - z_1'\rangle^{-N}\langle z_2 - n/2\rangle^{-N} \\
&\qquad\qquad \times \langle z_2' - n/2\rangle^{-N}\langle z_2 - z_2' + \zeta_1 - \zeta_1'\rangle^{-N}\langle z_1 + \zeta_2 - m\rangle^{-N} \\
&\qquad\qquad \times \langle -z_1' - \zeta_2' + m\rangle^{-N}\, dz\, d\zeta\, dz'\, d\zeta' \\
&\quad = (*).
\end{aligned}$$

We use the inequality $|\alpha(z,\zeta)||\alpha(z',\zeta')| \le 2^{-1}(|\alpha(z,\zeta)|^2 + |\alpha(z',\zeta')|^2)$ and use the symmetry of the variables (z,ζ) and (z',ζ') to see that

$$\begin{aligned}
(*) &\le c\int_{\mathbf{R}^{8d}} |\alpha(z,\zeta)|^2\langle z_1 - z_1'\rangle^{-N}\langle z_2 - n/2\rangle^{-N} \\
&\qquad \times \langle z_2' - n/2\rangle^{-N}\langle z_2 - z_2' + \zeta_1 - \zeta_1'\rangle^{-N} \\
&\qquad \times \langle z_1 + \zeta_2 - m\rangle^{-N}\langle -z_1' - \zeta_2' + m\rangle^{-N}\, dz\, d\zeta\, dz'\, d\zeta' \\
&= c\int_{\mathbf{R}^{6d}} |\alpha(z,\zeta)|^2\langle z_1 - z_1'\rangle^{-N}\langle z_2 - n/2\rangle^{-N} \\
&\qquad \times \langle z_2' - n/2\rangle^{-N}\langle z_1 + \zeta_2 - m\rangle^{-N}\, dz\, d\zeta\, dz' \\
&= c\int_{\mathbf{R}^{4d}} |\alpha(z,\zeta)|^2\langle z_1 + \zeta_2 - m\rangle^{-N}\langle z_2 - n/2\rangle^{-N}\, dz\, d\zeta.
\end{aligned}$$

Thus, we have proved the estimate

$$\begin{aligned}
\|\sigma(X,D)g_{mn}\|_{L^2}^2 \le c\int_{\mathbf{R}^{4d}} &|\alpha(z,\zeta)|^2\langle z_1 + \zeta_2 - m\rangle^{-N} \\
&\times \langle z_2 - n/2\rangle^{-N}\, dz\, d\zeta.
\end{aligned} \tag{3.5}$$

Let q be the real number defined by $p/2+1/q=1$. Then (3.5) and Hölder's inequality give

$$\begin{aligned}
&\sum_{m,n\in\mathbf{Z}^d} \|\sigma(X,D)g_{mn}\|_{L^2}^p \\
&\quad\le c\sum_{m,n\in\mathbf{Z}^d}\Big(\int_{\mathbf{R}^{4d}}|\alpha(z,\zeta)|^2\langle z_1+\zeta_2-m\rangle^{-N}\Big\langle z_2-\frac{n}{2}\Big\rangle^{-N}dz\,d\zeta\Big)^{p/2} \\
&\quad\le c\Big(\sum_{m,n\in\mathbf{Z}^d}(1+|m|+|n|)^{2s}\int_{\mathbf{R}^{4d}}|\alpha(z,\zeta)|^2\langle z_1+\zeta_2-m\rangle^{-N} \\
&\qquad\qquad\times\Big\langle z_2-\frac{n}{2}\Big\rangle^{-N}dz\,d\zeta\Big)^{p/2} \\
&\qquad\times\Big(\sum_{m,n\in\mathbf{Z}^d}(1+|m|+|n|)^{-spq}\Big)^{1/q} \\
&\quad= c\Big(\sum_{m,n\in\mathbf{Z}^d}(1+|m|+|n|)^{2s}\int_{\mathbf{R}^{4d}}|\alpha(z,\zeta)|^2\langle z_1+\zeta_2-m\rangle^{-N} \\
&\qquad\qquad\times\Big\langle z_2-\frac{n}{2}\Big\rangle^{-N}dz\,d\zeta\Big)^{p/2} \\
&\quad=(**),
\end{aligned}$$

where the first $=$ holds because $spq>2d$ by our assumption that $s>2d/p-d$. Since, for sufficiently large N,

$$\begin{aligned}
&\sum_{m,n\in\mathbf{Z}^d}(1+|m|+|n|)^{2s}\langle z_1+\zeta_2-m\rangle^{-N}\Big\langle z_2-\frac{n}{2}\Big\rangle^{-N} \\
&\quad\approx\sum_{n\in\mathbf{Z}^d}(1+|z_1+\zeta_2|+|n|)^{2s}\Big\langle z_2-\frac{n}{2}\Big\rangle^{-N}\approx(1+|z_1+\zeta_2|+|z_2|)^{2s},
\end{aligned}$$

we obtain

$$(**)\approx\Big(\int_{\mathbf{R}^{4d}}|\alpha(z,\zeta)|^2(1+|z_1+\zeta_2|+|z_2|)^{2s}\,dz\,d\zeta\Big)^{p/2}\approx\|\sigma\|_{M_m^{2,2}}^p,$$

where m is the weight of (1.4) with $t=0$. Thus, the claim for the weight (1.4) with $t=0$ is proved.

PART II. We prove the result for the weight (1.5) with $t=0$. To this end, we consider the adjoint operator $\sigma(X,D)^*$ of $\sigma(X,D)$. Take $\Phi,\Psi\in\mathcal{S}(\mathbf{R}^{2d})$ such that $\langle\Phi,\widetilde{\Psi}\rangle\neq 0$, and represent $\sigma(X,D)$ as

$$\sigma(X,D)f(x)=\frac{1}{\langle\Phi,\widetilde{\Psi}\rangle}\int_{\mathbf{R}^{5d}}\alpha(z,\zeta)T_zM_\zeta\Psi(x,\omega)\widehat{f}(\omega)e^{2\pi ix\omega}\,dz\,d\zeta\,d\omega,$$

where $\alpha(z,\zeta)=\sigma*M_\zeta\Phi(z)$. Then we have

$$\begin{aligned}
&\langle f,\sigma(X,D)^*\psi\rangle\\
&\quad=\langle\sigma(X,D)f,\psi\rangle=\int_{\mathbf{R}^d}\sigma(X,D)f(x)\overline{\psi(x)}\,dx\\
&\quad=\frac{1}{\langle\Phi,\widetilde{\Psi}\rangle}\int_{\mathbf{R}^{6d}}\alpha(z,\zeta)T_zM_\zeta\Psi(x,\omega)\widehat{f}(\omega)e^{2\pi ix\omega}\overline{\psi(x)}\,dz\,d\zeta\,d\omega\,dx\\
&\quad=\frac{1}{\langle\Phi,\widetilde{\Psi}\rangle}\int_{\mathbf{R}^d}\widehat{f}(\omega)\int_{\mathbf{R}^{5d}}\alpha(z,\zeta)T_zM_\zeta\Psi(x,\omega)e^{2\pi ix\omega}\overline{\psi(x)}\,dz\,d\zeta\,dx\,d\omega\\
&\quad=\Big\langle f,\overline{\mathcal{F}_\omega\Big[\frac{1}{\langle\Phi,\widetilde{\Psi}\rangle}\int_{\mathbf{R}^{5d}}\alpha(z,\zeta)T_zM_\zeta\Psi(x,\omega)e^{2\pi ix\omega}\overline{\psi(x)}\,dz\,d\zeta\,dx\Big]}\Big\rangle,
\end{aligned}$$

where $\mathcal{F}_\omega$ denotes the Fourier transform with respect to the variable ω. Thus,

$$\overline{\sigma(X,D)^*\psi}=\mathcal{F}_\omega\Big[\frac{1}{\langle\Phi,\widetilde{\Psi}\rangle}\int_{\mathbf{R}^{5d}}\alpha(z,\zeta)T_zM_\zeta\Psi(x,\omega)e^{2\pi ix\omega}\overline{\psi(x)}\,dz\,d\zeta\,dx\Big].\tag{3.6}$$

Since $\|\sigma(X,D)\|_{C_p}=\|\sigma(X,D)^*\|_{C_p}$ (see [2, p. 1092]), we have

$$\|\sigma(X,D)\|_{C_p}\lesssim\Big(\sum_{m,n\in\mathbf{Z}^d}\|\sigma(X,D)^*g_{mn}\|_{L^2}^p\Big)^{1/p}$$

by Lemma 2.3(i), where $\{g_{mn}\}$ is the frame of Example 2.2. So, we estimate $\|\sigma(X,D)^*g_{mn}\|_{L^2}$. Using the Plancherel theorem and (3.6) with $\psi=g_{mn}$, we have

$$\begin{aligned}
&\|\sigma(X,D)^*g_{mn}\|_{L^2}^2\\
&\quad\approx\Big\|\int_{\mathbf{R}^{5d}}\alpha(z,\zeta)T_zM_\zeta\Psi(x,\omega)e^{2\pi ix\omega}\overline{g_{mn}(x)}\,dz\,d\zeta\,dx\Big\|_{L^2_\omega}^2
\end{aligned}$$

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$$
\begin{aligned}
&= \int_{\mathbf{R}^{11d}} \alpha(z,\zeta)\overline{\alpha(z',\zeta')} e^{2\pi i(-z_1z_2-z_1\zeta_1-z_2\zeta_2)} e^{2\pi i(z_1'z_2'+z_1'\zeta_1'+z_2'\zeta_2')} \\
&\qquad \times \Theta(x-z_1,\omega-z_2)\overline{\Theta(x'-z_1',\omega-z_2')}\,\overline{g(x-m)} \\
&\qquad \times g(x'-m)e^{2\pi ix(z_2+\zeta_1-n/2)}e^{2\pi ix'(-z_2'-\zeta_1'+n/2)} \\
&\qquad \times e^{2\pi i\omega(z_1+\zeta_2-z_1'-\zeta_2')}\,dz\,d\zeta\,dx\,dz'\,d\zeta'\,dx'\,d\omega \\
&= (*),
\end{aligned}
$$

where $\Theta(x,\omega) = \Psi(x,\omega)e^{2\pi ix\omega}$. In the above integral, we first take the integration with respect to (x,x',ω). As in the proof of Part I, we obtain

$$
\begin{aligned}
&\Big|\int_{\mathbf{R}^{3d}} e^{2\pi ix(z_2+\zeta_1-n/2)}e^{2\pi ix'(-z_2'-\zeta_1'+n/2)}e^{2\pi i\omega(z_1+\zeta_2-z_1'-\zeta_2')} \\
&\qquad \times \Theta(x-z_1,\omega-z_2)\overline{\Theta(x'-z_1',\omega-z_2')}\,\overline{g(x-m)}g(x'-m)\,dx\,dx'\,d\omega\Big| \\
&\quad \lesssim \langle z_1-m\rangle^{-N}\langle z_1'-m\rangle^{-N}\langle z_2-z_2'\rangle^{-N} \\
&\qquad \times \langle z_2+\zeta_1-n/2\rangle^{-N}\langle -z_2'-\zeta_1'+n/2\rangle^{-N}\langle z_1+\zeta_2-z_1'-\zeta_2'\rangle^{-N},
\end{aligned}
$$

where N can be taken arbitrarily large. Thus,

$$
\begin{aligned}
(*) &\lesssim \int_{\mathbf{R}^{8d}} |\alpha(z,\zeta)||\alpha(z',\zeta')|\langle z_1-m\rangle^{-N}\langle z_1'-m\rangle^{-N}\langle z_2-z_2'\rangle^{-N} \\
&\qquad \times \langle z_2+\zeta_1-n/2\rangle^{-N}\langle -z_2'-\zeta_1'+n/2\rangle^{-N} \\
&\qquad \times \langle z_1+\zeta_2-z_1'-\zeta_2'\rangle^{-N}\,dz\,d\zeta\,dz'\,d\zeta' \\
&\lesssim \int_{\mathbf{R}^{8d}} |\alpha(z,\zeta)|^2\langle z_1-m\rangle^{-N}\langle z_1'-m\rangle^{-N}\langle z_2-z_2'\rangle^{-N} \\
&\qquad \times \langle z_2+\zeta_1-n/2\rangle^{-N}\langle -z_2'-\zeta_1'+n/2\rangle^{-N} \\
&\qquad \times \langle z_1+\zeta_2-z_1'-\zeta_2'\rangle^{-N}\,dz\,d\zeta\,dz'\,d\zeta' \\
&\approx \int_{\mathbf{R}^{6d}} |\alpha(z,\zeta)|^2\langle z_1-m\rangle^{-N}\langle z_1'-m\rangle^{-N}\langle z_2-z_2'\rangle^{-N} \\
&\qquad \times \langle z_2+\zeta_1-n/2\rangle^{-N}\,dz\,d\zeta\,dz' \\
&\approx \int_{\mathbf{R}^{4d}} |\alpha(z,\zeta)|^2\langle z_1-m\rangle^{-N}\langle z_2+\zeta_1-n/2\rangle^{-N}\,dz\,d\zeta.
\end{aligned}
$$

Hence, in the same way as in Part I, we obtain

$$\begin{aligned}
&\|\sigma(X,D)\|_{C_p}^p\\
&\quad\lesssim \sum_{m,n\in\mathbf{Z}^d} \|\sigma(X,D)^* g_{mn}\|_{L^2}^p\\
&\quad\lesssim \sum_{m,n\in\mathbf{Z}^d} \Big(\int_{\mathbf{R}^{4d}} |\alpha(z,\zeta)|^2 \langle z_1-m\rangle^{-N} \Big\langle z_2+\zeta_1-\frac{n}{2}\Big\rangle^{-N} dz\,d\zeta\Big)^{p/2}\\
&\quad\lesssim \Big(\sum_{m,n\in\mathbf{Z}^d} (1+|m|+|n|)^{2s}\\
&\qquad\quad\times \int_{\mathbf{R}^{4d}} |\alpha(z,\zeta)|^2 \langle z_1-m\rangle^{-N} \Big\langle z_2+\zeta_1-\frac{n}{2}\Big\rangle^{-N} dz\,d\zeta\Big)^{p/2}\\
&\quad\approx \Big(\int_{\mathbf{R}^{4d}} |\alpha(z,\zeta)|^2 (1+|z_1|+|z_2+\zeta_1|)^{2s}\,dz\,d\zeta\Big)^{p/2} \approx \|\sigma\|_{M_m^{2,2}}^p,
\end{aligned}$$

where $s>2d/p-d$ and where m is the weight of (1.5) with $t=0$. This proves the claim for the weight (1.5) with $t=0$.

PART III. Finally, we prove the result for general $t\in\mathbf{R}$. We use the fact that $\sigma_t(X,D)$ can be written as $\tau(X,D)$ by a simple transformation of symbols $\sigma\mapsto\tau$. In fact, if we define the symbol $\mathcal{U}_t\sigma$ by

$$(\mathcal{U}_t\sigma)^\wedge(\xi,\eta)=e^{2\pi it\xi\eta}\widehat{\sigma}(\xi,\eta),$$

then $\sigma_t(X,D)=(\mathcal{U}_t\sigma)(X,D)$. We have

$$(3.7)\qquad (\mathcal{U}_t\sigma)*M_\zeta\Phi(z)=e^{-2\pi it\zeta_1\zeta_2}\sigma*M_\zeta\mathcal{U}_t\Phi(z_1+t\zeta_2,z_2+t\zeta_1)$$

(see [9, Lemma 2.1], [7, Corollary 14.5.5]). Hence, the estimate obtained in Part I yields

$$\begin{aligned}
\|\sigma_t(X,D)\|_{C_p} &= \|(\mathcal{U}_t\sigma)(X,D)\|_{C_p}\\
&\lesssim \Big(\int_{\mathbf{R}^{4d}} |(\mathcal{U}_t\sigma)*M_\zeta\Phi(z)|^2 (1+|z_1+\zeta_2|+|z_2|)^{2s}\,dz\,d\zeta\Big)^{1/2}\\
&= \Big(\int_{\mathbf{R}^{4d}} |\sigma*M_\zeta\mathcal{U}_t\Phi(z_1+t\zeta_2,z_2+t\zeta_1)|^2\\
&\qquad\qquad \times (1+|z_1+\zeta_2|+|z_2|)^{2s}\,dz\,d\zeta\Big)^{1/2}
\end{aligned}$$

$$= \Big(\int_{\mathbf{R}^{4d}} |\sigma * M_\zeta \mathcal{U}_t \Phi(z_1, z_2)|^2 \times \big(1 + |z_1 + (1-t)\zeta_2| + |z_2 - t\zeta_1|\big)^{2s} \, dz \, d\zeta \Big)^{1/2}.$$

This implies the desired result for the weight (1.4), since the last quantity is equivalent to the norm of σ in $M_m^{2,2}$ (see Definition 1.1 and the paragraph just below it).

In the same way as above, the estimate obtained in Part II yields the result for the weight (1.5) for all $t \in \mathbf{R}$. □

To prove Theorem 1.4, we use the following lemma.

LEMMA 3.2 ([5, Lemma 2.3]). *Let $\varphi_0(z) = e^{-\pi z^2}$, $z \in \mathbf{R}^n$. Then for all $p > 0$, $r > 0$ and $(z_0, \zeta_0) \in \mathbf{R}^n \times \mathbf{R}^n$, we have*

$$|f * M_{\zeta_0} \varphi_0(z_0)|^p \le \frac{e^{p\pi r^2/2}}{|B(r)|} \int_{(z-z_0)^2 + (\zeta-\zeta_0)^2 < r^2} |f * M_\zeta \varphi_0(z)|^p \, dz \, d\zeta,$$

where $|B(r)|$ is the volume of the ball with radius r in $\mathbf{R}^{2n}$.

Proof of Theorem 1.4. We first prove the result for the case $t = 0$. Let $\{g_{mn}\}$ be the Gabor frame as given in Example 2.2. By the same reason as in the proof of Theorem 1.2, it is sufficient to prove the inequality

$$\Big(\sum_{m,n \in \mathbf{Z}^d} \|\sigma(X, D) g_{mn}\|_{L^2}^p \Big)^{1/p} \le C \|\sigma\|_{M^{p,p}}$$

for $\sigma \in \mathcal{S}(\mathbf{R}^{2d})$.

Taking $\Phi, \Psi \in \mathcal{S}(\mathbf{R}^{2d})$ as in the proof of Theorem 1.2, we have the estimate (3.5) and thus

$$\begin{aligned} &\sum_{m,n \in \mathbf{Z}^d} \|\sigma(X, D) g_{mn}\|_{L^2}^p \\ &\quad \lesssim \sum_{m,n \in \mathbf{Z}^d} \Big(\int_{\mathbf{R}^{4d}} |\sigma * M_\zeta \Phi(z)|^2 \langle z_1 + \zeta_2 - m \rangle^{-N} \Big\langle z_2 - \frac{n}{2} \Big\rangle^{-N} dz \, d\zeta \Big)^{p/2} \\ &\quad = (*). \end{aligned}$$

To proceed further, we take the special choice of Φ as

$$\Phi(z) = \Phi_0(z) = e^{-\pi z^2}, \quad z \in \mathbf{R}^{2d},$$

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and estimate $(*)$ with $\Phi = \Phi_0$ as follows:

$$\begin{aligned}(*) &\lesssim \sum_{m,n\in\mathbf{Z}^d} \\ &\quad \times \Big(\sum_{k,l\in\mathbf{Z}^{2d}} \sup_{\substack{z\in k+[0,1]^{2d}\\ \zeta\in l+[0,1]^{2d}}} |\sigma * M_\zeta \Phi_0(z)|^2 \langle z_1+\zeta_2-m\rangle^{-N} \Big\langle z_2-\frac{n}{2}\Big\rangle^{-N}\Big)^{p/2} \\ &\approx \sum_{m,n\in\mathbf{Z}^d} \\ &\quad \times \Big(\sum_{k,l\in\mathbf{Z}^{2d}} \sup_{\substack{z\in k+[0,1]^{2d}\\ \zeta\in l+[0,1]^{2d}}} |\sigma * (M_\zeta \Phi_0)(z)|^2 \langle k_1+l_2-m\rangle^{-N} \Big\langle k_2-\frac{n}{2}\Big\rangle^{-N}\Big)^{p/2} \\ &\le \sum_{m,n\in\mathbf{Z}^d} \\ &\quad \times \sum_{k,l\in\mathbf{Z}^{2d}} \sup_{\substack{z\in k+[0,1]^{2d}\\ \zeta\in l+[0,1]^{2d}}} |\sigma * M_\zeta \Phi_0(z)|^p \langle k_1+l_2-m\rangle^{-Np/2} \Big\langle k_2-\frac{n}{2}\Big\rangle^{-Np/2} \\ &\approx \sum_{k,l\in\mathbf{Z}^{2d}} \sup_{\substack{z\in k+[0,1]^{2d}\\ \zeta\in l+[0,1]^{2d}}} |\sigma * M_\zeta \Phi_0(z)|^p,\end{aligned}$$

where we used the assumption that $p \le 2$ to obtain the $\le$. By Lemma 3.2, the last sum is majorized by a constant times

$$\begin{aligned}&\sum_{k,l\in\mathbf{Z}^{2d}} \int_{(z-k)^2+(\zeta-l)^2\le 16d} |\sigma * M_\zeta \Phi_0(z)|^p \, dz\, d\zeta \\ &\qquad \approx \int_{\mathbf{R}^{4d}} |\sigma * M_\zeta \Phi_0(z)|^p \, dz\, d\zeta \approx \|\sigma\|^p_{M^{p,p}}.\end{aligned}$$

This proves the result for $\sigma(X,D)$.

Next we prove the result for general $t \in \mathbf{R}$. We use the operator $\mathcal{U}_t$ used in the proof of Theorem 1.2. Since $\sigma_t(X,D) = (\mathcal{U}_t\sigma)(X,D)$, it is sufficient to prove that the operator $\mathcal{U}_t$ is bounded in $M^{p,p}(\mathbf{R}^{2d})$.

Taking a $\Phi \in \mathcal{S}(\mathbf{R}^{2d})$, we have (3.7) and hence

$$\int_{\mathbf{R}^{4d}} |(\mathcal{U}_t\sigma) * M_\zeta \Phi(z)|^p \, dz\, d\zeta = \int_{\mathbf{R}^{4d}} |\sigma * M_\zeta \mathcal{U}_t \Phi(z)|^p \, dz\, d\zeta.$$

Then we have $\|\mathcal{U}_t\sigma\|_{M^{p,p}} \approx \|\sigma\|_{M^{p,p}}$, since $\mathcal{U}_t\Phi \in \mathcal{S}(\mathbf{R}^{2d}) \setminus \{0\}$. This completes the proof of Theorem 1.4. □

§4. Proof of Theorem 1.3

Proof. We will prove only (a) and (b), because we can treat (c) and (d) in the same way as (a) and (b). We take a $\Phi \in \mathcal{S}(\mathbf{R}^{2d}) \setminus \{0\}$.

(a) Let $m(z,\zeta) = (1+|z_1+t\zeta_2|)^s, t \neq 0$. What we have to prove is

$$\|\langle z_1+t\zeta_2\rangle^s \sigma * M_\zeta\Phi(z)\|_{L^2_{z,\zeta}(\mathbf{R}^{4d})} \approx \big\|(1+|\zeta_2|)^s F_2\tau_{1/t}(y_1,\zeta_2)\big\|_{L^2_{y_1,\zeta_2}(\mathbf{R}^{2d})}. \tag{4.1}$$

By complex interpolation with respect to the parameter s, it is sufficient to prove (4.1) for all nonnegative integers s. If s is a nonnegative integer, then the right-hand side of (4.1) is equivalent to

$$\sum_{|\alpha|\le s} \big\|\partial^\alpha_{y_2}\big(\tau_{1/t}(y_1,y_2)\big)\big\|_{L^2_{y_1,y_2}}.$$

Thus, we will prove that

$$\|\langle z_1+t\zeta_2\rangle^s \sigma * M_\zeta\Phi(z)\|_{L^2_{z,\zeta}} \approx \sum_{|\alpha|\le s} \big\|\partial^\alpha_{y_2}\big(\tau_{1/t}(y_1,y_2)\big)\big\|_{L^2_{y_1,y_2}} \tag{4.2}$$

for all nonnegative integers s.

We first prove that

$$\|\langle z_1+t\zeta_2\rangle^s \sigma * M_\zeta\Phi(z)\|_{L^2_{z,\zeta}} \lesssim \sum_{|\alpha|\le s} \big\|\partial^\alpha_{y_2}\big(\tau_{1/t}(y_1,y_2)\big)\big\|_{L^2_{y_1,y_2}}. \tag{4.3}$$

Set $\Theta(y) = e^{-2\pi i \frac{1}{t} y_1 y_2}\Phi(y)$. Then,

$$\begin{aligned}
&|\sigma * M_\zeta\Phi(z)| \\
&\quad = \Big|\int_{\mathbf{R}^{2d}} \sigma(y) e^{2\pi i\zeta(z-y)}\Phi(z-y)\,dy\Big| \\
&\quad = \Big|\int_{\mathbf{R}^{2d}} e^{-2\pi i\frac{1}{t}y_1y_2}\tau_{1/t}(y) e^{2\pi i(\zeta_1(z_1-y_1)+\zeta_2(z_2-y_2))} \\
&\qquad\qquad \times e^{2\pi i\frac{1}{t}(z_1-y_1)(z_2-y_2)}\Theta(z-y)\,dy\Big| \\
&\quad = \Big|\int_{\mathbf{R}^{2d}} \tau_{1/t}(y)\Theta(z-y) e^{-2\pi i(y_1(\frac{z_2}{t}+\zeta_1)+y_2(\frac{z_1}{t}+\zeta_2))}\,dy\Big|,
\end{aligned}$$

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and thus, by integration by parts,

$$\begin{aligned}
&\Big|(-2\pi i)^{|\alpha|}\Big(\frac{z_1}{t}+\zeta_2\Big)^{\alpha}\sigma * M_\zeta\Phi(z)\Big| \\
&\qquad = \Big|\int_{\mathbf{R}^{2d}} \tau_{1/t}(y)\Theta(z-y)\partial_{y_2}^{\alpha}[e^{-2\pi i(y_1(\frac{z_2}{t}+\zeta_1)+y_2(\frac{z_1}{t}+\zeta_2))}]\,dy\Big| \\
&\qquad = \Big|\int_{\mathbf{R}^{2d}} \partial_{y_2}^{\alpha}[\tau_{1/t}(y)\Theta(z-y)]e^{-2\pi i(y_1(\frac{z_2}{t}+\zeta_1)+y_2(\frac{z_1}{t}+\zeta_2))}\,dy\Big| \\
&\qquad = \Big|\int_{\mathbf{R}^{2d}} \sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}\tau_{1/t}^{(0,\alpha')}(y)(-1)^{|\alpha|''}\Theta^{(0,\alpha'')}(z-y) \\
&\qquad\qquad\qquad \times e^{-2\pi i(y_1(\frac{z_2}{t}+\zeta_1)+y_2(\frac{z_1}{t}+\zeta_2))}\,dy\Big|,
\end{aligned} \tag{4.4}$$

where

$$\tau_{1/t}^{(0,\alpha')}(y)=\partial_{y_2}^{\alpha'}\big(\tau_{1/t}(y)\big) \qquad \text{and} \qquad \Theta^{(0,\alpha'')}(y)=\partial_{y_2}^{\alpha''}\big(\Theta(y)\big).$$

Taking the sum of (4.4) over $|\alpha|\le s$, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^{4d}} |\sigma * M_\zeta\Phi(z)|^2(1+|z_1+t\zeta_2|)^{2s}\,dz\,d\zeta \\
&\quad \approx \sum_{|\alpha|\le s}\int_{\mathbf{R}^{4d}}\Big|\Big(\frac{z_1}{t}+\zeta_2\Big)^{\alpha}\sigma * M_\zeta\Phi(z)\Big|^2\,dz\,d\zeta \\
&\quad \lesssim \sum_{|\alpha'+\alpha''|\le s}\int_{\mathbf{R}^{4d}}\Big|\int_{\mathbf{R}^{2d}}\tau_{1/t}^{(0,\alpha')}(y)\Theta^{(0,\alpha'')}(z-y) \\
&\qquad\qquad\qquad \times e^{-2\pi i(y_1(\frac{z_2}{t}+\zeta_1)+y_2(\frac{z_1}{t}+\zeta_2))}\,dy\Big|^2\,dz\,d\zeta \\
&\quad = \sum_{|\alpha'+\alpha''|\le s}\int_{\mathbf{R}^{4d}}|\tau_{1/t}^{(0,\alpha')} * M_{(\frac{z_2}{t}+\zeta_1,\frac{z_1}{t}+\zeta_2)}\Theta^{(0,\alpha'')}(z)|^2\,dz\,d\zeta \\
&\quad = \sum_{|\alpha'+\alpha''|\le s}\int_{\mathbf{R}^{4d}}|\tau_{1/t}^{(0,\alpha')} * M_\zeta\Theta^{(0,\alpha'')}(z)|^2\,dz\,d\zeta \\
&\quad \approx \sum_{|\alpha|\le s}\|\tau_{1/t}^{(0,\alpha)}\|_{M^{2,2}}^2 \approx \sum_{|\alpha|\le s}\big\|\partial_{y_2}^{\alpha}\big(\tau_{1/t}(y_1,y_2)\big)\big\|_{L^2_{y_1,y_2}}^2,
\end{aligned}$$

where the last $\approx$ follows from the fact that $M^{2,2}=L^2$. This proves (4.3).

https://doi.org/10.1215/00277630-1543796 Published online by Cambridge University Press

Next, we prove the converse inequality

$$\sum_{|\alpha|\le s} \left\|\partial_{y_2}^{\alpha}(\tau_{1/t}(y_1,y_2))\right\|_{L^2_{y_1,y_2}} \lesssim \|\langle z_1+t\zeta_2\rangle^s \sigma * M_\zeta \Phi(z)\|_{L^2_{z,\zeta}}. \tag{4.5}$$

We prove this by induction on s.

When $s=0$, (4.5) is obvious since

$$\|\tau_{1/t}\|_{L^2} = \|\sigma\|_{L^2} \approx \|\sigma * M_\zeta \Phi(z)\|_{L^2_{z,\zeta}}.$$

Assume that (4.5) holds for a nonnegative integer s. Let α be a multi-index with $|\alpha| = s+1$. From (4.4), we have

$$\begin{aligned}
&\left|(-2\pi i)^{|\alpha|}\left(\frac{z_1}{t}+\zeta_2\right)^{\alpha}\sigma * M_\zeta\Phi(z)\right| \\
&\quad\ge \left|\int_{\mathbf{R}^{2d}} \tau_{1/t}^{(0,\alpha)}(y)\Theta(z-y)e^{-2\pi i(y_1(\frac{z_2}{t}+\zeta_1)+y_2(\frac{z_1}{t}+\zeta_2))}\,dy\right| \\
&\qquad - \sum_{\substack{\alpha'+\alpha''=\alpha\\ \alpha''\ne 0}} \binom{\alpha}{\alpha'} \\
&\qquad\quad \times \left|\int_{\mathbf{R}^{2d}} \tau_{1/t}^{(0,\alpha')}(y)\Theta^{(0,\alpha'')}(z-y)e^{-2\pi i(y_1(\frac{z_2}{t}+\zeta_1)+y_2(\frac{z_1}{t}+\zeta_2))}\,dy\right| \\
&\quad= |\tau_{1/t}^{(0,\alpha)} * M_{(\frac{z_2}{t}+\zeta_1,\frac{z_1}{t}+\zeta_2)}\Theta(z)| \\
&\qquad - \sum_{\substack{\alpha'+\alpha''=\alpha\\ \alpha''\ne 0}} \binom{\alpha}{\alpha'} |\tau_{1/t}^{(0,\alpha')} * M_{(\frac{z_2}{t}+\zeta_1,\frac{z_1}{t}+\zeta_2)}\Theta^{(0,\alpha'')}(z)|.
\end{aligned}$$

Thus, taking the $L^2_{z,\zeta}$-norm and using the fact that $M^{2,2}=L^2$, we obtain

$$\left\|(-2\pi i)^{|\alpha|}\left(\frac{z_1}{t}+\zeta_2\right)^{\alpha}\sigma * M_\zeta\Phi(z)\right\|_{L^2_{z,\zeta}} + \sum_{|\alpha'|<|\alpha|} \|\tau_{1/t}^{(0,\alpha')}\|_{L^2} \gtrsim \|\tau_{1/t}^{(0,\alpha)}\|_{L^2}.$$

Since, obviously,

$$\left\|\left\langle\frac{z_1}{t}+\zeta_2\right\rangle^{s+1}\sigma * M_\zeta\Phi(z)\right\|_{L^2_{z,\zeta}} \gtrsim \left\|(-2\pi i)^{|\alpha|}\left(\frac{z_1}{t}+\zeta_2\right)^{\alpha}\sigma * M_\zeta\Phi(z)\right\|_{L^2_{z,\zeta}},$$

and since the induction hypothesis implies that

$$\left\|\left\langle\frac{z_1}{t}+\zeta_2\right\rangle^{s}\sigma * M_\zeta\Phi(z)\right\|_{L^2_{z,\zeta}} \gtrsim \sum_{|\alpha'|<|\alpha|} \|\tau_{1/t}^{(0,\alpha')}\|_{L^2},$$

we obtain

$$\Big\| \Big\langle \frac{z_1}{t} + \zeta_2 \Big\rangle^{s+1} \sigma * M_\zeta \Phi(z) \Big\|_{L^2_{z,\zeta}} \gtrsim \|\tau^{(0,\alpha)}_{1/t}\|_{L^2}.$$

Thus, we proved (4.5).

(b) Let $m(z,\zeta) = (1+|z_1|)^s$, $s \geq 0$. We have

$$|\sigma * M_\zeta \Phi(z)| = \Big| \int_{\mathbf{R}^{2d}} \sigma(y) e^{2\pi i \zeta(z-y)} \Phi(z-y)\, dy \Big| = \big| \big(\sigma(\cdot)\Phi(z-\cdot)\big)^\wedge(\zeta) \big|.$$

Hence, by Plancherel's theorem,

$$\int_{\mathbf{R}^{2d}} |\sigma * M_\zeta \Phi(z)|^2 \, d\zeta = \int_{\mathbf{R}^{2d}} |\sigma(y)\Phi(z-y)|^2 \, dy,$$

and consequently,

$$\begin{aligned} \|\sigma\|^2_{M^{2,2}_m} &= \int_{\mathbf{R}^{4d}} |\sigma * M_\zeta \Phi(z)|^2 (1+|z_1|)^{2s} \, d\zeta \, dz \\ &= \int_{\mathbf{R}^{4d}} |\sigma(y)\Phi(z-y)|^2 (1+|z_1|)^{2s} \, dy \, dz = (I). \end{aligned}$$

By the inequality

$$(1+|z_1-y_1|)^{-2s} \leq \frac{(1+|z_1|)^{2s}}{(1+|y_1|)^{2s}} \leq (1+|z_1-y_1|)^{2s},$$

we obtain

$$\begin{aligned} (I) &\leq \int_{\mathbf{R}^{4d}} |\sigma(y)\Phi(z-y)|^2 (1+|z_1-y_1|)^{2s} (1+|y_1|)^{2s} \, dy \, dz \\ &\approx \int_{\mathbf{R}^{2d}} |\sigma(y)|^2 (1+|y_1|)^{2s} \, dy \end{aligned}$$

and

$$\begin{aligned} (I) &\geq \int_{\mathbf{R}^{4d}} |\sigma(y)\Phi(z-y)|^2 (1+|z_1-y_1|)^{-2s} (1+|y_1|)^{2s} \, dy \, dz \\ &\approx \int_{\mathbf{R}^{2d}} |\sigma(y)|^2 (1+|y_1|)^{2s} \, dy. \end{aligned}$$

Combining the above inequalities, we obtain

$$\|\sigma\|^2_{M^{2,2}_s} \approx \int_{\mathbf{R}^{2d}} |\sigma(y)|^2 (1+|y_1|)^{2s} \, dy.$$

□

§5. Alternate proofs of Theorems 1.2 and 1.4

An alternate proof of Theorem 1.2. We will see that Theorem 1.2 can be proved by a modification of the argument of Heil, Ramanathan, and Topiwala [12] or Gröchenig and Heil [9]. We follow [12] and give the proof of Theorem 1.2 for the case $t = 1/2$; the case of general $t \in \mathbf{R}$ can be reduced to the case $t = 1/2$ with the aid of the operator $\mathcal{U}_t$ as in Part III of the proof of Theorem 1.2 (Section 3). Since the essential argument is the same as [12, Sections 2–5], we omit some details. We use the notation

$$\sigma^W(X, D) = \sigma_{1/2}(X, D).$$

The argument of [12] (and [9] as well) is based on the fact that the inequality

$$\sum_{j > \mathrm{rank}(T)} s_j(A)^2 \le \|A - T\|_{C_2}^2 \tag{5.1}$$

holds for all compact operators A and for all finite-rank operators T. Thus, given a compact operator A, if we find a finite-rank operator T which approximates A well, then we can obtain an estimate of the singular values of A. To find a finite-rank approximation of $A = \sigma^W(X, D)$, we use a Gabor frame expansion of the symbol σ.

To begin with, we recall some basic facts on the Weyl correspondence (for details, see, e.g., [4]). For $\alpha = (\alpha_1, \alpha_2) \in \mathbf{R}^{2d}$, we define the unitary operator $\rho(\alpha)$ in $L^2(\mathbf{R}^d)$ by

$$\rho(\alpha) f(x) = e^{\pi i \alpha_1 \alpha_2} e^{2\pi i \alpha_2 x} f(x + \alpha_1).$$

For $\varphi, \psi \in L^2(\mathbf{R}^d)$, the function $W(\varphi, \psi)$ on $\mathbf{R}^{2d}$ is defined by

$$W(\varphi, \psi)(x, \omega) = \int_{\mathbf{R}^d} e^{-2\pi i p \omega} \varphi(x + p/2) \overline{\psi(x - p/2)}\, dp.$$

This function is called the *Wigner distribution*. (Notice that we are writing the variables x and ω in this order; in [4] and [12], the Wigner distribution is written as $W(\varphi, \psi)(\omega, x)$.) We have the formula

$$W\big(\rho(\alpha)\varphi, \rho(\beta)\psi\big)(x, \omega) \\ = e^{\pi i(-\alpha_2\beta_1 + \alpha_1\beta_2)} e^{2\pi i\{\zeta_1(x + z_1) + \zeta_2(\omega + z_2)\}} W(\varphi, \psi)(x + z_1, \omega + z_2),$$

where

$$(5.2)\quad z_1 = \frac{\alpha_1+\beta_1}{2}, \qquad z_2 = -\frac{\alpha_2+\beta_2}{2}, \qquad \zeta_1 = \alpha_2-\beta_2, \qquad \zeta_2 = \alpha_1-\beta_1.$$

Using the notation of translation and modulation on $\mathbf{R}^{2d}$, the above formula can be written as

$$W\bigl(\rho(\alpha)\varphi, \rho(\beta)\psi\bigr) = e^{\pi i(-\alpha_2\beta_1+\alpha_1\beta_2)} T_{-z} M_\zeta W(\varphi,\psi). \tag{5.3}$$

We write A to denote the linear mapping $\mathbf{R}^{4d} \to \mathbf{R}^{4d}$ defined by

$$A : (\alpha_1, \alpha_2, \beta_1, \beta_2) \mapsto (z_1, z_2, \zeta_1, \zeta_2) \quad \text{with (5.2)}.$$

For our purpose, the Wigner distribution is useful since the pseudodifferential operator corresponding to $W(\varphi,\psi)$ is given by the following simple formula:

$$W(\varphi,\psi)^W(X,D)f = \langle f,\psi\rangle\varphi. \tag{5.4}$$

We also need the formula

$$\|\sigma^W(X,D)\|_{C_2} = \|\sigma\|_{L^2}. \tag{5.5}$$

Next, we recall some facts from the frame theory (for details, see [7], [11]). We take a function $\phi \in \mathcal{S}(\mathbf{R}^d)$ and a discrete subgroup Λ of $\mathbf{R}^{2d}$ such that the functions

$$\phi_\alpha = \rho(\alpha)\phi, \quad \alpha \in \Lambda,$$

form a frame for $L^2(\mathbf{R}^d)$. We set

$$\Phi_{\alpha,\beta} = W(\phi_\alpha, \phi_\beta), \quad (\alpha,\beta) \in \Gamma = \Lambda \times \Lambda,$$
$$\Phi = W(\phi,\phi).$$

By (5.3), we have

$$\Phi_{\alpha,\beta} = e^{\pi i(-\alpha_2\beta_1+\alpha_1\beta_2)} T_{-z} M_\zeta \Phi, \qquad (z,\zeta) = A(\alpha,\beta). \tag{5.6}$$

The following are known in the theory of frames. The set $\{\Phi_{\alpha,\beta}\}_{(\alpha,\beta)\in\Gamma}$ is a frame for $L^2(\mathbf{R}^{2d})$. The dual frame of $\{\phi_\alpha\}_{\alpha\in\Lambda}$ is also of the form $\{\rho(\alpha)\widetilde{\phi}\}_{\alpha\in\Lambda}$ with a $\widetilde{\phi} \in \mathcal{S}(\mathbf{R}^d)$; we write $\widetilde{\phi}_\alpha = \rho(\alpha)\widetilde{\phi}$. The dual frame of $\{\Phi_{\alpha,\beta}\}_{(\alpha,\beta)\in\Gamma}$ is given by

$$\widetilde{\Phi}_{\alpha,\beta} = W(\widetilde{\phi}_\alpha, \widetilde{\phi}_\beta), \quad (\alpha,\beta) \in \Gamma. \tag{5.7}$$

Every $\sigma \in L^2(\mathbf{R}^{2d})$ can be decomposed as

$$\sigma = \sum_{(\alpha,\beta)\in\Gamma} \langle \sigma, \Phi_{\alpha,\beta}\rangle \widetilde{\Phi}_{\alpha,\beta} \tag{5.8}$$

with the series converging unconditionally in $L^2(\mathbf{R}^{2d})$.

Now, in order to get a finite-rank approximation of $\sigma^W(X,D)$, we take a subset $\Gamma_N \subset \Gamma$, $N \in \mathbf{N}$, and define

$$\sigma_N = \sum_{(\alpha,\beta)\in\Gamma_N} \langle \sigma, \Phi_{\alpha,\beta}\rangle \widetilde{\Phi}_{\alpha,\beta}. \tag{5.9}$$

By (5.4) and (5.7), the operator $\sigma_N^W(X,D)$ is given by

$$\sigma_N^W(X,D) : f \mapsto \sum_{(\alpha,\beta)\in\Gamma_N} \langle \sigma, \Phi_{\alpha,\beta}\rangle \langle f, \widetilde{\phi}_\beta\rangle \widetilde{\phi}_\alpha.$$

Thus, if $\{\alpha \mid (\alpha,\beta)\in\Gamma_N\}$ or $\{\beta \mid (\alpha,\beta)\in\Gamma_N\}$ is a finite set, then $\sigma_N^W(X,D)$ is a finite-rank operator, and we have

$$\operatorname{rank}\bigl(\sigma_N^W(X,D)\bigr) \le \min\bigl\{\sharp\{\alpha \mid (\alpha,\beta)\in\Gamma_N\}, \sharp\{\beta \mid (\alpha,\beta)\in\Gamma_N\}\bigr\}, \tag{5.10}$$

where $\sharp E$ denotes the cardinality of a set E. By (5.5) and by the expansions (5.8) and (5.9), we have

$$\begin{aligned} \|\sigma^W(X,D) - \sigma_N^W(X,D)\|_{C_2}^2 &= \|\sigma - \sigma_N\|_{L^2(\mathbf{R}^{2d})}^2 \\ &= \Bigl\| \sum_{(\alpha,\beta)\in\Gamma\setminus\Gamma_N} \langle \sigma, \Phi_{\alpha,\beta}\rangle \widetilde{\Phi}_{\alpha,\beta} \Bigr\|_{L^2(\mathbf{R}^{2d})}^2 \\ &\le c \sum_{(\alpha,\beta)\in\Gamma\setminus\Gamma_N} |\langle \sigma, \Phi_{\alpha,\beta}\rangle|^2, \end{aligned}$$

where the last inequality follows from the fact that $\{\widetilde{\Phi}_{\alpha,\beta}\}$ is a frame for $L^2(\mathbf{R}^{2d})$. We simply write

$$\gamma(z,\zeta) = \langle \sigma, T_z M_\zeta \Phi\rangle.$$

By (5.6),

$$|\langle \sigma, \Phi_{\alpha,\beta}\rangle| = |\langle \sigma, T_{-z} M_\zeta \Phi\rangle| = |\gamma(-z,\zeta)|, \qquad (z,\zeta) = A(\alpha,\beta),$$

and the above inequality can be written as

$$\|\sigma^W(X,D) - \sigma_N^W(X,D)\|_{C_2}^2 \le c \sum_{(z,\zeta)\in A(\Gamma)\setminus A(\Gamma_N)} |\gamma(-z,\zeta)|^2.$$

Applying the general principle (5.1), we obtain

$$\sum_{j>k_N} s_j\big(\sigma^W(X,D)\big)^2 \le c \sum_{(z,\zeta)\in A(\Gamma)\setminus A(\Gamma_N)} |\gamma(-z,\zeta)|^2,$$

where $k_N = \mathrm{rank}(\sigma_N^W(X,D))$.

To get an estimate of k_N, we take

$$\Gamma_N = \big\{(\alpha,\beta)\in\Gamma \,\big|\, |\alpha_1| + |\alpha_2| \le N\big\}.$$

Then, by (5.10),

$$k_N \le \sharp\big\{\alpha \,\big|\, (\alpha,\beta)\in\Gamma_N\big\} \le c_0 N^{2d},$$

where c_0 is a constant depending only on d. From (5.2), we have

$$A(\Gamma_N) = \big\{(z,\zeta)\in A(\Gamma) \,\big|\, |z_1 + \zeta_2/2| + |-z_2 + \zeta_1/2| \le N\big\}.$$

Thus,

$$\begin{aligned}
&\sum_{(z,\zeta)\in A(\Gamma)\setminus A(\Gamma_N)} |\gamma(-z,\zeta)|^2 \\
&\quad\le N^{-2s} \sum_{(z,\zeta)\in A(\Gamma)\setminus A(\Gamma_N)} |\gamma(-z,\zeta)|^2 (1 + |z_1+\zeta_2/2| + |-z_2+\zeta_1/2|)^{2s} \\
&\quad= N^{-2s} \sum_{(z,\zeta)\in A(\Gamma)\setminus A(\Gamma_N)} |\gamma(-z,\zeta)|^2 m(-z,\zeta)^2,
\end{aligned}$$

where m is the weight function of (1.5) with $t = 1/2$. Combining the above inequalities, we obtain

$$\sum_{j>c_0N^{2d}} s_j\big(\sigma^W(X,D)\big)^2 \le cN^{-2s} \sum_{(z,\zeta)\in A(\Gamma)\setminus A(\Gamma_N)} |\gamma(-z,\zeta)|^2 m(-z,\zeta)^2.$$

From this inequality and from the fact that $\{s_j(A)\}$ is a nonincreasing sequence, we obtain the estimate

$$s_k\big(\sigma^W(X,D)\big) \le ck^{-(s+d)/2d}\Big(\sum_{(z,\zeta)\in A(\Gamma)} |\gamma(-z,\zeta)|^2 m(-z,\zeta)^2\Big)^{1/2}.$$

Thus, if we have the inequality

$$(5.11)\qquad \Big(\sum_{(z,\zeta)\in A(\Gamma)} |\gamma(-z,\zeta)|^2 m(-z,\zeta)^2\Big)^{1/2} \le c\|\sigma\|_{M^{2,2}_m},$$

then we have

$$s_k\big(\sigma^W(X,D)\big) \le ck^{-(s+d)/2d}\|\sigma\|_{M^{2,2}_m}.$$

If in addition $(s+d)/2d > p$, then we have

$$\|\sigma^W(X,D)\|_{C_p} \le c\|\{k^{-(s+d)/2d}\}\|_{l^p}\|\sigma\|_{M^{2,2}_m} = c\|\sigma\|_{M^{2,2}_m},$$

which implies the claim of Theorem 1.2 for $t = 1/2$ with the weight (1.5).

If we take

$$\Gamma_N = \big\{(\alpha,\beta)\in\Gamma \;\big|\; |\beta_1| + |\beta_2| \le N\big\},$$

then we also have $k_N = \operatorname{rank}(\sigma_N^W(X,D)) \le c_0 N^{2d}$ and

$$A(\Gamma_N) = \big\{(z,\zeta)\in A(\Gamma) \;\big|\; |z_1 - \zeta_2/2| + |-z_2 - \zeta_1/2| \le N\big\}.$$

Hence, by the same argument as above, we obtain

$$\begin{aligned} s_k&\big(\sigma^W(X,D)\big) \\ &\le ck^{-(s+d)/2d} \\ &\quad\times\Big(\sum_{(z,\zeta)\in A(\Gamma)} |\gamma(-z,\zeta)|^2(1+|z_1-\zeta_2/2| + |-z_2-\zeta_1/2|)^{2s}\Big)^{1/2} \\ &= ck^{-(s+d)/2d}\Big(\sum_{(z,\zeta)\in A(\Gamma)} |\gamma(-z,\zeta)|^2 m(-z,\zeta)^2\Big)^{1/2}, \end{aligned}$$

where m is the weight function of (1.4) with $t = 1/2$. Hence, if inequality (5.11) holds, then the claim of Theorem 1.2 for $t = 1/2$ with weight (1.4) follows.

Thus, the rest of the proof is to show the inequality (5.11) for the weight functions of (1.4) and (1.5). This can be done at least for the following special choice of ϕ and Λ:

$$\phi(x) = 2^{d/4}e^{-\pi x^2} \quad (x\in\mathbf{R}^d),$$

$$\Lambda = a\mathbf{Z}^d \times b\mathbf{Z}^d, \quad a,b>0,\ ab<1.$$

In fact, for the above ϕ and Λ, it is known that $\{\rho(\alpha)\phi\}_{\alpha\in\Lambda}$ is a frame for $L^2(\mathbf{R}^d)$; this fact is due to Seip and Wallstén ([21], [22]). We have

$$\Phi(x,\omega) = W(\phi,\phi)(x,\omega) = 2^d e^{-2\pi(x^2+\omega^2)},$$

and for this Φ, using the inequality of Lemma 3.2 and arguing in a similar way as in the proof of Theorem 1.4, we can prove inequality (5.11). This completes the proof. $\square$

Finally, we give a sketch of an alternate proof of Theorem 1.4 for the case $0 < p \leq 1$.

An alternate proof of Theorem 1.4 for $0 < p \leq 1$. With an appropriate pair $\Phi, \Psi \in \mathcal{S}(\mathbf{R}^{2d})$ and with appropriate $\alpha, \beta \in (0,\infty)$, we have the representation

$$\sigma = \sum_{k,l\in\mathbf{Z}^{2d}} \langle \sigma, T_{k\alpha}M_{l\beta}\Phi\rangle T_{k\alpha}M_{l\beta}\Psi$$

with

$$\|\sigma\|_{M^{p,p}} \approx \Big(\sum_{k,l\in\mathbf{Z}^{2d}} |\langle \sigma, T_{k\alpha}M_{l\beta}\Phi\rangle|^p\Big)^{1/p}$$

(see [5, Theorem 3.7]). We take $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)$ and take Ψ to be the Rihaczek distribution

$$\Psi(x,\omega) = R(\varphi_1,\varphi_2)(x,\omega) = \varphi_1(x)\overline{\widehat{\varphi_2}(\omega)}e^{-2\pi i x\omega}, \quad x,\omega\in\mathbf{R}^d.$$

Then $T_z M_\zeta \Psi(X,D)$ has the following simple form:

$$T_z M_\zeta \Psi(X,D)f(x) = e^{-2\pi i\zeta_1 z_1}\langle f, M_{z_2}T_{z_1+\zeta_2}\varphi_2\rangle M_{z_2+\zeta_1}T_{z_1}\varphi_1(x)$$

(see [8, Lemma 8.35]). In particular, $T_z M_\zeta\Psi(X,D)$ is an operator of rank 1, and

$$\|T_z M_\zeta \Psi(X,D)\|_{C_p} \leq c \tag{5.12}$$

with c independent of z and ζ. We use the fact that $\|\cdot\|_{C_p}^p$ is subadditive for $0 < p \leq 1$:

$$\|T+S\|_{C_p}^p \leq \|T\|_{C_p}^p + \|S\|_{C_p}^p, \quad 0 < p \leq 1 \tag{5.13}$$

(this fact is due to McCarthy [15, Theorem 2.8]). Now combining the above results, we obtain, for $0 < p \le 1$,

$$\begin{aligned}\|\sigma(X,D)\|_{C_p}^p &= \Big\| \sum_{k,l \in \mathbf{Z}^{2d}} \langle \sigma, T_{k\alpha} M_{l\beta} \Phi \rangle T_{k\alpha} M_{l\beta} \Psi(X,D) \Big\|_{C_p}^p \\ &\le \sum_{k,l \in \mathbf{Z}^{2d}} |\langle \sigma, T_{k\alpha} M_{l\beta} \Phi \rangle|^p \|T_{k\alpha} M_{l\beta} \Psi(X,D)\|_{C_p}^p \\ &\lesssim \sum_{k,l \in \mathbf{Z}^{2d}} |\langle \sigma, T_{k\alpha} M_{l\beta} \Phi \rangle|^p \approx \|\sigma\|_{M^{p,p}}^p .\end{aligned}$$

In the above proof, the use of the Rihaczek distribution makes the argument simple but is not essential since, as the argument in the proof of Theorem 1.2 shows, the estimate (5.12) holds for arbitrary $\Psi \in \mathcal{S}(\mathbf{R}^{2d})$.

If we use Lemma 2.3, we can also avoid using the subadditivity (5.13). In fact, the argument in the proof of Theorem 1.2 shows that, for the frame $\{g_{mn}\}$ of Example 2.2,

$$\|T_z M_\zeta \Psi(X,D) g_{mn}\|_{L^2}^2 \le c \langle z_1 + \zeta_2 - m \rangle^{-N} \langle z_2 - n/2 \rangle^{-N}$$

(see (3.5)), which implies that

$$\sum_{m,n \in \mathbf{Z}^d} \|T_z M_\zeta \Psi(X,D) g_{mn}\|_{L^2}^p \le c.$$

Hence, if $0 < p \le 1$, we have

$$\begin{aligned}\|\sigma(X,D)\|_{C^p}^p &\lesssim \sum_{m,n \in \mathbf{Z}^d} \|\sigma(X,D) g_{mn}\|_{L^2}^p \\ &\le \sum_{m,n \in \mathbf{Z}^d} \Big(\sum_{k,l \in \mathbf{Z}^d} |\langle \sigma, T_{k\alpha} M_{l\beta} \Phi \rangle| \|T_{k\alpha} M_{l\beta} \Psi(X,D) g_{mn}\|_{L^2} \Big)^p \\ &\le \sum_{m,n \in \mathbf{Z}^d} \sum_{k,l \in \mathbf{Z}^d} |\langle \sigma, T_{k\alpha} M_{l\beta} \Phi \rangle|^p \|T_{k\alpha} M_{l\beta} \Psi(X,D) g_{mn}\|_{L^2}^p \\ &\lesssim \sum_{k,l \in \mathbf{Z}^d} |\langle \sigma, T_{k\alpha} M_{l\beta} \Phi \rangle|^p \approx \|\sigma\|_{M^{p,p}}^p ,\end{aligned}$$

where the assumption that $p \le 1$ is used to obtain the last $\le$. □

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