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# ON THE BILINEAR SQUARE FOURIER MULTIPLIER OPERATORS ASSOCIATED WITH $g_\lambda^*$ FUNCTION

ZHENGYANG LI AND QINGYING XUE*

**Abstract.** This paper will be devoted to study a class of bilinear square-function Fourier multiplier operator associated with a symbol $m$ defined by

$$\begin{aligned}\mathfrak{T}_{\lambda,m}(f_1,f_2)(x) &= \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|x-z|+t}\Big)^{n\lambda} \\ &\quad\times\Big|\int_{(\mathbb{R}^n)^2} e^{2\pi i x\cdot(\xi_1+\xi_2)} m(t\xi_1,t\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\xi_1\,d\xi_2\Big|^2\frac{dz\,dt}{t^{n+1}}\Big)^{1/2}.\end{aligned}$$

A basic fact about $\mathfrak{T}_{\lambda,m}$ is that it is closely associated with the multilinear Littlewood–Paley $g_\lambda^*$ function. In this paper we first investigate the boundedness of $\mathfrak{T}_{\lambda,m}$ on products of weighted Lebesgue spaces. Then, the weighted endpoint $L\log L$ type estimate and strong estimate for the commutators of $\mathfrak{T}_{\lambda,m}$ will be demonstrated.

## §1. Introduction

### 1.1 Background

It is well known that the $N$-linear $(N\geqslant 1)$ Fourier multiplier operator $T_m$ is defined as follows:

$$T_m(f_1,\ldots,f_N)(x)=\frac{1}{(2\pi)^{nN}}\int_{(\mathbb{R}^n)^N} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi)\widehat{f_1}(\xi_1)\cdots\widehat{f_N}(\xi_N)\,d\xi,$$

for $m\in L^\infty(\mathbb{R}^N)$ and $f_1,\ldots,f_N\in\mathcal{S}$, where $x\in\mathbb{R}^n,\xi=(\xi_1,\ldots,\xi_m)\in(\mathbb{R}^n)^N$. By using paraproducts, Coifman and Meyer [8] proved that if $m$ is a bounded function on $\mathbb{R}^{nN}\backslash\{0\}$ and it satisfies that

$$|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_N}^{\alpha_N} m(\xi_1,\ldots,\xi_N)|\leqslant C_\alpha(|\xi_1|+\cdots+|\xi_m|)^{-(|\alpha_1|+\cdots+|\alpha_N|)},$$

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*Corresponding author.

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away from the origin for $|\alpha_1| + \cdots + |\alpha_N| \leqslant L$ with $L$ sufficiently large, then $T_m$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. By decreasing the smoothness condition of $T_m$ in [8], Tomita [25] gave a Hörmander type theorem for $T_m$. Later on, Fujita and Tomita [14] demonstrated a weighted version of the results in [25] for $T_m$ under the Hörmander condition with classical $A_p$ weights. In 2013, Bui and Duong [2] established the norm inequality for a class of Calderón–Zygmund singular integral operators with kernels satisfying some mild regularity condition. As an application, they [2] obtained the multiple weighted norm inequality of multilinear Fourier multipliers. For more works about multilinear Fourier multipliers, we refer the reader to [15, 20, 21]. Recently, Si, Xue and Yabuta [28] considered the bilinear square-function Fourier multiplier operator defined as follows,

$$\mathfrak{T}_m(f_1, f_2)(x) = \left(\int_0^\infty |T_m^t(f_1, f_2)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$T_m^t(f_1, f_2)(x) = \int_{(\mathbb{R}^n)^2} e^{2\pi i x\cdot(\xi_1+\xi_2)} m(t\xi_1, t\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\, d\xi_1\, d\xi_2.$$

By studying a class of multilinear square functions, the authors in [28] demonstrated the multiple weighted norm inequality for $\mathfrak{T}_m$ and obtained some weighted estimates for the commutators of $\mathfrak{T}_m$ with $BMO$ functions.

REMARK 1.1. Note that if $t = 1$, then $T_m^1$ coincides with the well-known bilinear Fourier multiplier operator defined and studied in [8]. The operator studied in [28] can be looked as a vector valued or square version of bilinear Fourier multiplier operator $T_m^1$.

In this paper, we investigate the boundedness of the following bilinear square-function Fourier multiplier operator $\mathfrak{T}_{\lambda,m}$, which is associated with the multilinear $g_\lambda^*$-function defined in [23].

$$\mathfrak{T}_{\lambda,m}(f_1, f_2)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{|x-z|+t}\right)^{n\lambda} \left|T_m^t(f_1, f_2)(x)|^2 \frac{dt}{t}\right|^2 \frac{dz\, dt}{t^{n+1}}\right)^{1/2}.$$

Let $\vec{f} = (f_1, f_2)$ and $K_t(x, y_1, y_2) = \frac{1}{t^{2n}}\check{m}\left(\frac{x-y_1}{t}, \frac{x-y_2}{t}\right)$. Then, $\mathfrak{T}_{\lambda,m}$ can be written as

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$$\mathfrak{T}_{\lambda,m}(\vec{f})(x) = \bigg(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{|x-z|+t}\Big)^{n\lambda} \times \bigg|\int_{(\mathbb{R}^n)^2} K_t(x,y_1,y_2) f_1(y_1) f_2(y_2)\,dy_1\,dy_2\bigg|^2 \frac{dz\,dt}{t^{n+1}}\bigg)^{1/2}.$$

The commutator of $\mathfrak{T}_{\lambda,m}$ is defined by

$$\mathfrak{T}^{\vec{b}}_{\lambda,m}(\vec{f})(x) = \sum_{i=1}^{2} \bigg(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{|x-z|+t}\Big)^{n\lambda} \bigg|\int_{(\mathbb{R}^n)^2} (b_i(x)-b_i(y)) \times K_t(x,y_1,y_2) f_1(y_1) f_2(y_2)\,dy_1\,dy_2\bigg|^2 \frac{dz\,dt}{t^{n+1}}\bigg)^{1/2}, \quad \vec{b} = (b_1,b_2).$$

REMARK 1.2. In [23], the authors studied a class of multilinear $g^*_\lambda$ function associated with convolution type kernels. The endpoint $L^1 \times \cdots \times L^1 \to L^{1/m,\infty}$ boundedness, and multiple weighted boundedness for the multilinear $g^*_\lambda$ function were established. Later, in [27] the same results were extended to kernels of nonconvolution type. For more previous nice works of the classical $g^*_\lambda$ function, one may see the famous works of Stein [24], Fefferman [13], Muckenhoupt and Wheeden [22].

Our object of investigation in this paper is the multilinear square-function Fourier multiplier operator associated with the multilinear $g^*_\lambda$ function. Before stating our main results, we need to introduce some more notations and definitions. For $m$ exponents $p_1,\ldots,p_m$, denote by $p$ the number given by $1/p = 1/p_1 + \cdots + 1/p_m$, and $\vec{P}$ for the vector $\vec{P} = (p_1,\ldots,p_m)$. For any real number $r > 1$, the vector $\vec{P}/r$ is given by $\vec{P}/r = (p_1/r,\ldots,p_m/r)$. The following multiple weights classes $A_{\vec{P}}$ were introduced and studied by Lerner *et al.* [19].

DEFINITION 1.3. (Multiple weights [19].) Let $1 \leqslant p_1,\ldots,p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$. Given $\vec{\omega} = (\omega_1,\ldots,\omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{P}}$ condition if

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q \prod_{i=1}^m \omega_i^{p/p_i}\Big)^{1/p} \prod_{i=1}^m \Big(\frac{1}{|Q|}\int_Q \omega_i^{1-p_i'}\Big)^{1/p_i'} < \infty,$$

when $p_i = 1$, $((1/|Q|)\int_Q \omega_i^{1-p_i'})^{1/p_i'}$ is understood as $(\inf_Q \omega_i)^{-1}$.

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Throughout this paper, we always assume that $m \in L^\infty((\mathbb{R}^n)^2)$ and satisfies the conditions

$$(1.1)\qquad |\partial^\alpha m(\xi_1,\xi_2)| \lesssim \frac{(|\xi_1|+|\xi_2|)^{-|\alpha|+\varepsilon_1}}{(1+|\xi_1|+|\xi_2|)^{\varepsilon_1+\varepsilon_2}}$$

and

$$(1.2)\qquad |m(\xi_1,\xi_2)| \lesssim \frac{(|\xi_1|+|\xi_2|)^{-s+\varepsilon_1}}{(1+|\xi_1|+|\xi_2|)^{\varepsilon_1+\varepsilon_2}}$$

for some $\varepsilon_1, \varepsilon_2 > 0$, $|\alpha| \leqslant s$ and $n+1 \leqslant s \leqslant 2n$ for some integer $s$.

REMARK 1.4. Note that, for the same $\varepsilon_1, \varepsilon_2$, $\alpha$ and $s$, conditions (1.1) and (1.2) are more weaker than the following condition:

$$(1.3)\qquad |\partial^\alpha m(\xi_1,\xi_2)| \lesssim \begin{cases} (|\xi_1|+|\xi_2|)^{-s-|\alpha|-\varepsilon_2}, & |\xi_1|+|\xi_2|>1;\\ (|\xi_1|+|\xi_2|)^{-|\alpha|+\varepsilon_1}, & 0<|\xi_1|+|\xi_2|\leqslant 1. \end{cases}$$

Moreover, for $|\alpha| \leqslant s$, condition (1.3) is equivalent with

$$(1.4)\qquad |\partial^\alpha m(\xi_1,\xi_2)| \lesssim \begin{cases} \dfrac{(|\xi_1|+|\xi_2|)^{-|\alpha|+\varepsilon_1}}{(1+|\xi_1|+|\xi_2|)^{s+\varepsilon_1+\varepsilon_2}}, & |\xi_1|+|\xi_2|>1;\\ \dfrac{(|\xi_1|+|\xi_2|)^{-|\alpha|+\varepsilon_1}}{(1+|\xi_1|+|\xi_2|)^{s+\varepsilon_1+\varepsilon_2}}, & 0<|\xi_1|+|\xi_2|\leqslant 1. \end{cases}$$

These facts show that conditions (1.1) and (1.2) are still more weaker than (1.4), which, in turn, also indicates that our conditions (1.1) and (1.2) are reasonable.

The main results of this paper are:

THEOREM 1.1. *Let $s$ be an integer with $s \in [n+1, 2n]$ and $\lambda > 2s/n+1$, $p_0$ be a number satisfying $2n/s \leqslant p_0 \leqslant 2$. Let $p_0 \leqslant p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, and $\vec{\omega} \in A_{\vec{P}/p_0}$. Suppose that $m \in L^\infty((\mathbb{R}^n)^2)$ satisfies (1.1) and (1.2) and that the bilinear square Fourier multiplier operator $\mathfrak{T}_{\lambda,m}$ is bounded from $L^{q_1} \times L^{q_2}$ into $L^{q,\infty}$, for any $p_0 < q_1, q_2$ and $1/q = 1/q_1 + 1/q_2$. Then the following weighted estimates hold.*

(i) *If $p_1, p_2 > p_0$, then $||\mathfrak{T}_{\lambda,m}(\vec{f})||_{L^p(\nu_{\vec{\omega}})} \leqslant C||f_1||_{L^{p_1}(\omega_1)}||f_2||_{L^{p_2}(\omega_2)}$.*
(ii) *If $p_0 > 2n/s$ and $p_1 = p_0$ or $p_2 = p_0$, then*

$$||\mathfrak{T}_{\lambda,m}(\vec{f})||_{L^{p,\infty}(\nu_{\vec{\omega}})} \leqslant C||f_1||_{L^{p_1}(\omega_1)}||f_2||_{L^{p_2}(\omega_2)}.$$

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THEOREM 1.2. *Let $s$, $\lambda$, $p_0, p_1, p_2, p$, $\vec{\omega}$, $m$ and $\mathfrak{T}_{\lambda,m}$ be the same as in Theorem 1.1. Then the following weighted estimates hold for the commutators of $\mathfrak{T}_{\lambda,m}(\vec{f})$.*

(i) *If $p_1, p_2 > p_0$, then for any $\vec{b} \in BMO^2$, it holds that*

$$||\mathfrak{T}^{\vec{b}}_{\lambda,m}(\vec{f})||_{L^p(\nu_{\vec{\omega}})} \leqslant C||\vec{b}||_{BMO}||f_1||_{L^{p_1}(\omega_1)}||f_2||_{L^{p_2}(\omega_2)},$$

*where $||\vec{b}||_{BMO} = \max_j ||b_j||_{BMO}$.*

(ii) *Let $\vec{\omega} \in A_{(1,1)}$ and $\vec{b} \in BMO^2$. Then, there exists a constant $C$ (depending on $\vec{b}$) such that*

$$\nu_{\vec{\omega}}\big(\{x \in \mathbb{R}^n : |\mathfrak{T}^{\vec{b}}_{\lambda,m}(\vec{f})(x)| > t^2\}\big) \leqslant C \prod_{j=1}^{2} \Big(\int_{\mathbb{R}^n} \Phi\Big(\frac{|f_j(x)|}{t}\Big)\omega_j(x)\Big)^{1/2},$$

*where $\Phi(t) = t^{p_0}(1 + \log^+ t)^{p_0}$, and the function $\log^+ t$ is defined by $\log^+ t = \log t$, if $t > 1$, otherwise $\log^+ t = 0$.*

The article is organized as follows. Proof of Theorems 1.1 and 1.2 will be shown in Section 2. In Section 3, we give an example to show that the assumption that $\mathfrak{T}_{\lambda,m}$ is bounded from $L^{q_1} \times L^{q_2}$ into $L^{q,\infty}$ in Theorems 1.1 and 1.2 is reasonable.

## §2. Proofs of Theorems 1.1 and 1.2

This section will be devoted to prove Theorems 1.1 and 1.2. The following two propositions provide a foundation for our proofs.

### 2.1 Two key propositions

PROPOSITION 2.1. *Let $s \in \mathbb{N}$ satisfy $n + 1 \leqslant s \leqslant 2n$. Suppose $m \in L^\infty((\mathbb{R}^n)^2)$ satisfies (1.1) and (1.2). Then, for any $2n/s < p \leqslant 2$, $\lambda > 2s/n + 1$, there exist $C > 0$ and $\delta > n/p$, such that*

$$\begin{aligned}\Big(\int_{S_j(Q)} \int_{S_k(Q)} \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{|z| + t}\Big)^{n\lambda} \Big|\check{m}\Big(\frac{x - z - y_1}{t}, \frac{x - z - y_2}{t}\Big) \\ - \check{m}\Big(\frac{\bar{x} - y_1}{t}, \frac{\bar{x} - y_2}{t}\Big)\Big|^2 \frac{dz\, dt}{t^{5n+1}}\Big)^{p'/2} dy_1\, dy_2\Big)^{1/p'} \\ \leqslant C\frac{|x - \bar{x}|^{2(\delta - n/p)}}{|Q|^{2\delta/n}} 2^{-2\delta \max(j,k)} \qquad (2.1)\end{aligned}$$

*for all balls $Q$, all $x, \bar{x} \in (1/2)Q$ and $(j, k) \neq (0, 0)$.*

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*Proof.* For convenience, we denote the left-hand side of (2.1) by $A_{j,k}(m, Q)(x, \bar{x})$. Let $u = ax$ $(a > 0)$, $Q = B(x_0, R)$, $v = az$ and $\tau = at$, we may get

$$\begin{aligned} A_{j,k}(m, Q)(x, \bar{x}) = a^{1/2-2n/p'}\Big(\int_{S_j(Q^a)}\int_{S_k(Q^a)}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{at}{|v|+at}\Big)^{n\lambda} \\ \times\Big|\check{m}\Big(\frac{x^a - v - u_1}{at}, \quad \frac{x^a - v - u_2}{at}\Big) \\ - \check{m}\Big(\frac{\bar{x}^a - u_1}{at}, \frac{\bar{x}^a - u_2}{at}\Big)\Big|^2 \frac{dv dt}{t^{5n+1}}\Big)^{p'/2} du_1\, du_2\Big)^{1/p'} \\ = a^{2n/p}\Big(\int_{S_j(Q^a)}\int_{S_k(Q^a)}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{\tau}{|v|+\tau}\Big)^{n\lambda}\Big| \\ \times\check{m}\Big(\frac{x^a - v - u_1}{\tau}, \frac{x^a - v - u_2}{\tau}\Big) \\ - \check{m}\Big(\frac{\bar{x}^a - u_1}{\tau}, \frac{\bar{x}^a - u_2}{\tau}\Big)\Big|^2 \frac{dv d\tau}{\tau^{4n+1}}\Big)^{p'/2} du_1\, du_2\Big)^{1/p'} \\ = a^{2n/p} A_{j,k}(m, Q^a)(x^a, \bar{x}^a), \end{aligned}$$

where $Q^a = B(ax_0, aR)$, $x^a = ax$ and $\bar{x}^a = a\bar{x}$. Thus, if we take $a = 1/(2^{\max(j,k)}R)$, it is easy to see that the following estimate implies the desired one.

$$\begin{aligned} A_{j,k}(m, Q^a)(x^a, \bar{x}^a) &\lesssim \frac{|x^a - \bar{x}^a|^{2(\delta-n/p)}}{|Q^a|^{2\delta/n}} 2^{-2\delta\max(j,k)} \\ &= |x^a - \bar{x}^a|^{2(\delta-n/p)}. \end{aligned} \tag{2.2}$$

Since $x^a, \bar{x}^a \in (1/2)Q^a$, $aR = 1/2^{\max(j,k)}$. Therefore, in order to prove (2.2), we only need to show (2.1) is true for all balls $Q$ with radius $R = 1/2^{\max(j,k)}$. Without loss of generality, we may assume $|h| = |x - \bar{x}| < 1/2$ and $k \geqslant j$ (hence $k \geqslant 1$). Thus, the proof of Proposition 2.1 is reduced to show that

$$A_{j,k}(m, Q)(x, \bar{x}) \lesssim |x - \bar{x}|^{2(\delta-n/p)}, \tag{2.3}$$

where $Q = B(x_0, 2^{-k})$ and $\delta > n/p$.

Let $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfying $\operatorname{supp}\Psi \in \{(\xi, \eta) : 1/2 \leqslant |\xi| + |\eta| \leqslant 2\}$ and

$$\sum_{j\in\mathbb{Z}} \Psi(2^{-j}\xi, 2^{-j}\eta) = 1, \quad \text{for all } (\xi, \eta) \in (\mathbb{R}^{2n}) \setminus \{0\}.$$

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Thus, we can write

$$m(\xi,\eta)=\sum_{j\in\mathbb{Z}} m_j(\xi,\eta):=\sum_{j\in\mathbb{Z}} \Psi(2^{-j}\xi,2^{-j}\eta)m(\xi,\eta)$$

and hence $\operatorname{supp} m_j \subseteq \{(\xi,\eta): 2^{j-1}\leqslant |\xi|+|\eta|\leqslant 2^{j+1}\}$.

Using the change of variables, (2.3) is equivalent to that

$$\bigg(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\bigg(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|z|+t}\Big)^{n\lambda}\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big) - \check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}\leqslant C|h|^{2(\delta-n/p)},$$

for $Q=B(x_0,2^{-k})$, $h=x-\bar{x}$ and $Q_{\bar{x}}=Q-\bar{x}$. We prove this in the following three cases.

**(a) The case** $2n/p<s<2n/p+1$. Since (1.1) and (1.2) remain valid for any smaller positive number than $\varepsilon_1$, we may take $\varepsilon_1$ sufficiently close to $s-2n/p$ so that $0<\varepsilon_1<s-2n/p$.

First we introduce $A_\ell$ and $A_\ell(I)$ as follows,

$$A_\ell := \bigg(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\bigg(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|z|+t}\Big)^{n\lambda}\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big) - \check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'};$$

$$A_\ell(I) := \bigg(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\bigg(\int_{\mathbb{R}^n}\int_I\Big(\frac{t}{|z|+t}\Big)^{n\lambda}\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big) - \check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'},$$

where $I$ is any interval in $\mathbb{R}_+$, in particular, $I$ could be right half-infinite.

In addition, we denote

$$E_1=\{z\in\mathbb{R}^n:|z|<t,|z|<1/8\},\qquad E_2=\{z\in\mathbb{R}^n:|z|<t,1/8\leqslant|z|<3\},$$
$$E_3=\{z\in\mathbb{R}^n:|z|<t,|z|\geqslant 3\},\qquad E_4=\{z\in\mathbb{R}^n:|z|\geqslant t,|z|<1/8\},$$
$$E_5=\{z\in\mathbb{R}^n:|z|\geqslant t,1/8\leqslant|z|<3\},\qquad E_6=\{z\in\mathbb{R}^n:|z|\geqslant t,|z|\geqslant 3\},$$

then we have $A_\ell(I)\leqslant\sum_{i=1}^{6}A_\ell^i(I)$, where

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$$A_\ell^i(I) := \Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{E_i}\int_I\Big(\frac{t}{|z|+t}\Big)^{n\lambda}\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big) - \check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}.$$

Now, we begin with the estimate of $A_\ell^1(I)$.

**Estimate for** $A_\ell^1(I)$. Since $Q_{\bar{x}} = B(x_0-\bar{x}, 1/2^k)$, then $2^{-2}\leqslant |y_1+h|\leqslant 2$ and $|y_2+h|\leqslant 2^{j-k+1}$ for all $y_1\in S_k(Q_{\bar{x}})$ and $y_2\in S_j(Q_{\bar{x}})$. Note that $|z|<1/8$, we have $1/8<|y_1+h-z|\leqslant 17/8$. This implies that

$$\begin{aligned}A_\ell^1(I) &\leqslant \Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{|z|\leqslant\min\{1/8,t\}}\int_I\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big)\\&\qquad - \check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}\\&\lesssim \Big(\int_{|y_2|\leqslant 2^{j-k+2}}\int_{1/8<|y_1|\leqslant 17/8}\Big(\int_{|z|\leqslant\min\{1/8,t\}}\int_I\\&\qquad\times\Big|\check{m}\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}\\&\leqslant \Big(\int_{|y_2|\leqslant 2^{j-k+2}}\int_{1/8<|y_1|\leqslant 17/8}\Big(\int_I\Big|\check{m}\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dt}{t^{4n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}.\end{aligned}$$

Note that $|y_1|\sim 1$ in the last integration above, by the Minkowski inequality and the Hausdorff–Young inequality, for $|\alpha|=s$, we have

$$\begin{aligned}A_\ell^1(I) &\lesssim \Big(\int_{|y_2|\leqslant 2^{j-k+2}}\int_{1/8<|y_1|\leqslant 17/8}\\&\qquad\times\Big(\int_I|y_1^\alpha|^2\Big|\check{m}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dt}{t^{4n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}\\&\leqslant \Big(\int_I\Big(\int_{|y_2|\leqslant 2^{j-k+2}}\int_{1/8<|y_1|\leqslant 17/8}\\&\qquad\times\Big|y_1^\alpha\check{m}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^{p'}dy_1\,dy_2\Big)^{2/p'}\frac{dt}{t^{4n+1}}\Big)^{1/2}\\&= \Big(\int_I\Big(\int_{|ty_2|\leqslant 2^{j-k+2}}\int_{1/8<|ty_1|\leqslant 17/8}\\&\qquad\times|y_1^\alpha\check{m}_\ell(y_1,y_2)|^{p'}dy_1\,dy_2\Big)^{2/p'}t^{2|\alpha|+4n/p'}\frac{dt}{t^{4n+1}}\Big)^{1/2}\\&\leqslant \Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{2/p}t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}\end{aligned}$$

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$$\lesssim \Big(\int_I t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{1/p}. \tag{2.4}$$

Hence, we obtain

$$A_\ell^1(I)\lesssim \frac{(2^\ell)^{\varepsilon_1-|\alpha|+2n/p}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}}\Big(\int_I t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}. \tag{2.5}$$

Now, setting $\varphi_\ell(\xi,\eta)=m_\ell(\xi,\eta)(e^{2\pi i t^{-1}h\cdot(\xi+\eta)}-1)$, we have

$$\begin{aligned}&\check{m}_\ell\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big)-\check{m}_\ell\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\\&=\check{\varphi}_\ell\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big).\end{aligned}$$

Proceeding the same argument as before, we have

$$\begin{aligned}A_\ell^1(I)&\lesssim\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{|z|\leqslant\min\{1/8,t\}}\\&\quad\times\int_I\Big|(y_1-z)^\alpha\Big(\check{m}_\ell\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big)\\&\quad-\check{m}_\ell\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}\\&\leqslant\Big(\int_I\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big|y_1^\alpha\check{\varphi}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^{p'}dy_1\,dy_2\Big)^{2/p'}\frac{dt}{t^{4n+1}}\Big)^{1/2}\\&=\Big(\int_I\Big(\int_{S_j(t^{-1}Q_{\bar{x}})}\int_{S_k(t^{-1}Q_{\bar{x}})}|y_1^\alpha\check{\varphi}_\ell(y_1,y_2)|^{p'}dy_1\,dy_2\Big)^{2/p'}\\&\quad\times t^{2|\alpha|+4n/p'}\frac{dt}{t^{4n+1}}\Big)^{1/2}\\&\leqslant\Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha\varphi_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{2/p}t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}\\&=\Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha[m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)]|^p\,d\xi\,d\eta\Big)^{2/p}\\&\quad\times t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}.\end{aligned} \tag{2.6}$$

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**Estimate for $A_\ell^2(I)$.**

$$\begin{aligned}A_\ell^2(I) &\leqslant \bigg(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\bigg(\int_{1/8\leqslant|z|\leqslant\min\{3,t\}}\int_I \frac{1}{t^{5n+1}}\\ &\quad\times\bigg|\check{m}\bigg(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\bigg)\\ &\quad-\check{m}\bigg(\frac{y_1-z}{t},\frac{y_2-z}{t}\bigg)\bigg|^2 dzdt\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}\\ &\leqslant \bigg(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\bigg(\int_{1/8\leqslant|z|\leqslant\min\{3,t\}}\\ &\quad\times\int_I\bigg|\check{m}\bigg(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\bigg)\bigg|^2\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}\\ &\quad+\bigg(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\bigg(\int_{1/8\leqslant|z|\leqslant\min\{3,t\}}\int_I\bigg|\check{m}\bigg(\frac{y_1-z}{t},\frac{y_2-z}{t}\bigg)\bigg|^2\\ &\quad\times\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}\\ &=: A_\ell^{2,1}(I)+A_\ell^{2,2}(I).\end{aligned}$$

We observe that if $z\in E_2$, then $t\geqslant 1/8$. The Minkowski inequality and the Hausdorff–Young inequality yield that

$$\begin{aligned}A_\ell^{2,1}(I) &\lesssim \bigg(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\bigg(\int_{|z|\leqslant t}\int_{\{t\in I:t\geqslant 1/8\}}\\ &\quad\times\bigg|\check{m}_\ell\bigg(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\bigg)\bigg|^2\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}\\ &\leqslant \bigg(\int_{\{t\in I:t\geqslant 1/8\}}\int_{|z|\leqslant t}\bigg(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\\ &\quad\times\bigg|\check{m}_\ell\bigg(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\bigg)\bigg|^{p'}dy_1\,dy_2\bigg)^{2/p'}\frac{dt}{t^{5n+1}}\bigg)^{1/2}\\ &= \bigg(\int_{\{t\in I:t\geqslant 1/8\}}\int_{|z|\leqslant t}\bigg(\int_{S_j(t^{-1}Q_{\bar{x}})}\int_{S_k(t^{-1}Q_{\bar{x}})}\\ &\quad\times\bigg|\check{m}_\ell\bigg(\frac{u_1}{t},\frac{u_2}{t}\bigg)\bigg|^{p'}du_1\,du_2\bigg)^{2/p'}\frac{dt}{t^{5n+1-4n/p'}}\bigg)^{1/2}\\ &\leqslant \bigg(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\bigg)^{1/p}\bigg(\int_{\{t\in I:t\geqslant 1/8\}}t^{2s-4n/p-1}\,dt\bigg)^{1/2}\end{aligned}$$

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$$\leqslant \frac{(2^\ell)^{\varepsilon_1-s+2n/p}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}}\Big(\int_I t^{2s-4n/p-1}\,dt\Big)^{1/2}.$$

Repeating the same estimates above, we may obtain

$$A_\ell^{2,2}(I) \lesssim \frac{(2^\ell)^{\varepsilon_1-s+2n/p}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}}\Big(\int_I t^{2s-4n/p-1}\,dt\Big)^{1/2}.$$

On the other hand, similar to inequality (2.6), we have

$$\begin{aligned}
A_\ell^2(I) &\lesssim \Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{|z|\leqslant t}\int_{\{t\in I: t\geqslant 1/8\}}\frac{1}{t^{5n+1}} \\
&\quad\times\Big|\Big(\check{m}_\ell\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big) \\
&\quad-\check{m}_\ell\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big)\Big|^2 dz dt\Big)^{p'/2} dy_1\,dy_2\Big)^{1/p'} \\
&\leqslant \Big(\int_{|z|\leqslant t}\int_{\{t\in I: t\geqslant 1/8\}}\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})} \\
&\quad\times\Big|\check{\varphi}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^{p'} dy_1\,dy_2\Big)^{2/p'}\frac{dz\,dt}{t^{2s+5n+1}}\Big)^{1/2} \\
&= \Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)|^p\,d\xi\,d\eta\Big)^{2/p} \\
&\quad\times t^{2s-4n/p-1}\,dt\Big)^{1/2}.
\end{aligned}$$

**Estimate for $A_\ell^3(I)$.**

$$\begin{aligned}
A_\ell^3(I) &\leqslant \Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{3<|z|\leqslant t}\int_I\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big) \\
&\quad-\check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2} dy_1\,dy_2\Big)^{1/p'} \\
&\leqslant \Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{|z|\leqslant t} \\
&\quad\times\int_I\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2} dy_1\,dy_2\Big)^{1/p'} \\
&\quad+\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{|z|\leqslant t} \\
&\quad\times\int_I\Big|\check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2} dy_1\,dy_2\Big)^{1/p'} \\
&=: A_\ell^{3,1}(I)+A_\ell^{3,2}(I).
\end{aligned}$$

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Note that $z > 3$ and $1/2^2 \leqslant |y_1 + h| \leqslant 2$, then $|y_1 + h - z| > |z| - |y_1 + h| > 1$ and $|y_1 - z| > |z| - |y_1| > 2$. Similar to the estimate for $A_\ell^1(I)$, we get

$$A_\ell^3(I) \lesssim \Big(\int_I t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\partial_\xi^\alpha m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{1/p}$$

and

$$A_\ell^3(I) \lesssim \Big(\int_I \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\partial_\xi^\alpha\big[m_\ell(\xi,\eta) \times (e^{-2\pi i t^{-1} h\cdot(\xi+\eta)} - 1)\big]|^p\,d\xi\,d\eta\Big)^{2/p} t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}.$$

**Estimate for $A_\ell^4(I)$.** Note that $|y_1 + h - z| \sim 1$, $|y_1 - z| \sim 1$ and $\lambda > 2s/p + 1$, employ the Minkowski inequality and the Hausdorff–Young inequality, we may obtain

$$\begin{aligned}
A_\ell^4(I) &\leqslant \sum_{i=1}^{\infty}\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{2^{i-1}t\leqslant|z|\leqslant\min\{2^i t,1/8\}}\int_I\Big(\frac{t}{|z|+t}\Big)^{n\lambda} \\
&\quad\times\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big) \\
&\quad - \check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'} \\
&\leqslant \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2}\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{2^{i-1}t\leqslant|z|\leqslant\min\{2^i t,1/8\}} \\
&\quad\times\int_I\Big|\check{m}\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big) \\
&\quad - \check{m}\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'} \\
&\lesssim \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2}\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{2^{i-1}t\leqslant|z|\leqslant\min\{2^i t,1/8\}} \\
&\quad\times\int_I\Big|\check{m}\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'} \\
&\lesssim \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2}\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{2^{i-1}t\leqslant|z|\leqslant\min\{2^i t,1/8\}} \\
&\quad\times\int_I\Big|(y_1)^\alpha\check{m}\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}
\end{aligned}$$

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$$\leqslant \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2}\Big(\int_I \int_{2^{i-1}t\leqslant|z|\leqslant\min\{2^i t,1/8\}} \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}$$
$$\times |\partial_\xi^\alpha m_\ell(\xi,\eta)|^p \,d\xi\,d\eta\Big)^{2/p} t^{-5n-1+2|\alpha|+4n/p'}\,dzdt\Big)^{1/2}$$
$$\leqslant \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2-in/2}\Big(\int_I \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}$$
$$\times |\partial_\xi^\alpha m_\ell(\xi,\eta)|^p \,d\xi\,d\eta\Big)^{2/p} t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}$$
$$\lesssim \Big(\int_I t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\partial_\xi^\alpha m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{1/p}.$$

Recall that

$$\check{m}_\ell\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big)-\check{m}_\ell\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)=\check{\varphi}_\ell\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big).$$

Similarly,

$$A_\ell^4(I)$$
$$\lesssim \sum_{i=1}^{\infty}\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\int_{2^{i-1}t\leqslant|z|\leqslant\min\{2^i t,1/8\}}\int_I\Big(\frac{t}{|z|+t}\Big)^{n\lambda}\Big|(y_1-z)^\alpha$$
$$\times \check{\varphi}_\ell\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}\,dy_1\,dy_2\Big)^{1/p'}$$
$$\leqslant \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2-in/2}\Big(\int_I\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}$$
$$\times \Big|y_1^\alpha\check{\varphi}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^{p'}\,dy_1\,dy_2\Big)^{2/p'}\frac{dt}{t^{4n+1}}\Big)^{1/2}$$
$$\leqslant \Big(\int_I\Big(\int_{S_j(t^{-1}Q_{\bar{x}})}\int_{S_k(t^{-1}Q_{\bar{x}})}|y_1^\alpha\check{\varphi}_\ell(y_1,y_2)|^{p'}\,dy_1\,dy_2\Big)^{2/p'}$$
$$\times t^{2|\alpha|+4n/p'}\frac{dt}{t^{4n+1}}\Big)^{1/2}$$
$$\leqslant \Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha\varphi_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{2/p}t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}$$
$$=\Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha[m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)]|^p\,d\xi\,d\eta\Big)^{2/p}$$
$$\times t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}.$$

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**Estimate for $A_\ell^5(I)$.** Denote $F = \{2^{i-1}t, 1/8\} \leqslant |z| \leqslant \min\{2^i t, 3\}$, we get

$$\begin{aligned}
A_\ell^5(I) &\leqslant \sum_{i=1}^{\infty} \Big( \int_{S_j(Q_{\bar{x}})} \int_{S_k(Q_{\bar{x}})} \Big( \int_F \int_I \Big( \frac{t}{|z|+t} \Big)^{n\lambda} \\
&\quad \times \Big| \check{m}\Big( \frac{y_1+h-z}{t}, \frac{y_2+h-z}{t} \Big) \\
&\quad - \check{m}\Big( \frac{y_1-z}{t}, \frac{y_2-z}{t} \Big) \Big|^2 \frac{dz\,dt}{t^{5n+1}} \Big)^{p'/2} dy_1\,dy_2 \Big)^{1/p'} \\
&\leqslant \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2} \Big( \int_{S_j(Q_{\bar{x}})} \int_{S_k(Q_{\bar{x}})} \Big( \int_F \int_I \\
&\quad \times \Big| \check{m}\Big( \frac{y_1+h-z}{t}, \frac{y_2+h-z}{t} \Big) \Big|^2 \\
&\quad \times \frac{dz\,dt}{t^{5n+1}} \Big)^{p'/2} dy_1\,dy_2 \Big)^{1/p'} + \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2} \Big( \int_{S_j(Q_{\bar{x}})} \int_{S_k(Q_{\bar{x}})} \\
&\quad \times \Big( \int_F \int_I \Big| \check{m}\Big( \frac{y_1-z}{t}, \frac{y_2-z}{t} \Big) \Big|^2 \frac{dz\,dt}{t^{5n+1}} \Big)^{p'/2} dy_1\,dy_2 \Big)^{1/p'} \\
&=: A_\ell^{5,1}(I) + A_\ell^{5,2}(I).
\end{aligned}$$

We observe that if $\{2^{i-1}t, 1/8\} \leqslant |z| \leqslant \min\{2^i t, 3\}$, then $t \sim 2^{-i}$. By the Minkowski inequality and the Hausdorff–Young inequality, we have

$$\begin{aligned}
&A_\ell^{5,1}(I) \\
&\quad \leqslant \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2} \Big( \int_F \int_I \Big( \int_{S_j(Q_{\bar{x}})} \int_{S_k(Q_{\bar{x}})} \\
&\qquad \times \Big| \check{m}_\ell\Big( \frac{y_1+h-z}{t}, \frac{y_2+h-z}{t} \Big) \Big|^{p'} dy_1\,dy_2 \Big)^{2/p'} \frac{dt}{t^{5n+1}} \Big)^{1/2} \\
&\quad = \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2} \Big( \int_F \int_I \Big( \int_{S_j(t^{-1}Q_{\bar{x}})} \int_{S_k(t^{-1}Q_{\bar{x}})} \\
&\qquad \times \Big| \check{m}_\ell\Big( u_1, u_2 \Big) \Big|^{p'} du_1\,du_2 \Big)^{2/p'} \frac{dz\,dt}{t^{5n+1-4n/p'}} \Big)^{1/2} \\
&\quad \leqslant \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2+in/2+2s} \Big( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |m_\ell(\xi,\eta)|^p \, d\xi\,d\eta \Big)^{1/p} \\
&\qquad \times \Big( \int_I t^{2s-4n/p-1}\,dt \Big)^{1/2} \\
&\quad \leqslant \frac{(2^\ell)^{\varepsilon_1-s+2n/p}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}} \Big( \int_I t^{2s-4n/p-1}\,dt \Big)^{1/2}.
\end{aligned}$$

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Repeating the same estimates above, we may obtain

$$A_\ell^{5,2}(I)\lesssim\frac{(2^\ell)^{\varepsilon_1-s+2n/p}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}}\Big(\int_I t^{2s-4n/p-1}\,dt\Big)^{1/2}.$$

On the other hand, similar to inequality (2.6), we have

$$\begin{aligned}
&A_\ell^5(I)\\
&\lesssim\sum_{i=1}^\infty 2^{-(i-1)n\lambda/2}\Big(\int_{S_j(Q_{\bar x})}\int_{S_k(Q_{\bar x})}\Big(\int_F\int_I\Big|\Big(\check m_\ell\Big(\frac{y_1+h-z}{t},\frac{y_2+h-z}{t}\Big)\\
&\quad-\check m_\ell\Big(\frac{y_1-z}{t},\frac{y_2-z}{t}\Big)\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}\\
&\leqslant\sum_{i=1}^\infty 2^{-(i-1)n\lambda/2}\Big(\int_F\int_I\Big(\int_{S_j(Q_{\bar x})}\int_{S_k(Q_{\bar x})}\\
&\quad\times\Big|\check\varphi_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^{p'}dy_1\,dy_2\Big)^{2/p'}\frac{dz\,dt}{t^{2s+5n+1}}\Big)^{1/2}\\
&\leqslant\sum_{i=1}^\infty 2^{-(i-1)n\lambda/2+in/2+2s}\Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\\
&\quad\times|m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)|^p\,d\xi\,d\eta\Big)^{2/p}t^{2s-4n/p-1}\,dt\Big)^{1/2}\\
&\leqslant\Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)|^p\,d\xi\,d\eta\Big)^{2/p}\\
&\quad\times t^{2s-4n/p-1}\,dt\Big)^{1/2}.
\end{aligned}$$

Finally, we consider for $A_\ell^6(I)$.

**Estimate for** $A_\ell^6(I)$. Since $|z|>3$, then $|y_1+h-z|>1$, $|y_1-z|>2$. Repeating the similar estimate for $A_\ell^4(I)$, the Minkowski inequality and the Hausdorff–Young inequality yield

$$A_\ell^6(I)\lesssim\Big(\int_I t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{1/p}$$

and

$$\begin{aligned}
A_\ell^6(I)\lesssim&\Big(\int_I\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha[m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)]|^p\,d\xi\,d\eta\Big)^{2/p}\\
&\times t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}.
\end{aligned}$$

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Combining all estimates of these six terms, it yields that

$$\begin{aligned} A_\ell(I) &\lesssim \Big(\int_I t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{1/p} \\ &\quad + \Big(\int_I t^{2s-4n/p-1}\,dt\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\Big)^{1/p} \\ &\leqslant \frac{(2^\ell)^{\varepsilon_1-s+2n/p}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}}\Big(\int_I t^{2s-4n/p-1}\,dt\Big)^{1/2} \\ &\quad + \frac{(2^\ell)^{\varepsilon_1-|\alpha|+2n/p}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}}\Big(\int_I t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2} \end{aligned}$$

and

$$\begin{aligned} A_\ell(I) &\lesssim \Big(\int_I \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha[m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)]|^p\,d\xi\,d\eta\Big)^{2/p} \\ &\quad \times t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2} \\ &\quad + \Big(\int_I \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)|^p\,d\xi\,d\eta\Big)^{2/p} \\ &\quad \times t^{2s-4n/p-1}\,dt\Big)^{1/2}. \end{aligned}$$

By the following fact

$$\begin{aligned} &|\partial_\xi^\alpha[m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)]| \\ &\quad \lesssim \frac{2^\ell|h|}{t}\frac{(2^\ell)^{\varepsilon_1-|\alpha|}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}} + \sum_{\beta=1}^{|\alpha|}\Big(\frac{|h|}{t}\Big)^\beta\frac{(2^\ell)^{\varepsilon_1-|\alpha|+\beta}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}} \end{aligned}$$

and

$$|m_\ell(\xi,\eta)(e^{-2\pi i t^{-1}h\cdot(\xi+\eta)}-1)| \lesssim \frac{2^\ell|h|}{t}\frac{(2^\ell)^{\varepsilon_1-s}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}},$$

it follows that

$$\begin{aligned} A_\ell(I) &\lesssim \Big(\int_I \Big(\frac{2^\ell|h|}{t}\frac{(2^\ell)^{\varepsilon_1-|\alpha|}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}} + \sum_{\beta=1}^{|\alpha|}\Big(\frac{|h|}{t}\Big)^\beta\frac{(2^\ell)^{\varepsilon_1-|\alpha|+\beta}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}}\Big)^2 \\ &\quad \times 2^{4n\ell/p}t^{2|\alpha|-4n/p-1}\,dt\Big)^{1/2} \end{aligned} \tag{2.7}$$

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$$\lesssim \sum_{\beta=0}^{|\alpha|} |h|^{\max(\beta,1)} \frac{2^{\ell(-|\alpha|+2n/p+\max(\beta,1)+\varepsilon_1)}}{(1+2^\ell)^{\varepsilon_1+\varepsilon_2}}$$

$$\times \left(\int_I t^{2(|\alpha|-2n/p-\max(\beta,1))-1}\, dt\right)^{1/2}. \tag{2.8}$$

Now, we fix sufficiently small $\varepsilon > 0$ so that $\varepsilon(s-2n/p) < \min\{\varepsilon_1, \varepsilon_2\}$. Then, if $2^\ell|h| \geqslant 1$, noting $2n/p < s < 2n/p + 1$ and using (2.5) for $I = (0, (2^\ell|h|)^{1+\varepsilon}]$, we have

$$A_\ell((0, (2^\ell|h|)^{1+\varepsilon}]) \lesssim 2^{-\ell(s+\varepsilon_2-2n/p)}(2^\ell|h|)^{(1+\varepsilon)(s-2n/p)} = |h|^{(1+\varepsilon)(s-2n/p)} 2^{\ell(\varepsilon(s-2n/p)-\varepsilon_2)}.$$

By (2.8) for $I = [(2^\ell|h|)^{1+\varepsilon}, \infty)$, we have

$$A_\ell([(2^\ell|h|)^{1+\varepsilon}, \infty)) \lesssim \sum_{\beta=0}^{|\alpha|} |h|^{\max(\beta,1)} 2^{\ell(-|\alpha|+2n/p+\max(\beta,1))}(2^\ell|h|)^{(1+\varepsilon)(s-2n/p-\max(\beta,1))}$$

$$= \sum_{\beta=0}^{|\alpha|} |h|^{-\varepsilon\max(\beta,1)+(1+\varepsilon)(s-2n/p)} 2^{\ell\varepsilon((s-2n/p)-\max(\beta,1))}.$$

Thus, noting $\varepsilon(s-2n/p) - \varepsilon_2 < 0$ and $|h| < 1$, we obtain

$$\sum_{2^\ell|h|\geqslant 1} A_\ell \lesssim \sum_{2^\ell|h|\geqslant 1} |h|^{(1+\varepsilon)(s-2n/p)} 2^{\ell(\varepsilon(s-2n/p)-\varepsilon_2)}$$

$$+ \sum_{2^\ell|h|\geqslant 1} \sum_{\beta=0}^{|\alpha|} |h|^{-\varepsilon\max(\beta,1)+(1+\varepsilon)(s-2n/p)} 2^{\ell\varepsilon((s-2n/p)-\max(\beta,1))}$$

$$\leqslant |h|^{s-2n/p+\varepsilon_2} + \sum_{\beta=0}^{|\alpha|} |h|^{s-2n/p} \lesssim |h|^{s-2n/p}. \tag{2.9}$$

In the case $2^\ell|h| < 1$, using (2.5) for $I = (0, (2^\ell|h|)^{1-\varepsilon}]$, we have

$$A_\ell((0, (2^\ell|h|)^{1-\varepsilon}]) \lesssim 2^{\ell(-s+2n/p+\varepsilon_1)}(2^\ell|h|)^{(1-\varepsilon)(s-2n/p)} = |h|^{(1-\varepsilon)(s-2n/p)} 2^{\ell(-\varepsilon(s-2n/p)+\varepsilon_1)}.$$

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Furthermore, by using (2.8) for $I = [(2^\ell|h|)^{1-\varepsilon}, \infty)$, we have

$$\begin{aligned}
&A_\ell([(2^\ell|h|)^{1-\varepsilon}, \infty)) \\
&\lesssim \sum_{\beta=0}^{|\alpha|} |h|^{\max(\beta,1)} 2^{\ell(-s+2n/p+\max(\beta,1))} (2^\ell|h|)^{(1-\varepsilon)(s-2n/p-\max(\beta,1))} \\
&= \sum_{\beta=0}^{|\alpha|} |h|^{\varepsilon \max(\beta,1)+(1-\varepsilon)(s-2n/p)} 2^{-\varepsilon\ell(s-2n/p-\max(\beta,1))}.
\end{aligned}$$

By the fact that $\varepsilon(s-2n/p)-\varepsilon_1 < 0$ and $|h| < 1$, we obtain

$$\begin{aligned}
\sum_{2^\ell|h|<1} A_\ell &\lesssim \sum_{2^\ell|h|<1} |h|^{(1-\varepsilon)(s-2n/p)} 2^{\ell(-\varepsilon(s-2n/p)+\varepsilon_1)} \\
&\quad + \sum_{2^\ell|h|<1} \sum_{\beta=0}^{|\alpha|} |h|^{\varepsilon \max(\beta,1)+(1-\varepsilon)(s-2n/p)} 2^{-\varepsilon\ell(s-2n/p-\max(\beta,1))} \\
&\leqslant |h|^{s-2n/p-\varepsilon_1} + \sum_{\beta=0}^{|\alpha|} |h|^{s-2n/p} \lesssim |h|^{s-2n/p-\varepsilon_1} + |h|^{s-2n/p}.
\end{aligned} \tag{2.10}$$

Noting that $0 < \varepsilon_1 < s - 2n/p$ and taking $\delta = (s-\varepsilon_1)/2$, by (2.9) and (2.10), it holds that

$$\begin{aligned}
&\Big(\int_{S_j(Q_{\bar{x}})} \int_{S_k(Q_{\bar{x}})} \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{|z|+t}\Big)^{n\lambda} \Big|\check{m}\Big(\frac{y_1+h}{t}, \frac{y_2+h}{t}\Big) \\
&\qquad - \check{m}\Big(\frac{y_1}{t}, \frac{y_2}{t}\Big)\Big|^2 \frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2} dy\,dz\Big)^{1/p'} \\
&\qquad \leqslant \sum_{\ell\in\mathbb{Z}} A_\ell \lesssim |h|^{2(\delta-n/p)}.
\end{aligned}$$

This leads to the conclusion of Proposition 2.1 in the case $2n/p < s < 2n/p + 1$.

**(b) The case** $2n/p < s = 2n/p + 1$. First, we choose $1 < p_0 < p$ such that $2n/p_0 < s$. Then $p_0$ satisfies $2n/p_0 < s = 2n/p + 1 < 2n/p_0 + 1$. Hence, for

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all balls $Q$, all $x, \bar{x} \in \frac{1}{2}Q$ and $(j,k) \neq (0,0)$, by step (a), we have

$$\begin{aligned}
&\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|z|+t}\Big)^{n\lambda}\Big|\check{m}\Big(\frac{y_1+h}{t},\frac{y_2+h}{t}\Big)-\check{m}\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2 \\
&\quad\times\frac{dz\,dt}{t^{5n+1}}\Big)^{p_0'/2}\,dy\,dz\Big)^{1/p_0'} \\
&\leqslant C\frac{|h|^{2\delta-2n/p_0}}{|Q|^{2\delta/n}}2^{-2\delta\max(j,k)}.
\end{aligned}$$

By the Hölder inequality, it yields that

$$\begin{aligned}
&\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|z|+t}\Big)^{n\lambda}\Big|\check{m}\Big(\frac{y_1+h}{t},\frac{y_2+h}{t}\Big)-\check{m}\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2 \\
&\quad\times\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}\,dy\,dz\Big)^{1/p'} \\
&\leqslant (2^{n(j+k)}|Q|^2)^{(1/p_0)-(1/p)}\Big(\int_{S_j(Q_{\bar{x}})}\int_{S_k(Q_{\bar{x}})}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|z|+t}\Big)^{n\lambda} \\
&\quad\times\Big|\check{m}\Big(\frac{y_1+h}{t},\frac{y_2+h}{t}\Big)-\check{m}\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}\,dy\,dz\Big)^{1/p'} \\
&\lesssim (2^{2n\max(j,k)}|Q|^2)^{(1/p_0)-(1/p)}\frac{|h|^{2\delta-2n/p_0}}{|Q|^{2\delta/n}}\frac{1}{2^{2\delta\max(j,k)}} \\
&= \frac{|h|^{(2\delta-2n/p_0+2n/p)-2n/p}}{|Q|^{(2\delta-2n/p_0+2n/p)/n}}2^{-(2\delta-2n/p_0+2n/p)\max(j,k)}.
\end{aligned}$$

Therefore, taking $\delta - n/p_0 + n/p > n/p$ as $\delta$ newly, we obtain the desired estimate.

**(c) The case** $2n/p+1 < s \leqslant 2n$. In this case there is an integer $l$ such that $2n/p+l < s \leqslant 2n/p+1+l$. Then it follows that $2n/p < s-l \leqslant 2n/p+1$. Thus, regarding $s-l$ as $s$, we may deduce this case to the previous case (a) or case (b). This completes the proof of Proposition 2.1.

PROPOSITION 2.2. *Let $s \in \mathbb{N}$ with $n+1 \leqslant s \leqslant 2n$. Let $m \in L^\infty((\mathbb{R}^n)^2)$ and satisfy (1.1) and (1.2). Then, for $2n/s < p \leqslant 2$, $\lambda > 2s/n+1$, there exists a constant $C > 0$, such that the following inequality holds for all balls*

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*$Q$ with center at $x$ and $(j,k)\neq(0,0)$.*

$$\Big(\int_{S_j(Q)}\int_{S_k(Q)}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|x-z|+t}\Big)^{n\lambda} \times\Big|\check{m}\Big(\frac{x-y_1}{t},\frac{x-y_2}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'} \leqslant C\frac{1}{|Q|^{2/p}}2^{-2n\max(j,k)/p}. \tag{2.11}$$

*Proof.* Let $Q=B(x,R)$, $u=ax$ $(a>0)$ and $s=at$, we have

$$\begin{aligned} &B_{j,k}(m,Q)(x):\\ &=\Big(\int_{S_j(Q)}\int_{S_k(Q)}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|x-z|+t}\Big)^{n\lambda}\Big|\check{m}\Big(\frac{x-y_1}{t},\frac{x-y_2}{t}\Big)\Big|^2\\ &\quad\times\frac{dz\,dt}{t^{4n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'}\\ &=a^{2n/p}\Big(\int_{S_j(Q^a)}\int_{S_k(Q^a)}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{s}{|x^a-v|+s}\Big)^{n\lambda}\\ &\quad\times\Big|\check{m}\Big(\frac{x^a-u_1}{t},\frac{x^a-u_2}{s}\Big)\Big|^2\frac{dvds}{s^{5n+1}}\Big)^{p'/2}du_1\,du_2\Big)^{1/p'}\\ &=a^{2n/p}B_{j,k}(m,Q^a)(x^a), \end{aligned}$$

where $Q^a=B(ax,aR)$, $x^a=ax$. So, taking $a=1/(2^{\max(j,k)}R)$, the estimate $B_{j,k}(m,Q^a)(x^a)\lesssim 1$ implies the desired estimate. Thus, we only need to show (2.11) in the case $R=1/2^{\max(j,k)}$. We may also assume $k\geqslant j$ and hence $k\geqslant 1$. Then, for $Q=B(x,2^{-k})$, it is sufficient to show that

$$B_{j,k}(m,Q)(x)\lesssim 1.$$

By changing variables, it is enough to show that

$$\Big(\int_{S_j(Q_x)}\int_{S_k(Q_x)}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|x-z|+t}\Big)^{n\lambda} \times\Big|\check{m}\Big(\frac{y}{t},\frac{z}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\Big)^{p'/2}dy_1\,dy_2\Big)^{1/p'} \leqslant C\frac{1}{|Q|^{2/p}}2^{-2n\max(j,k)/p}, \tag{2.12}$$

where $Q_x=Q-x$.

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For every interval $I$ in $\mathbb{R}_+$, let

$$\begin{aligned}
&B_{j,k}(m_\ell, Q, I)(x)\\
&\quad = B^1_{j,k}(m_\ell, Q, I)(x) + B^2_{j,k}(m_\ell, Q, I)(x)\\
&\quad := \bigg(\int_{S_j(Q_x)}\int_{S_k(Q_x)}\bigg(\int_{\mathbb{R}^n}\int_I\Big(\frac{t}{|x-z|+t}\Big)^{n\lambda}\\
&\qquad \times \Big|\check{m}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}\\
&\quad = \bigg(\int_{S_j(Q_x)}\int_{S_k(Q_x)}\bigg(\int_{|x-z|<t}\int_I\Big|\check{m}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}\\
&\qquad + \sum_{i=1}^{\infty}2^{-(i-1)n\lambda}\bigg(\int_{S_j(Q_x)}\int_{S_k(Q_x)}\bigg(\int_{2^{i-1}t\leqslant|x-z|<2^it}\int_I\Big|\check{m}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\\
&\qquad \times\frac{dz\,dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}.
\end{aligned}$$

Note that $y_1 \sim 1$. The Minkowski inequality, together with the Hausdorff–Young inequality implies that

$$\begin{aligned}
&B^1_{j,k}(m_\ell, Q, I)(x)\\
&\quad \lesssim (2^kR)^{-|\alpha|}\bigg(\int_{S_j(Q_x)}\int_{S_k(Q_x)}\bigg(\int_{|x-z|<t}\int_I\\
&\qquad \times\Big|y_1^\alpha\check{m}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^2\frac{dt}{t^{5n+1}}\bigg)^{p'/2}dy_1\,dy_2\bigg)^{1/p'}\\
&\quad \lesssim (2^kR)^{-|\alpha|}\bigg(\int_I\bigg(\int_{S_j(Q_x)}\int_{S_k(Q_x)}\\
&\qquad \times\Big|y_1^\alpha\check{m}_\ell\Big(\frac{y_1}{t},\frac{y_2}{t}\Big)\Big|^{p'}dy_1\,dy_2\bigg)^{2/p'}\frac{dt}{t^{4n+1}}\bigg)^{1/2}\\
&\quad = C(2^kR)^{-|\alpha|}\bigg(\int_I\bigg(\int_{S_j(t^{-1}Q_x)}\int_{S_k(t^{-1}Q_x)}|y_1^\alpha\check{m}_\ell(y_1,y_2)|^{p'}\,dy_1\,dy_2\bigg)^{2/p'}\\
&\qquad \times t^{2|\alpha|+4n/p'}\frac{dt}{t^{4n+1}}\bigg)^{1/2}\\
&\quad \lesssim (2^kR)^{-|\alpha|}\bigg(\int_I\bigg(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\bigg)^{2/p}t^{2|\alpha|-4n/p-1}\,dt\bigg)^{1/2}\\
&\quad \lesssim (2^kR)^{-|\alpha|}\bigg(\int_I t^{2|\alpha|-4n/p-1}\,dt\bigg)^{1/2}\bigg(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\partial_\xi^\alpha m_\ell(\xi,\eta)|^p\,d\xi\,d\eta\bigg)^{1/p}.
\end{aligned}$$

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Similarly,

$$
\begin{aligned}
&B^2_{j,k}(m_\ell, Q, I)(x) \\
&\lesssim (2^k R)^{-|\alpha|} \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2} \Big( \int_{S_j(Q_x)} \int_{S_k(Q_x)} \Big( \int_{2^{i-1}t \leqslant |x-z| < 2^i t} \int_I \\
&\quad \times \Big| y_1^\alpha \check{m}_\ell\Big(\frac{y_1}{t}, \frac{y_2}{t}\Big)\Big|^2 \frac{dz\, dt}{t^{5n+1}} \Big)^{p'/2} dy_1\, dy_2 \Big)^{1/p'} \\
&\lesssim (2^k R)^{-|\alpha|} \sum_{i=1}^{\infty} 2^{-(i-1)n\lambda/2 + in/2} \Big( \int_{S_j(Q_x)} \int_{S_k(Q_x)} \Big( \int_I \Big| y_1^\alpha \check{m}_\ell\Big(\frac{y_1}{t}, \frac{y_2}{t}\Big)\Big|^2 \\
&\quad \times \frac{dt}{t^{4n+1}} \Big)^{p'/2} dy_1\, dy_2 \Big)^{1/p'} \\
&\lesssim (2^k R)^{-|\alpha|} \Big( \int_I t^{2|\alpha| - 4n/p - 1}\, dt \Big)^{1/2} \Big( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\partial_\xi^\alpha m_\ell(\xi, \eta)|^p \, d\xi\, d\eta \Big)^{1/p}.
\end{aligned}
$$

Next, we consider two cases according to the value of $\ell$.

**Case (a).** $\ell < 0$. In this case, taking $|\alpha| = 0$ and $I = [2^{\ell(1+\varepsilon)}, \infty)$, the estimate in (1.2) implies that

$$
B_{j,k}(m_\ell, Q, [2^{\ell(1+\varepsilon)}, \infty)) \lesssim 2^{\ell(1+\varepsilon)(-2n/p)} 2^{\ell\varepsilon_1} 2^{\ell(2n/p)} = 2^{\ell(\varepsilon_1 - 2\varepsilon n/p)}.
$$

In virtue of $2^k R = 1$, taking $|\alpha| = s$ and $I = [0, 2^{\ell(1+\varepsilon)}]$, the estimate in (1.1) implies that

$$
B_{j,k}(m_\ell, Q, [0, 2^{\ell(1+\varepsilon)}]) \lesssim 2^{\ell(1+\varepsilon)(s-2n/p)} 2^{-\ell(s-2n/p)} = 2^{\ell\varepsilon(s-2n/p)}.
$$

Hence,

$$
B_{j,k}(m_\ell, Q, [0, \infty)) \lesssim 2^{\ell(\varepsilon_1 - 2\varepsilon n/p)} + 2^{\ell\varepsilon(s-2n/p)}.
$$

**Case (b).** $\ell \geqslant 0$. By repeating the same arguments as in case (a), we get

$$
B_{j,k}(m_\ell, Q, [2^{\ell(1-\varepsilon)}, \infty)) \lesssim 2^{\ell(1-\varepsilon)(-2n/p)} 2^{-\ell\varepsilon_2} 2^{\ell(2n/p)} = 2^{\ell(2\varepsilon n/p - \varepsilon_2)}
$$

and

$$
B_{j,k}(m_\ell, Q, [0, 2^{\ell(1-\varepsilon)}]) \lesssim 2^{\ell(1-\varepsilon)(s-2n/p)} 2^{-\ell(s-2n/p)} = 2^{-\ell\varepsilon(s-2n/p)}.
$$

Therefore,

$$
B_{j,k}(m_\ell, Q, [0, \infty)) \lesssim 2^{\ell(2\varepsilon n/p - \varepsilon_2)} + 2^{-\ell\varepsilon(s-2n/p)}.
$$

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Choosing $\varepsilon > 0$ so that $2n\varepsilon/p < \min(\varepsilon_1, \varepsilon_2)$, we obtain from case (a) and case (b)

$$\begin{aligned} B_{j,k}(m, Q)(x) &\leqslant \sum_{\ell<0} B_{j,k}(m_\ell, Q, [0,\infty)) + \sum_{\ell\geqslant 0} B_{j,k}(m_\ell, Q, [0,\infty)) \\ &\lesssim \sum_{\ell<0}[2^{\ell(\varepsilon_1-2\varepsilon n/p)} + 2^{\ell\varepsilon(s-2n/p)}] \\ &\quad + \sum_{\ell\geqslant 0}[2^{\ell(2\varepsilon n/p-\varepsilon_2)} + 2^{-\ell\varepsilon(s-2n/p)}] \lesssim 1. \end{aligned}$$

This completes the proof of Proposition 2.2.

### 2.2 Related multilinear square function

In order to finish our proof, we need to introduce some definitions and necessary Lemmas.

Definition 2.1. (multilinear square function $T_\lambda$) Let $K$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ and $K_t = t^{-mn}K(\cdot/t)$. Then, the multilinear square function $T_\lambda$ is defined as follows

(2.13)

$$T_\lambda(\vec{f})(x) = \Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|x-z|+t}\Big)^{n\lambda}\Big|\int_{\mathbb{R}^{mn}} K_t(z,\vec{y})\prod_{j=1}^m f_j(y_j)\,d\vec{y}\Big|^2\frac{dz\,dt}{t^{n+1}}\Big)^{1/2},$$

where $\vec{f} = (f_1, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)$ and all $x \notin \bigcap_{j=1}^m \mathtt{supp} f_j$.

For $x \in \mathbb{R}^n$, $r, a > 0$, we set $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$ and $aB(x,r) = \{y \in \mathbb{R}^n : |y-x| < ar\}$. For all balls $Q$, let $S_j(Q) = 2^jQ \setminus 2^{j-1}Q$ if $j \geqslant 1$, and $S_0(Q) = Q$.

Definition 2.2. (kernel condition) Let $1 \leqslant p_0 < \infty$. Then, assume that

(H1) For all $p_0 \leqslant q_1, q_2, \ldots, q_m < \infty$ and $0 < q < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$, $T$ maps $L^{q_1}\times\cdots\times L^{q_m}$ into $L^{q,\infty}$.

(H2) There exists $\delta > n/p_0$ so that for the conjugate exponent $p_0'$ of $p_0$, one has

$$\Big(\int_{S_{j_m}(Q)}\cdots\int_{S_{j_1}(Q)}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|z|+t}\Big)^{n\lambda} \times \big|K_t(x-z,\vec{y}) - K_t(x'-z,\vec{y})\big|^2\frac{dz\,dt}{t^{n+1}}\Big)^{p_0'/2}d\vec{y}\Big)^{1/p_0'}$$

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$$\leqslant C\frac{|x-x'|^{m(\delta-n/p_0)}}{|Q|^{m\delta/n}}2^{-m\delta j_0}$$

for all balls $Q$, all $x,z\in(1/2)Q$ and $(j_1,\ldots,j_m)\neq(0,\ldots,0)$, where $j_0=\max_{k=1,\ldots,m}\{j_k\}$.

(H3) There exists some positive constant $C>0$ such that

$$\begin{aligned}\Big(\int_{S_{j_m}(Q)}\cdots\int_{S_{j_1}(Q)}\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|x-z|+t}\Big)^{n\lambda}\\ \times\big|K_t(x,\vec{y})\big|^2\frac{dz\,dt}{t^{n+1}}\Big)^{p_0'/2}d\vec{y}\Big)^{1/p_0'}\\ \leqslant C\frac{2^{-mnj_0/p_0}}{|Q|^{m/p_0}}\end{aligned}$$

for all balls $Q$ with center at $x$ and $(j_1,\ldots,j_m)\neq(0,\ldots,0)$, where $j_0=\max_{k=1,\ldots,m}\{j_k\}$.

DEFINITION 2.3. (Commutators of multilinear square operator) The commutators of multilinear square operator $T_\lambda$ with BMO functions $\vec{b}=(b_1,b_2,\ldots,b_m)$ are defined by

$$\begin{aligned}T_{\lambda,\vec{b}}(\vec{f})(x)&=\Big(\iint_{\mathbb{R}^{n+1}_+}\Big(\frac{t}{|x-z|+t}\Big)^{n\lambda}\\ &\quad\times\Big|\int_{\mathbb{R}^{mn}}(b_i(x)-b_i(y_i))K_t(z,\vec{y})\prod_{j=1}^m f_j(y_j)\,d\vec{y}\Big|^2\frac{dz\,dt}{t^{n+1}}\Big)^{1/2},\end{aligned}\tag{2.14}$$

for any $\vec{f}=(f_1,\ldots,f_m)\in\mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)$ and all $x\notin\bigcap_{j=1}^m\mathtt{supp}f_j$.

We may obtain the following weighted estimates.

LEMMA 2.3. *Let $T_\lambda$ be the multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some $1\leqslant p_0<\infty$. Then, for any $p_0\leqslant p_1,\ldots,p_m<\infty$, $1/p=1/p_1+\cdots+1/p_m$ and $\vec{\omega}\in A_{\vec{P}/p_0}$, the following weighted estimates hold.*

(i) *If there is no $p_i=p_0$, then $\|T_\lambda(\vec{f})\|_{L^p(\nu_{\vec{w}})}\leqslant C\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}$.*
(ii) *If there is a $p_i=p_0$, then $\|T_\lambda(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})}\leqslant C\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}$.*

As for the commutators of $T$, we obtain the following weighted estimates.

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LEMMA 2.4. *Let* $T_\lambda$ *be the multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some* $1 \leqslant p_0 < \infty$. *Let* $\vec{b} \in BMO^m$. *Then, for any* $p_0 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ *and* $\vec{\omega} \in A_{\vec{P}/p_0}$, *we have*

$$||T_{\lambda,\vec{b}}\vec{f}||_{L^p(\nu_{\vec{\omega}})} \leqslant C||\vec{b}||_{BMO} \prod_{i=1}^{m} ||f_i||_{L^{p_i}(\omega_i)},$$

*where* $||\vec{b}||_{BMO} = \max_j ||b_j||_{BMO}$.

LEMMA 2.5. *Let* $T_\lambda$ *be the multilinear square function with a kernel satisfying conditions (H1), (H2) and (H3) for some* $1 \leqslant p_0 < \infty$. *Let* $\vec{b} \in BMO^m$. *Let* $\vec{\omega} \in A_{(1,\ldots,1)}$ *and* $\vec{b} \in BMO^m$. *Then, there exists a constant* $C$ *(depending on* $\vec{b}$*) such that*

$$\nu_{\vec{\omega}}\big(\big\{x \in \mathbb{R}^n : |T_{\lambda,\vec{b}}\vec{f}(x)| > t^m\big\}\big) \leqslant C \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^n} \Phi\Big(\frac{|f_j(x)|}{t}\Big)\omega_j(x)\,dx\Big)^{1/m},$$

*where* $\Phi(t) = t^{p_0}(1 + \log^+ t)^{p_0}$.

REMARK 2.4. The proofs of Lemmas 2.3–2.5 are almost the same as the [28, proofs of Theorems 1.3–1.5] with few modifications, so we omit them here.

With Propositions 2.1–2.2 and Lemmas 2.3–2.5 in hand, the proofs of Theorems 1.1 and 1.2 will be quite direct.

### 2.3 Proofs of Theorems 1.1 and 1.2.

*Proof.* (a) The case $p_0 > 2n/s$. By Proposition 2.1 and Proposition 2.2, it is easy to see that the associated kernel of $\mathfrak{T}_{\lambda,m}$ satisfies the conditions (H2) and (H3). Since we have supposed (H1) from the beginning, applying Lemmas 2.3–2.5, we obtain the desired conclusions in Theorems 1.1 and 1.2.

(b) The case $p_0 = 2n/s$. By the property of $A_p$ weights, there exists a real number $\tilde{p}_0$ satisfying $p_0 = 2n/s < \tilde{p}_0 < \min(p_1, p_2, 2)$ and $\vec{\omega} \in A_{\vec{p}/\tilde{p}_0}$ (see [1] or [19]). Therefore, by step (a), we finish the proofs of Theorems 1.1 and 1.2.

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## §3. An example

In this section, an example will be given to show that there are some multilinear Fourier multiplier operators $\mathfrak{T}_{\lambda,m}$ which are bounded from $L^{q_1} \times L^{q_2}$ to $L^q$. Thus, the assumption that $\mathfrak{T}_{\lambda,m}$ is bounded from $L^{q_1} \times L^{q_2}$ into $L^{q,\infty}$ in Theorems 1.1–1.2 is reasonable.

Denote $h_t(x,u) = (1 + |x-u|/t)^{-n\lambda}$. The bilinear Fourier multiplier operators $\mathfrak{T}_{\lambda,m}$ can be written as

$$\mathfrak{T}_{\lambda,m}(f_1,f_2)(x) = \bigg(\iint_{\mathbb{R}^{n+1}_+}\bigg|\int_{(\mathbb{R}^n)^2} e^{2\pi i x\cdot(\xi_1+\xi_2)} \times m(t\xi_1,t\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,d\xi_1\,d\xi_2\bigg|^2 h_t(x,u)\frac{du\,dt}{t^{n+1}}\bigg)^{1/2}.$$

First, we consider its bilinearization in the following form:

$$\begin{aligned}
&\widetilde{\mathcal{T}}_{\lambda,m}(\vec{f})(x)\\
&= \iint_{\mathbb{R}^{n+1}_+}\int_{(\mathbb{R}^n)^4} e^{2\pi i x\cdot(\xi_1+\xi_2+\xi_3+\xi_4)} m(t\xi_1,t\xi_2)m(t\xi_3,t\xi_4)\\
&\quad \times \prod_{i=1}^{4}\hat{f}_i(\xi_i)\,d\xi_i\,h_t(x,u)\frac{du\,dt}{t^{n+1}}\\
&= \int_{(\mathbb{R}^n)^4}\iint_{\mathbb{R}^{n+1}_+} e^{2\pi i x\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}\\
&\quad \times m(t\xi_1,t\xi_2)m(t\xi_3,t\xi_4)\left(1+\frac{|x-u|}{t}\right)^{-n\lambda}\frac{du\,dt}{t^{n+1}}\prod_{i=1}^{4}\hat{f}_i(\xi_i)\,d\xi_i\\
&= \int_{(\mathbb{R}^n)^4}\bigg(\iint_{\mathbb{R}^{n+1}_+} e^{-2\pi i t u\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}\\
&\quad \times m(t\xi_1,t\xi_2)m(t\xi_3,t\xi_4)\,(1+|u|)^{-n\lambda}\frac{du\,dt}{t^{n+1}}\bigg)\\
&\quad \times e^{2\pi i x\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}\prod_{i=1}^{4}\hat{f}_i(\xi_i)\,d\xi_i\\
&= \int_{(\mathbb{R}^n)^4} e^{2\pi i x\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}\tilde{m}(\xi_1,\xi_2,\xi_3,\xi_4)\prod_{i=1}^{4}\hat{f}_i(\xi_i)\,d\xi_i
\end{aligned}$$

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where

$$\tilde{m}(\xi_1, \xi_2, \xi_3, \xi_4) = \iint_{\mathbb{R}_+^{n+1}} e^{-2\pi i t u\cdot(\xi_1+\xi_2+\xi_3+\xi_4)} m(t\xi_1, t\xi_2) m(t\xi_3, t\xi_4)\,(1+|u|)^{-n\lambda} \frac{du\,dt}{t^{n+1}}.$$

EXAMPLE 3.1. Suppose that $m(0,0)=0$ and there exists some $\varepsilon > 0$ such that

$$|\partial^\alpha m(\xi_1, \xi_2)| \leqslant (1+|\xi_1|+|\xi_2|)^{-2n-1-\varepsilon}, \quad \text{for all } |\alpha| \leqslant 2n+1. \tag{3.1}$$

Then, there exists a constant $\delta$, with $0 < \delta \leqslant 1$, such that

(i) $\widetilde{\mathcal{T}}_{\lambda,m}$ is bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n) \times L^{q_3}(\mathbb{R}^n) \times L^{q_4}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $2-\delta < q_1, q_2, q_3, q_4 < \infty$ with $1/q = 1/q_1 + 1/q_2 + 1/q_3 + 1/q_4$.

(ii) $\mathfrak{T}_{\lambda,m}$ is bounded from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $2-\delta < q_1, q_2 < \infty$ with $1/q = 1/q_1 + 1/q_2$.

*Proof.* (i) The assumption $m(0,0)=0$ and the mean-value theorem gives that $|m(\xi_1,\xi_2)| \leqslant |\xi_1|+|\xi_2|$. This together with (3.1) implies that $|m(\xi_1,\xi_2)| \leqslant (|\xi_1|+|\xi_2|)^{1/4}/(1+|\xi_1|+|\xi_2|)^{3/4}$ for $\xi_1, \xi_2 \in \mathbb{R}^n$. Note that $\lambda > 1$, we have

$$\begin{aligned}
&|\tilde{m}(\xi_1, \xi_2, \xi_3, \xi_4)| \\
&\quad = \left| \iint_{\mathbb{R}_+^{n+1}} e^{-2\pi i t u\cdot(\xi_1+\xi_2+\xi_3+\xi_4)} m(t\xi_1, t\xi_2) m(t\xi_3, t\xi_4)\,(1+|u|)^{-n\lambda} \frac{du\,dt}{t^{n+1}} \right| \\
&\quad \leqslant \int_{\mathbb{R}^n} (1+|u|)^{-n\lambda}\,du \int_0^\infty \frac{(|t\xi_1|+|t\xi_2|)^{1/4}}{(1+|t\xi_1|+|t\xi_2|)^{3/4}} \frac{(|t\xi_3|+|t\xi_4|)^{1/4}}{(1+|t\xi_3|+|t\xi_4|)^{3/4}} \frac{dt}{t} \\
&\quad \lesssim \int_0^\infty \frac{(t(|\xi_1|+|\xi_2|))(t(|\xi_3|+|\xi_4|))^{1/4}}{(1+t(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|))^{3/4}} \frac{dt}{t} \\
&\quad \leqslant \int_0^\infty \frac{s^{1/2}}{(1+s)^{3/4}} \frac{ds}{s} < \infty.
\end{aligned}$$

Next we consider the case $0 < s \leqslant 2n+1$. We get

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$$
\begin{aligned}
&|\partial^{\alpha}\tilde{m}(\xi_1,\xi_2,\xi_3,\xi_4)| \\
&\quad = \Big|\iint_{\mathbb{R}^{n+1}_+} \partial^{\alpha}\big(e^{-2\pi i t u\cdot(\xi_1+\xi_2+\xi_3+\xi_4)} \\
&\qquad \times m(t\xi_1,t\xi_2)m(t\xi_3,t\xi_4)\big)\,(1+|u|)^{-n\lambda}\frac{du\,dt}{t^{n+1}}\Big| \\
&\quad \leqslant \sum_{0\leqslant|\beta|\leqslant s}\Big|\iint_{\mathbb{R}^{n+1}_+}\partial^{\alpha-\beta}\big(m(t\xi_1,t\xi_2)m(t\xi_3,t\xi_4)\big)\partial^{\beta}\big(e^{-2\pi i t u\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}\big) \\
&\qquad \times (1+|u|)^{-n\lambda}\frac{du\,dt}{t^{n+1}}\Big| \\
&\quad \lesssim \sum_{0\leqslant|\beta|\leqslant s}\int_{\mathbb{R}^n}(1+|u|)^{-n\lambda}\,du\int_0^{\infty} \\
&\qquad \times \frac{t^{|\alpha|}}{(1+|t\xi_1|+|t\xi_2|)^{2n+1+\varepsilon}(1+|t\xi_3|+|t\xi_4|)^{2n+1+\varepsilon}}\frac{dt}{t} \\
&\quad \lesssim \int_0^{\infty}\frac{t^{|\alpha|}}{(1+t(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|))^{2n+1+\varepsilon}}\frac{dt}{t} \\
&\quad = \frac{1}{(|\xi_1|+|\xi_2|+|\xi_3|+|\xi_4|)^{|\alpha|}}\int_0^{\infty}\frac{s^{|\alpha|}}{(1+s)^{2n+1+\varepsilon}}\frac{ds}{s}.
\end{aligned}
$$

By Theorem 1 in [15], we may obtain that there exists $0<\delta\leqslant 1$ such that $\widetilde{\mathcal{T}}_{\lambda,m}$ is bounded from $L^{q_1}(\mathbb{R}^n)\times L^{q_2}(\mathbb{R}^n)\times L^{q_3}(\mathbb{R}^n)\times L^{q_4}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $2-\delta<q_1,q_2,q_3,q_4$ with $1/q=1/q_1+1/q_2+1/q_3+1/q_4$.

(ii) Note that

$$
\begin{aligned}
\mathfrak{T}_{\lambda,m}(\vec{f})(x)^2 &= \iint_{\mathbb{R}^{n+1}_+}\int_{(\mathbb{R}^n)^4} e^{2\pi i x\cdot(\xi_1+\xi_2-\xi_3-\xi_4)} \\
&\quad \times m(t\xi_1,t\xi_2)\overline{m(t\xi_3,t\xi_4)}\hat{f}_1(\xi_1)\hat{f}_2(\xi_2) \\
&\quad \times \overline{\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)}\,d\xi_1\,d\xi_2\,d\xi_3\,d\xi_4\,h_t(x,u)\frac{du\,dt}{t^{n+1}} \\
&= \iint_{\mathbb{R}^{n+1}_+}\int_{(\mathbb{R}^n)^4} e^{2\pi i x\cdot(\xi_1+\xi_2+\xi_3+\xi_4)}m(t\xi_1,t\xi_2)\overline{m(-t\xi_3,-t\xi_4)} \\
&\quad \times \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\overline{\hat{f}_1(-\xi_1)\hat{f}_2(-\xi_2)}\,d\xi_1\,d\xi_2\,d\xi_3\,d\xi_4\,h_t(x,u)\frac{du\,dt}{t^{n+1}}.
\end{aligned}
$$

Then, as a consequence of (i), we obtain that $\mathfrak{T}_{\lambda,m}$ is bounded from $L^{q_1}(\mathbb{R}^n)\times L^{q_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $2-\delta<q_1,q_2<\infty$ with $1/q=1/q_1+1/q_2$.

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Zhengyang Li
*School of Mathematics and Computing Sciences*
*Hunan University of Science and Technology*
*Xiangtan 411201*
*People's Republic of China*
zhengyli@mail.bnu.edu.cn

Qingying Xue
*School of Mathematical Sciences*
*Beijing Normal University*
*Laboratory of Mathematics and Complex Systems*
*Ministry of Education*
*Beijing 100875*
*People's Republic of China*
qyxue@bnu.edu.cn

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