Can. J. Math., Vol. XXXV, No. 3, 1983, pp. 454-477

LOCAL STABILITY AND SATURATION IN SPACES OF ORDERINGS

NIELS SCHWARTZ

If k is a f.r. (= formally real) field which is partially ordered with positive cone, P, X_P denotes the space of total orders T of k with $P \subset T$. Suppose you have a subset $A \subset X_P$ and an element $T \in X_P$, $T \notin A$. Then the main question investigated in this paper is the following: How can T be separated from A by using elements of k? To be more specific, this is split up into two different questions.

Question 1. Suppose A is closed. Then there is an $n \in \mathbf{N}$ and elements $a_1, \ldots, a_n \in k$ such that the basic open set $H = H(a_1, \ldots, a_n)$ is a neighborhood of T and has empty intersection with A. Now, if T is given, what is the least $n \in \mathbf{N}$ (if it exists) such that T has a neighborhood basis consisting of basic open sets of the form $H(a_1, \ldots, a_n)$? It is a well-known fact that this number is $\leqq n$, if k (with space of orderings X_P) has finite stability $\leqq n$ (see for example [**2**], [**3**], [**7**]). In this paper this problem is treated locally, without any global stability hypothesis. But, of course, the various descriptions of what it means that k has global stability index n serve as guides for the local investigations. These descriptions are in terms of the size of fans in X_P ([**2**]), in terms of the stability index of Henselizations of k with respect to valuations compatible with orderings in X_P ([**3**]), and in terms of the fundamental ideal of the Witt ring and the reduced Witt ring of k ([**2**], [**3**], [**7**], [**9**]). (Most of this has been carried over to abstract spaces of orderings by Marshall, [**17**].) With the exception of the Witt ring description, all of these characterizations of global stability carry over to the local case. In any attempt to carry over the Witt ring description to the local case the fundamental ideal must be replaced by some other ideal. An obvious candidate to replace the fundamental ideal is the minimal prime ideal of the Witt ring belonging to the ordering under consideration ([**15**, Chapter 8]). Now the problem is that many statements on the minimal prime ideals of the Witt ring have global stability

Received September 21, 1981 and in revised form June 3, 1982 and November 24, 1982.

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

consequences. Thus, with respect to Witt rings, the answer of Question 1 remains unsatisfactory.

Question 2. When is it possible to separate T from A by using just one element of k, i.e., when does there exist an $a \in k$ such that $T \in H(a)$ and $A \cap H(a) = \emptyset$? This is possible if and only if the partial order $\cap A$ of k is not contained in T. The set

$$S(A) = \{S \in X_P; S \supset \cap A\}$$

is called the *saturated hull* or *saturation* of A. This notion was first introduced in [**14**] and further investigated in [**10**], [**11**], [**12**], [**13**]. In the terminology of abstract spaces of orderings, $S(A)$ is the *subspace* of X_P generated by A ([**16**]). A is called *saturated* if $A = S(A)$. Saturated sets of orderings are used extensively in [**2**] and [**3**], just to name a few. Now Question 2 can be rephrased by asking for a description of the saturation $S(A)$ of the set $A \subset X_P$.

Every ordering $T \notin X_P$ gives a character $\chi_T: k^*/P^* \to \{+1, -1\}$ by defining $\chi_T(aP^*) = +1$ or -1 according as $a \in T$ or $a \in -T$. Thus, X_P will be considered as a subset of the $\mathbf{Z}_2$-vector space $\chi(k^*/P^*)$ of characters $k^*/P^* \to \{+1, -1\}$. For $A \subset \chi(K^*/P^*)$, let $L(A)$ be the affine linear subspace generated by A and $\bar{A}$ the topological closure of A. Now for a subset $A \subset X_P$ there is a trivial description of the saturation of $A: S(A) = \overline{L(A)} \cap X_P$. This is just a restatement of the definition of the saturated hull. But if one wants to give a description of $S(A)$ without leaving X_P things get much more complicated. For $A \subset X_P$, let $\psi(A) = L(A) \cap X_P$ and $\phi(A) = \bar{A}$. Since saturated sets of orderings must always be closed, A is assumed to be closed. To find an internal construction of $S(A)$ in X_P one considers the following (possibly transfinite) sequence

$$A \subset \Psi(A) \subset \Psi\, \phi\, \Psi(A) \subset \ldots \subset S(A).$$

In contrast to the external description of $S(A)$ in $\chi(k^*/P^*)$, this sequence does not always reach $S(A)$ after the first two steps. In fact, it is an open question whether the union of this sequence equals $S(A)$. In particular, it is not known whether A is saturated whenever $A = \phi\, \psi(A)$. Because of cardinality reasons the sequence must break off. For a space X_P of orderings, let $s(X_P)$ be the least ordinal α such that the above sequence breaks off after at most α steps for all closed subsets $A \subset X_P$. This number is determined for a few types of spaces of orderings.

The paper starts with a short section which serves to reduce separation questions in X_P to separation questions in the connected components of X_P ([**16**]). Section 2 deals with Question 1, Section 3 with Question 2.

Most of the proofs in this paper are done by using two tools, namely the theory of real holomorphy rings ([**20**]) and the theory of real valuations, in particular results which may be found in [**1**], [**3**], [**4**], [**5**], [**19**]. Both of these methods can be unified by considering the real spectrum ([**6**]) of the real holomorphy ring of the field under consideration. But for the purpose of this paper this would only mean a change of language. Therefore I will not use the real spectrum.

I wish to thank the referee for numerous suggestions on earlier versions of the paper. His remarks definitely helped improve the paper.

1. Connected components. Call two orderings $T_1, T_2 \in X_P$ *related* if they are either equal or there is a nontrivial fan ([**2**]) containing both T_1 and T_2. Marshall ([**16**]) showed that this is an equivalence relation on X_P. It is slightly different from the Verwandtschaftsrelation in ([**5**]). Following Marshall, the classes of the relationship relation are called *connected components* of X_P. For $T \in X_P$, $V(T)$ denotes the connected component containing T.

THEOREM 1. *Let $A \subset X_P$ be a closed subset, $T \in X_P \setminus A$. Suppose there is a neighborhood $H(a_1, \dots, a_l)$ of T in X_P such that*

$$V(T) \cap H(a_1, \dots, a_l) \cap A = \emptyset.$$

Then there is a neighborhood $H(b_1, \dots, b_l) \subset X_P$ of T (same l as before) such that

$$H(b_1, \dots, b_l) \cap A = \emptyset.$$

($H(a_1, \dots, a_l)$ is defined to be $\{T \in X_P; -a_1, \dots, -a_l \in T\}$.)

Proof. Let $\widetilde{X}_p$ be the space of real places of k into $\mathbf{R}$ which are compatible ([**19**]) with some $T \in X_P$. This is a closed subspace of the space of all real places of k into $\mathbf{R}$ ([**8**], [**20**]). $\lambda: X_p \to \widetilde{X}_P$ is the canonical mapping. For $T \in X_P$, $F(T)$ denotes the set $\lambda^{-1}\lambda(T)$. If φ is a real place of k, v_φ is the valuation belonging to φ. Γ_v is the value group of the valuation v, A_v is the valuation ring, k_v the residue field. Set $B = A \cap H(a_1, \dots, a_1)$ and $\widetilde{B} = \lambda(B)$.

Case 1. Suppose $\lambda(T) \in \widetilde{B}$.

(a) First assume that $a_1, \dots, a_l \in A^*_{v_{\lambda(T)}}P^*$. Then one may assume that $a_1, \dots, a_l \in A^*_{v_{\lambda(T)}}$. Suppose there is some $S \in A$ such that $\lambda(S) = \lambda(T)$. Then $a_1, \dots, a_l < 0$ with respect to S, i.e., $S \in B$. But, by hypothesis, $\lambda(T) \notin \widetilde{B}$. Thus, $\lambda(T) \notin \widetilde{A} = \lambda(A)$. By [**20**, 1.9] there is some $a \in A_k$, the real holomorphy ring of k ([**20**]) such that $\varphi(a) > 0$ for $\varphi \in \widetilde{A}$ and $\lambda(T)(a) < 0$. Hence, $a > 0$ with respect to all $S \in A$ and $a < 0$ with respect to T.

(b) Next assume that there is some i such that $a_i \notin A^*_{v_{\lambda(T)}}P$. Then there exists a largest convex subgroup $\Delta \subset \Gamma_{v_{\lambda(T)}}$ such that

$$v_{\lambda(T)}(a_i) \notin \Delta\, v\lambda(T)(P^*) \quad \text{for some } i.$$

Let v be the coarsening of $v_{\lambda(T)}$ belonging to Δ. In particular:

$$v(P^*) \neq \Gamma_{v_{\lambda(T)}}/\Delta.$$

Pick any $\varphi \in \widetilde{B}$. Then v is not coarser than v_φ. For, otherwise T and some $S \in B$ induce orderings $\overline{T}$, $\overline{S}$ on k_v. Since $\lambda(T) \notin \widetilde{B}$, $\overline{T} \neq \overline{S}$. But then there is a fan in X_P containing both T and S ([**2**, Lemma 7]). This contradicts $B \cap V(T) = \emptyset$. Thus, there is some $x_\varphi \in A^*_{v_\varphi} \cap \mathfrak{M}_v$. Now consider

$$p_\varphi = x_\varphi{}^2/(1 + x_\varphi{}^2).$$

This is in A_k ([**20**]). $\widetilde{B}$ is covered by the open sets

$$U_\varphi = \{\psi \in \widetilde{X}_p;\ \psi(p_\varphi) > 0\}.$$

By compactness, there is a finite subcover U_{φ_i}, $i = 1, \dots, r$. Now set

$$p = p_{\varphi_1} + \dots + p_{\varphi_r}.$$

Then $p \in \mathfrak{M}_v$ and $p \in A^*_{v_\varphi}$ for all $\varphi \in \widetilde{B}$. By the choice of Δ,

$$v(P^*)\, v(a_i) \cap \Delta' \neq \emptyset$$

for all convex subgroups $\Delta \subsetneq \Delta' \subset \Gamma_{v_{\lambda(T)}}$. Thus, for all $i = 1, \dots, l$ there is some $p_i \in P^*$ such that $1 > v(a_ip_i) \geqq v(p)$. Replacing p_i by $p_i' = p_i/(1 + a_i{}^2p_i{}^2)$ we still have $1 > v(a_ip_i') \geqq v(p)$. Moreover, $a_ip_i' \in A_k$. Hence, we may assume that p_i is chosen such that $a_ip_i \in A_k$. One sees that $a_ip_i + np^2 < 0$ with respect to T for all i and all $n \in \mathbf{N}$. For each $\varphi \in B$ there is some $n_\varphi \in \mathbf{N}$ such that

$$n_\varphi\, \varphi(p) > |\varphi(a_ip_i)| \quad \text{for all } i$$

(since φ maps into $\mathbf{R}$ and $\varphi(p) > 0$). By compactness of $\widetilde{B}$, there is some $n \in \mathbf{N}$ which works for all $\varphi \in \widetilde{B}$ simultaneously. Now consider

$$H = H(a_1p_1 + np, \dots, a_lp_l + np).$$

Clearly, this is a neighborhood of T. On the other hand,

$$H \cap A \subset H(a_1, \dots, a_l) \cap A = B.$$

Thus, $H \cap A = H \cap B$. But

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

$$B \subset H(-(a_1p_1 + np), \ldots, -(a_1p_1 + np)),$$

so that $H \cap B = \emptyset$.

Case 2. Suppose $\lambda(T) \in \widetilde{B}$. Then there is some $S \in B \cap F(T)$. By $B \cap V(T) = \emptyset$, $S \notin V(T)$, i.e., $F(T) \not\subset V(T)$. But this is possible only if

$$V(T) = \{T\} \quad \text{and} \quad [\Gamma_{v_{\lambda(T)}}:v_{\lambda(T)}(P^*)] = 2.$$

Then $F(T) = \{T, S\}$. Hence there is some $a \in k$ such that $a < 0$ with respect to T and $a > 0$ with respect to S. By $V(T) \cap B = \emptyset$, $T \notin A$. Therefore,

$$V(T) \cap H(a) \cap A = \emptyset.$$

Let $C = H(a) \cap A$. Since $S \notin H(a)$ it follows that $\lambda(T) \notin \lambda(C)$. Now we are back in Case 1.

This theorem has a number of corollaries which will be used later in the paper.

COROLLARY 2. *For an ordering $T \in X_P$, the following are equivalent*:

(i) *T does not lie in a nontrivial fan of X_P.*

(ii) *T has a neighborhood basis in X_P which consists of sets of the form $H(a)$, $a \in k$.*

Proof. (ii) $\Rightarrow$ (i) is trivial. For (i) $\Rightarrow$ (ii), note that $V(T) = \{T\}$.

COROLLARY 3. *If $\lambda:X_P \to \widetilde{X}_P$ is the canonical mapping used in the proof of Theorem 1 and $C \subset \widetilde{X}_P$ is closed, then $\lambda^{-1}(C) \subset X_P$ is saturated.*

This can also be found in [**18**, §3]. A special case of this result is proved in [**1**, Proof of 6.6].

Proof. See Case 1 (a) in the proof of the theorem.

COROLLARY 4. *Let $A \subset X_P$ be closed, $T \in X_P$. Then the following are equivalent*:

(i) *$T \in S(A)$, the saturated hull of A.*

(ii) *$T \in S(V(T) \cap A)$.*

In particular, $S(A) = \bigcup_{T \in X_P} S(V(T) \cap A)$.

COROLLARY 5: *Let $A \subset X_p$ be a subset, $T \in X_p$. Then the following are equivalent*:

(i) *$T \in \psi(A)$ (for the notation, see the introduction).*

(ii) *$T \in \psi(V(T) \cap A)$.*

Proof. (ii) $\Rightarrow$ (i) is clear. For the proof of (i) $\Rightarrow$ (ii), let $T = T_1 \ldots T_r$ with

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

$T_i \in A$ (X_P is considered as a subset of $\chi(k^*/P^*)$!). Let X be the finite subspace of X_P generated by $T_1, \ldots, T_r$. In finite spaces of orderings $\psi(A) = S(A)$ is always true. Hence, the claim follows from Corollary 4.

2. Local stability. The (global) *stability index* $\mathrm{st}(k, X_P)$ of the field k with space of orderings X_P is the logarithm of the exponent of the cokernel of the total signature function $W_P(k) \to C(X_P, \mathbf{Z})$ ([**2**], [**3**]). The following descriptions of the stability index are well known:

$$\mathrm{st}(k, X_P) = \sup\{n \in \mathbf{N}_0;\ \text{there is a fan } F \subset X_P \text{ with } |F| = 2^n\}$$

([**2**], [**4**]). (Here, $\mathbf{N}_0 = \mathbf{N} \cup \{O\}$.)

$$\mathrm{st}(k, X_P) = \inf\{n \in \mathbf{N}_0;\ X_P \text{ has a basis of sets of the form } H(a_1, \ldots, a_n),\ a_1, \ldots, a_n \in k\}$$

([**2**], [**3**], [**7**]).

$$\mathrm{st}(k, X_P) \leqq 1 \quad \text{or}$$
$$\mathrm{st}(k, X_P) = \sup\{\mathrm{st}(\tilde{k}_v, X_v);\ v \text{ valuation compatible with some } T \in X_P,\ [\Gamma_v : v(P^*)] > 1\},$$

where $\tilde{k}_v$ denotes the Henselization of k with respect to v and X_v is the space of orderings of $\tilde{k}_v$ extending some $T \in X_P$ ([**3**]). $\mathrm{st}(k, X_P) \leqq n$ if and only if $I_P(k)^{n+1} \subset 2W_P(k)$ if and only if $I_P(k)^{n+1} = 2I_P(k)^n$, where $W_P(k)$ is the reduced Witt ring of k determined by P and $I_P(k)$ is its fundamental ideal ([**2**], [**3**], [**7**], [**9**]).

Now, for $T \in X_P$, call

$$\mathrm{st}(T) = \sup\{n \in \mathbf{N}_0;\ \text{there exists a fan } F \subset X_P \text{ such that } T \in F \text{ and } |F| = 2^n\}$$

the *local stability index* of X_P in T. This raises the question whether the above characterizations of the global stability index or some analogues of these carry over to the local stability index.

THEOREM 6. *Suppose that* $|X_P| > 1$. *For* $T \in X_P$, *each of the following numbers is equal to* $\mathrm{st}(T)$:

$a = \inf\{n \in \mathbf{N};$ *T has a neighborhood basis of sets of the form* $H(a_1, \ldots, a_n),\ a_1, \ldots, a_n \in k\}$,

$b = \sup\{\mathrm{st}_{\tilde{k}_v}(T);$ *v valuation compatible with T,* $[\Gamma_v : v(P^*)] > 1\}$ *if this set is nonempty, and* $b = 1$ *otherwise.*

(Here $\mathrm{st}_{\tilde{k}_v}(T)$ is the local stability index of the unique extension of T to an ordering of $\tilde{k}_v$.)

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

Proof. $\mathrm{st}(T) = b$ is trivial and so is $a \geqq \mathrm{st}(T)$. To show $a \leqq \mathrm{st}(T)$: If $\mathrm{st}(T) = \infty$ there is nothing to prove. So, suppose $\mathrm{st}(T) < \infty$. The proof is done by induction on $\mathrm{st}(T)$:

$\mathrm{st}(T) = 1$: This is Corollary 2.

$\mathrm{st}(T) = n > 1$: Let $A \subset X_P$ be closed such that $T \notin A$. If $A \cap F(T) = 0$ (the notations are the same as in the proof of Theorem 1), then there is some $a \in k^*$ such that $T \in H(a)$ and $H(a) \cap A = \emptyset$ (see Case 1 (a) in the proof of Theorem 1). Thus, suppose that $F(T) \cap A \neq \emptyset$. From [**4**] and [**5**] one knows that there is some $S \in F(T)$, $S \neq T$ such that S is contained in any maximal fan containing T. Pick $a \in k^*$ such that $T \in H(a)$, $S \notin H(a)$. $H(a)$ is the space $X_{P[-a]}$ of all orderings containing $P[-a]$ the partial order generated by P and $-a$. Now, for any fan $F \subset X_P$ with $T \in F$, $F \cap X_{P[-a]}$ is a fan in $X_{P[-a]}$ containing T. Moreover, if F is a maximal fan,

$$|F \cap X_{P[-a]}| = |F|/2.$$

Thus, in $X_{P[-a]}$, the local stability index of T is $< n$. By induction, there are $a_1, \ldots, a_{n-1} \in k$ such that

$$T \in H(a_1, \ldots, a_{n-1}) \cap X_{P[-a]} \quad \text{and}$$
$$A \cap H(a_1, \ldots, a_{n-1}) \cap X_{P[-a]} = \emptyset.$$

This proves the claim since $X_{P[-a]} = H(a)$.

$\mathrm{st}(T) = \infty$ if T lies in fans of arbitrary size. It is an immediate consequence of [**4**] that, in fact, T belongs to an infinite fan.

In analogy with various characterizations of SAP fields (see for example [**2**], [**3**], [**9**]) one obtains the following:

COROLLARY 7. *For $T \in X_P$, the following are equivalent*:

(i) $\mathrm{st}(T) \leqq 1$.

(ii) *T has a neighborhood basis consisting of sets of the form $H(a)$, $a \in k$.*

(iii) *For the valuation $v_{\lambda(T)}$ (notation as in the proof of Theorem 1) and all of its coarsenings v the following holds: either $v(P^*) = \Gamma_v$ or $[\Gamma_v : v(P^*)] = 2$ and P induces a total order on the residue field k_v.*

This, of course, provides alternative proofs for some of the characterizations of SAP fields.

The notion of SAP fields was first introduced in [**14**]. In [**9**] Elman and Lam established the connections of SAP with the fundamental ideal of the Witt ring and with the linkage of Pfister forms. Bröcker ([**3**]) extended the connections of SAP and the fundamental ideal to higher stability indices.

Finally, Becker and Köpping ([**2**]) carried this over to reduced Witt rings. In any attempt to carry these results over to the local situation the first question is which ideal shall take the place of the fundamental ideal. An obvious choice is to relate the local stability index in T to the minimal prime ideal $\mathfrak{P}_T$ of the reduced Witt ring $W_P(k)$ belonging to the total order T ([**15**, Chapter 8]). The next proposition shows that the relation of the stability index with the fundamental ideal does not carry over to a similar connection between the local stability index in T and the prime ideal $\mathfrak{P}_T \subset W_P(k)$. The proof requires the following:

LEMMA 8. *Suppose $T \in X_p$ and $\mathfrak{P}_T \subset W_P(k)$ is the corresponding minimal prime ideal. Let $\varphi \in \mathfrak{P}_T$ be an n-fold Pfister form. Then there are $a_1, \ldots, a_n \in k^*$ such that*

$$\varphi = \ll a_1, \ldots, a_n \gg$$

and, for all i, $\ll a_i \gg \in \mathfrak{P}_T$.

Proof. Let $\varphi = \ll b_1, \ldots, b_n \gg$ be a representation of φ. Suppose that

$$\ll b_1 \gg, \ldots, \ll b_r \gg \notin \mathfrak{P}_T, \quad \ll b_{r+1} \gg, \ldots, \ll b_n \gg \in \mathfrak{P}_T.$$

Since $\mathfrak{P}_T$ is prime, $r < n$. Using $\ll x, y \gg = \ll x, xy \gg$, one finds

$$\varphi = \ll b_1b_n, \ldots, b_rb_n, b_{r+1}, \ldots, b_n \gg.$$

Morever, $\ll b_1b_n \gg, \ldots, \ll b_rb_n \gg \in \mathfrak{P}_T$.

PROPOSITION 9. *For $T \in X_P$ and the corresponding minimal prime ideal $\mathfrak{P}_T \subset W_P(k)$, the following are equivalent*:

(i) $\operatorname{st}(k, X_P) \leqq n$.

(ii) $2\, \mathfrak{P}_T^n = \mathfrak{P}_T^{n+1}$.

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (i): Suppose there is a fan $F \subset X_P$, $|F| = 2^{n+1}$. Set $Q = T \cap (\cap F)$. X_Q is a finite space of orderings, $\operatorname{st}(k, X_Q) \leqq n + 2$. Pick $S \in F$, $S \neq T$. Then, in X_Q, $\operatorname{st}(S) \leqq n + 2$. Hence there are $a_1, \ldots, a_{n+2} \in k$ such that

$$X_Q \cap H(-a_1, \ldots, -a_{n+2}) = \{S\}.$$

The Pfister form $\varphi = \ll a_1, \ldots, a_{n+2} \gg \in W_Q(k)$ is in the minimal prime ideal $\mathfrak{P}_{Q,T} \subset W_Q(k)$ corresponding to T. By Lemma 8,

$$\varphi \in \mathfrak{P}_{Q,T}^{n+2}.$$

From hypothesis (ii) it follows that also

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

$$2\,\mathfrak{P}^n_{Q,T} = \mathfrak{P}_{Q,T}{}^{n+1}.$$

By [**2**, Satz 15] and Lemma 8, there is some n-fold Pfister form $\psi \in \mathfrak{P}^n_{Q,T}$ such that $4\,\psi = \varphi$. Consider $W_Q(k)$ as a subring of $C(X_Q, \mathbf{Z})$ and define

$$f{:}C(X_Q, \mathbf{Z}) \to \mathbf{Z}{:}\rho \mapsto \sum_{R \in F} \rho(R).$$

Then $f(\mathfrak{P}^n_{Q,T}) \subset 2^{n+1}\,\mathbf{Z}$. But $f(\psi) = 2^n$. This is a contradiction.

Next, connections between linkage of Pfister forms and the stability index are investigated. Compare the first result with [**9**, Theorem 3.5].

PROPOSITION 10. *For $T \in X_P$, let all two n-fold Pfister forms $\varphi_1, \varphi_2 \in \mathfrak{P}_T$ be linked for some fixed $n \geqq 2$. Then* $\mathrm{st}(k,X_P) \leqq 1$.

Proof. Assume there is a fan $F \subset X_P$, $|F| = 4$. Then there are $T_1, T_2, T_3 \in F \setminus \{T\}$. Then there are two n-fold Pfister forms $\varphi_1, \varphi_2 \in \mathfrak{P}_T$ such that

$$s(\varphi_1) \cap F = \{T_1, T_2\} \quad \text{and} \quad s(\varphi_2) \cap F = \{T_3\}$$

where $W_P(k)$ is considered as a subring of $C(X_P, \mathbf{Z})$ and $s(\psi)$ is the support of the form ψ. By hypothesis, φ_1 and φ_2 are linked, i.e., there is some $(n\text{-}1)$-fold Pfister form ψ and there are $a_1, a_2 \in k$ such that $\varphi_i = \ll a_i \gg \chi$. Since $\{T_1, T_2, T_3\} \subset s(\psi) \cap F$, it follows that $s(\psi) \cap F = F$. But then

$$s(\ll a_2 \gg) \cap F = \{T_3\}, \text{ which is impossible.}$$

A similar argument shows that $\mathrm{st}(k, X_P) \leqq n - m$ if for fixed $n \in \mathbf{N}$ and m, $1 \leqq m \leqq n-1$, any two n-fold Pfister forms in $\mathfrak{P}_T$ are m-linked. Thus, conditions on the existence of linkages of Pfister forms in $\mathfrak{P}_T$ do not give a description of the local stability index in T. Things look different if one asks about properties of existing linkages:

THEOREM 11. *For $T \in X_P$, the following are equivalent*:

(i) $\mathrm{st}(T) \leqq 1$.

(ii) *If the n-fold Pfister forms ($n \geqq 2$) $\varphi_1, \ldots, \varphi_r \in \mathfrak{P}_T$ are m-linked then there is an m-linkage in $\mathfrak{P}_T$, i.e., there is an m-fold Pfister form $\psi \in \mathfrak{P}_T$ and there are $(n - m)$-fold Pfister forms $\psi_1, \ldots, \psi_r \in \mathfrak{P}_T$ such that $\varphi_i = \psi\,\psi_i$.*

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

(iii) *Any two linked n-fold Pfister forms* ($n \geqq 2$) *have linkage in* $\mathfrak{P}_T$ (*as in condition* (ii)).

Proof. (iii) is a special case of (ii).

(iii) $\Rightarrow$ (i): Suppose there is a fan $F \subset X_P$, $T \in F$, $|F| = 4$. If $F = \{T, T_1, T_2, T_3\}$ then there are 1-fold Pfister forms $\ll a \gg, \ll b \gg \in \mathfrak{P}_T$ such that

$$s(\ll a \gg) \cap F = \{T_1, T_2\} \quad \text{and} \quad s(\ll b \gg) \cap F = \{T_2, T_3\}$$

where again $W_P(k)$ is embedded in $C(X_P, \mathbf{Z})$ and $s(\psi)$ is the support of ψ. The 2-fold Pfister forms $\ll 1, a \gg$ and $\ll 1, b \gg$ are linked. Thus, there exist $\ll a' \gg, \ll b' \gg, \ll c \gg \in \mathfrak{P}_T$ such that $\ll 1, a \gg = \ll a', c \gg$, $\ll 1, b \gg = \ll b', c \gg$. But then

$$s(\ll c \gg) \cap F = \{T_1, T_2, T_3\},$$

which is impossible.

(i) $\Rightarrow$ (ii): Let $\varphi_i = \psi \psi_i$, $i = 1, \ldots, r$. Let $\psi = \ll a_1, \ldots, a_m \gg$. If $\psi \in \mathfrak{P}_T$, Lemma 8 shows that it is possible to choose the $\psi_i \in \mathfrak{P}_T$. Thus, suppose that $\psi \notin \mathfrak{P}_T$. Since $\mathfrak{P}_T$ is prime, $\psi_i \in \mathfrak{P}_T$ for all i. Let

$$A = s(\psi_1) \cup \ldots \cup s(\psi_r).$$

This is a closed subset of X_P and $T \notin A$. By st $(T) \leqq 1$ and Corollary 2, there is some $a \in k$ such that $T \in H(a)$ and $H(a) \cap A = \emptyset$. Then

$$\psi' = \ll aa_1, \ldots, aa_m \gg \in \mathfrak{P}_T$$

is the desired linkage.

3. Saturation. In this section the sequence

$$A \subset \psi(A) \subset \phi\psi(A) \subset \psi\phi\psi(A) \subset \ldots \subset S(A)$$

introduced in the introduction is investigated for closed subsets $A \subset X_P$. This attempt to internally construct and describe $S(A)$ in X_P is closely related to Marshall's result [**18**, Theorem 3.16].

The following is an immediate consequence of Corollary 2:

COROLLARY 12. *The following are equivalent*:

(i) $\mathrm{st}(k, X_P) \leqq 1$.

(ii) $\mathrm{s}(X_P) = 0$. (*For the definition of* $s(X_P)$, *see the introduction.*)

This is a well-known result, see [**12**], [**10**, Corollary 6.6].

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

The following examples exhibit spaces of orderings X_P with $s(X_P) = 1$, 2 and 3, respectively.

Example 1. Let $k = \mathbf{R}((X))((Y))$, the formal power series field over $\mathbf{R}$ in two variables. Let P be the set of sums of squares. So X_P is the space of all orderings of k. Then $s(X_P) = 1$.

Example 2. Let $k = \mathbf{R}((X_1))((X_2))\ldots$ be the countably many times iterated formal power series field over $\mathbf{R}$. Let P be the sums of squares in k again. There is a canonical homeomorphism

$$f: X_P \rightarrow \{+1, -1\}^{\mathbf{N}}: T \mapsto (n \mapsto \operatorname{sign}_T(X_n)),$$

where $\{+1, -1\}^{\mathbf{N}}$ carries the product topology ([**1**, Proposition 4.1]). For every subset $N \subset \mathbf{N}$ define $\chi_N \in \{+1, -1\}^{\mathbf{N}}$ by

$$\chi_N(n) = \begin{cases} -1 \text{ if } n \in N \\ +1 \text{ if } n \notin N. \end{cases}$$

Set $\chi_n = \chi_{\{n\}}$. Then $A = \{\chi_\emptyset, \chi_1, \chi_2, \ldots\}$ is a closed subset of X_P. It is easy to verify that $\phi\,\psi(A) = X_P$. Obviously, $s(X_P) \leqq 2$. Thus, the set A shows that $s(X_P) = 2$.

Example 3. Let K be the field of Example 2. Set $k = \mathbf{Q}(X_1, X_2, \ldots)$ and embed k canonically into K. The set of orderings induced by K on k is precisely the set of orderings of k which are compatible with the restriction of the natural valuation of K to k. Thus, this is a saturated set by Corollary 3 and [**19**, 7.3]. Call this space of orderings X. Let T be any archimedean ordering of K. Set $Y = X \cup \{T\}$. Let $L = k((X_0))$. Each of the orderings of k induces two orderings of L ([**1**, Proposition 4.1], [**19**, Chapter 7]). Let X_P be the orderings of L induced by elements of Y. Let $\widetilde{X}$ be those orderings in X_P which are induced by some element of X. Then $X_P = \widetilde{X} \cup \{T_1, T_2\}$ where T_1 and T_2 are induced by T. As in Example 2, there is a homeomorphism

$$f: \widetilde{X} \rightarrow \{+1, -1\}^{\mathbf{N}_0}: S \mapsto (n \mapsto \operatorname{sign}_S(X_n)).$$

For $N \subset \mathbf{N}_0$, define χ_N as in Example 2. Moreover, set $\rho_N = \chi_{\mathbf{N}_0 \setminus N}$. Let

$$A = \{\chi_\emptyset, \chi_1, \ldots\} \cup \{\rho_\emptyset, \rho_1, \ldots\} \cup \{T_1\}.$$

Then $A \subset X_P$ is closed. It is easy to verify that

$$A \subsetneqq \psi(A) \subsetneqq \phi\,\psi(A) \subsetneqq \psi\,\phi\,\psi(A) = X_P.$$

Since $s(X_P) \leqq 3$ is clear, it follows that $s(X_P) = 3$.

With the notations introduced in Section 1, suppose that $\Delta \subset \Gamma_{v_{\lambda(T)}}$ is a convex subgroup. Then Δ_P denotes the subgroup $\Delta \cdot v_{\lambda(T)}(P^*)$. Since the convex subgroups of $\Gamma_{v_{\lambda(T)}}$ form a chain which is closed under arbitrary unions and intersections, it follows that, for any $a \in k^* \setminus P^*U$ (U is the group of $v_{\lambda(T)}$-units), there is a largest convex subgroup $\Delta(a)$ such that $a \notin \Delta(a)_P$. $\Delta(a)$ determines a coarsening of $v_{\lambda(T)}$ which is denoted by v_a. For any valuation v, let X_v be the set of orderings $T \in X_P$ compatible with v. This is a saturated subset of X_P (Corollary 3 and [**19**, 7.3]).

The next result, which is again a corollary of the proof of Theorem 1, gives a sufficient condition for a given subset of X_P to be saturated. This will be applied later in the paper.

COROLLARY 13. *Let $A \subset X_P$ be a closed subset and $T \in X_P$. Suppose that there is some $a \in k^* \setminus P^*A^*_{v\,\lambda(T)}$ such that $T \in H(a)$ and*

$$H(a) \cap A \cap X_{v_a} = \emptyset.$$

Then $T \notin S(A)$.

Proof. Set $B = H(a) \cap A$. For $S \in B$, v_a cannot be coarser than $v_{\lambda(S)}$. Now the same argument as in the proof of Theorem 1, Case 1 (b) provides $p, q \in P^*$ and $n \in \mathbf{N}$ such that

$$T \in H(ap + nq) \quad \text{and} \quad A \cap H(ap + nq) = \emptyset.$$

Before starting to determine the number $s(X_P)$ for some spaces of orderings, it is useful to note that, for closed and open A, $\psi(A) = S(A)$ always holds. This is a consequence of the following:

LEMMA 14. *For any closed and open subset $A \subset X_P$, define*

$$m(A) = \min\{n \in \mathbf{N};\ \exists\, m \in \mathbf{N}\ \forall\, i = 1, \dots, m\ \ \exists\, a_{i1}, \dots, a_{in_i} \in k:$$
$$A = \bigcup_{i=1}^{m} H(a_{i1}, \dots, a_{in}) \text{ and } n = \sum_{i=1}^{i} n_i\}.$$

There is a function $M : \mathbf{N} \to \mathbf{N}$ such that for any closed and open subset $A \subset X_P$ and any $T \in \psi(A)$ there are $T_1, \dots, T_l \in A$, $l \leqq Mm(A)$, such that $T = T_1 \dots T_l$ (where X_P is considered as a subset of the space $\chi(k^*/P^*)$ of characters of k^*/P^*).

Remark. Lemma 14 is true for any abstract space of orderings in the sense of Marshall ([**16**]). For, suppose that $A \subset X$ is a closed and open subset of an abstract space X of orderings and that $T \in \psi(A)$. Let $T = T_1 \dots T_r$ be a representation of T with $T_i \in A$. Now consider the finite subspace $Y \subset X$ generated by $T_1, \dots, T_r$. Then Y is isomorphic to the space $X(k)$ of all orderings of some field k ([**16**]). Thus, Lemma 14 is applicable, and one finds a representation of T with at most $Mm(A)$ factors from A. (The referee pointed this out to me.)

Proof of Lemma 14. The proof is by induction on $m(A)$. First note that A itself is saturated if A is of the form $H(a_1, \dots, a_n)$. In particular, this is the case if $m(A) = 1$. So, define $M(1) = 1$. Now suppose that $m(A) > 1$, and represent

$$A = A_1 \cup \dots \cup A_m \quad \text{with}$$
$$A_i = H(a_{i1}, \dots, a_{in_i}), \quad m(A) = n_1 + \dots + n_m.$$

As noted above, one may also assume that $m > 1$. Now pick $T \in \psi(A)$, and let $T = T_1 \dots T_r$ ($T_i \in A$) be a shortest representation. Let X be the subspace of X_P generated by $T_1, \dots, T_r$. By Corollary 3,

$$X \subset \lambda^{-1} \lambda(\{T_1, \dots, T_r\}).$$

By Corollary 5, one may assume that X is connected. Without loss of generality assume that $X_P = X$. Let v be the finest valuation of k compatible with every $S \in X_P$. As before, Γ_v, A_v, k_v are the value group, the valuation ring, the residue field of v, respectively. Since X_P has only one connected component,

$$[\Gamma_v : v(P^*)] > 1.$$

Let $\overline{X_P}$ be the space of orderings of k_v induced by orderings of k in X_P. The exact sequence

$$1 \to k_v^*/\bar{P}^* \to k^*/P^* \to \Gamma_v/v(P^*) \to 1$$

gives the following exact sequence of character spaces:

$$1 \to \chi(\Gamma_v/v(P^*)) \to \chi(k^*/P^*) \xrightarrow{p} \chi(k_v^*/\bar{P}^*) \to 1.$$

Therefore, $\chi(\Gamma_v/v(P^*))$ will be considered as a subspace of $\chi(k^*/P^*)$. If a section is chosen for p this gives a bijection

$$X_P \to \overline{X_P} \times \chi(\Gamma_v/v(P^*)) \quad ([\mathbf{1}, \text{Proposition 4.1}]).$$

By $H(a, b) = H(a, -ab)$, one may assume that for each $i = 1, \dots, m$ there is some l_i, $0 \leqq l_i \leqq n_i$ such that $v(a_{i1})v(P^*), \dots, v(a_{il_i})v(P^*)$ are linearly independent in $\Gamma_v/v(P^*)$ and such that $a_{i,l_i+1}, \dots, a_{in_i} \in A_v^*$.

First assume that $l_i = 0$ for all $i = 1, \dots, m$. Then

$$p(A) = \bigcup_{i=1}^{m} p(A_i) = \bigcup_{i=1}^{m} H(\overline{a_{i1}}, \dots, \overline{a_{in_i}})$$

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

($\overline{a} \in k_v$ is the image of $a \in A_v{}^*$) and $p(T) = p(T_1) \dots p(T_r)$. By definition of v, not all of $p(T_1), \dots, p(T_r)$ are in the same connected component of $\overline{X_P}$. It follows from Corollary 5 that (without loss of generality) there is some $s < r$ such that

$$p(T) = p(T_1) \dots p(T_s).$$

Let $T' = T_1 \dots T_s$. Then $p(T) = p(T')$, and, by $l_i = 0$ for all i, there is some $T_1' \in A$ with $p(T_1') = p(T_1)$ and $T_1' \, T_1 = T' \, T$. This gives a representation

$$T = T' \, TT' = T_1' \, T_1 \, T_1 \dots T_s = T_1' \, T_2 \dots T_s$$

of length $s < r$, contradicting the definition of r.

Next assume that $l_i = 0$ for some i and $l_j > 0$ for some j. In this case

$$p(A) = \bigcup_{i=1}^{m} H(\overline{a_{i,l_i+1}}, \dots, \overline{a_{in_i}}),$$

so that $m(p(A)) < m(A)$. Therefore, $p(T) \in p(\psi(A)) = \psi(p(A))$ has a representation $p(T) = p(S_1) \dots p(S_t)$ with $S_1, \dots, S_t \in A$ and $t \leqq Mm(p(A))$. Thus,

$$T_1 \dots T_r \, S_1 \dots \, S_t \in \chi(\Gamma_v/v(P^*)).$$

If $l_i = 0$ then $R \, \chi(\Gamma_v/v(P^*)) \subset X_P$ for all $R \in A_i$. Hence there are $R_1, R_2 \in A_i$ such that

$$R_1R_2 = T_1 \dots T_rS_1 \dots S_t.$$

This shows that $T = R_1R_2S_1 \dots S_t$, and T has a representation of length $t + 2 \leqq Mm(p(A)) + 2$.

Finally, assume that $l_i > 0$ for all i. Let

$$b = M(1) + \dots + M(m(A) - 1),$$

and let B be the set of all products of elements of A with not more than b factors. We are done if $B = X_P$. So, suppose that $B \neq X_P$. By [**18**, Theorem 3.16] there are $T_1, T_2, T_3 \in B$ with $T_4 = T_1T_2T_3 \notin B$. Now there are two different cases to consider:

(a) Suppose that $p(T_4) = p(T_i)$ for some $i = 1, 2, 3$, say $i = 1$. T_1 has a representation $T_{11} \dots T_{1u}$ with $u \leqq b$ and $T_{1i} \in A$. Consider the product $T_5 = T_{11}T_1T_4 = T_{11}T_2T_3$. If $T_5 \in A$, then

$$T_4 = T_5T_{11}T_1 = T_5T_{12} \dots T_{1u}$$

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

with all factors in A and $u \leqq b$. This contradicts $T_4 \notin B$. Therefore, $T_5 \notin A$. $T_{11} \in A_i$ for some i, say for $i = 1$. By $T_5 \notin A_1$ and $T_5 \in p^{-1}(T_{11})$ and $H(a, b) = H(a, -ab)$ one may assume that

$$T_5 \in A_1' = H(a_{12}, \ldots, a_{1n_1}).$$

Now consider $A' = A \cup A_1'$. Since $m(A') < m(A)$, every element from X_P can be represented as a product of at most b factors from A'. Let $S \in A' \setminus A \subset A_1' \setminus A_1$. There is some $R \in A_1$ with $p(S) = p(R)$. Since $p^{-1}(\{p(S), p(T_5)\})$ is a fan, $T_6 = RST_5 \in X_P$. Moreover, an easy verification shows that $T_6 \in A_1$. Now

$$S = RT_6T_5 = RT_6T_{11}T_2T_3$$

is a product of at most $2b + 3$ elements from A. Putting this together with the fact that every element of X_P is a product of at most b elements from A', one sees that every element of X_P is a product of at most $b(2b + 3)$ elements from A.

(b) Suppose that $p(T_4) \neq p(T_i)$ for all $i = 1, 2, 3$. By induction, there is some $S \in B$ with $p(S) = p(T_4)$, say $S = S_1 \ldots S_u$ with $u \leqq b$ and $S_i \in A$. Set $T_5 = S_1ST_4$. This is in X_P since $p^{-1}(\{p(S_1), p(T_4)\})$ is a fan. Because of $T_5 = S_1ST_1T_2T_3$, T_5 is a product of at most $4b + 1$ factors from A. Now the proof continues exactly as in case (a), and every element of X_P is representable as a product of at most $b(4b + 3)$ factors from A.

Altogether, one may set $Mm(A) = b(4b + 3)$.

THEOREM 15. *Let $A \subset X_P$ be closed and open. Then $\psi(A) = S(A)$.*

Proof. We use induction on $m(A)$. The case $m(A) = 1$ is clear. So, assume now that $m(A) > 1$ and that

$$A = A_1 \cup \ldots \cup A_m, \quad A_i = H(a_{i1}, \ldots, a_{in_i})$$

is a representation of A with $m(A) = n_1 + \ldots + n_m$. By Lemma 13, $\psi(A) = A^l \cap X_P$ for some $l \in \mathbf{N}$. This implies that $\psi(A)$ is topologically closed. Now let $T \notin \psi(A)$. If $F(T) \cap A = \emptyset$, by Corollary 3, there is some $a \in k^*$ such that $T \in H(a)$ and $H(a) \cap A = \emptyset$, hence $T \notin S(A)$. Suppose that $F(T) \cap A \neq \emptyset$. Let v be the coarsest of the valuations $v_{a_{ij}}$ ($i = 1, \ldots, m, j = 1, \ldots, n_i$), say $v = v_{a_{11}}$. To see that v is defined note that at least one $v_{a_{ij}}$ is defined. This is the case if $a_{ij} \notin P^*A^*_{v_{\lambda(T)}}$. But $F(T) \cap A_i \neq \emptyset$ for some i. Now, if $a_{ij} \in P^*A^*_{v_{\lambda(T)}}$ for all j then one may assume that $a_{ij} \in A^*_{v_{\lambda(T)}}$ for all j. Then each a_{ij} is positive or negative with respect to S for $S \in F(T)$ simultaneously. Since there is some j such that $a_{ij} > 0$ with respect to T, this leads to $F(T) \cap A_i = \emptyset$, a contradiction. By $H(a, b) = H(a, -ab)$, every A_i is the intersection of

$$B_i = H(b_{i1}, \dots, b_{il_i}) \quad \text{and} \quad C_i = H(c_{i1}, \dots, c_{im_i})$$

where $0 \leqq l_i \leqq n_, m_i = n_i - l_i$ and, for each i, $v(b_{i1})v(P^*), \dots, v(b_{il_i})v(P^*)$ are linearly independent in $\Gamma_v/v(P^*)$ and $v = v_{b_{ij}}$ and the c_{ij} are v-units. Let X_v be the space of those orderings in X_p compatible with v, $\overline{X_v}$ the orderings of the residue field k_v induced by X_v, $p:X_v \to \overline{X_v}$ the natural mapping. Note that

$$p(A_i \cap X_v) = H(\overline{c_{i1}}, \dots, \overline{c_{im_i}})$$

so that

$$p(A \cap X_v) = \bigcup_{i=1}^{m} H(\overline{c_{i1}}, \dots, \overline{c_{im_i}}).$$

Since $m_1 < n_1$, it follows by induction that $\psi\, p(A \cap X_v)$ is saturated. If $p(T) \notin \psi\, p(A \cap X_v)$, then there is some $z \in A_v^* \subset k$ such that

$$p(T) \in H(\bar{z}) \quad \text{and} \quad H(\bar{z}) \cap \psi\, p(A \cap X_v) = \emptyset.$$

Then, of course, $T \in H(z)$ and $H(z) \cap A \cap X_v = \emptyset$. Hence, for any $S \in A \cap F(T)$, $S \notin H(z)$. From $T \in H(z)$ one obtains

$$\emptyset \neq F(T) \cap H(z) \neq F(T).$$

This shows that $z \notin P^*A^*_{v_{\lambda(T)}}$ and v_z is defined. Since v_z is finer than v we also have $T \in H(z) \cap X_{v_z}$ and $H(z) \cap A \cap X_{v_z} = \emptyset$.

Next suppose that $p(T) \in \psi\, p(A \cap X_v)$. Since $\psi(A \cap X_v)$ is closed and $p^{-1}\, p(T)$ is a fan, there is some $z \in k^*$ such that

$$T \in H(z) \quad \text{and} \quad H(z) \cap p^{-1}\, p(T) \cap \psi(A \cap X_v) = \emptyset.$$

Since $p(T) \in \psi\, p(A \cap X_v) = p\, \psi(A \cap X_v)$, v_z is defined. By definition of the b_{ij}, z is a product of some elements b_{ij} modulo $P^*A_v{}^*$. In particular this implies $v_z = v$. Therefore, for any $S \in \psi(A \cap X_v)$, one of

$$p^{-1}\, p(S) \cap \psi(A \cap X_v) \cap H(z) = \emptyset,$$
$$p^{-1}\, p(S) \cap \psi(A \cap X_v) \cap H(-z) = \emptyset$$

holds. This shows that

$$p\, \psi(A \cap X_v) = p(\psi(A \cap X_v) \cap H(z)) \cup p(\psi(A \cap X_v) \cap H(-z))$$

partitions $\psi\, p(A \cap X_v)$ into closed and open subsets. Hence the function

$$\alpha:\psi\, p(A \cap X_v) \to \mathbf{Z}:p(S) \mapsto \begin{cases} +1 \text{ if } S \in \psi(A \cap X_v) \cap H(-z) \\ -1 \text{ if } S \in \psi(A \cap X_v) \cap H(z) \end{cases}$$

is continuous. An easy verification shows that [**1**, Theorem 5.3] is applicable. Thus, there exists some $a \in A_v^*$ such that

$$\psi\, p(A \cap X_v) \to \mathbf{Z}{:}p(S) \mapsto \mathrm{sign}_{p(S)}(\bar{a})$$

is the function α. Now it follows that $T \in H(az)$, $H(az) \cap A \cap X_v = \emptyset$. Moreover, $v = v_z = v_{az}$. In any event, it has been shown that the hypotheses of Corollary 13 are fulfilled, so $T \notin S(A)$.

Hence $\psi(A)$ is saturated.

In general, $\psi(A) \subsetneqq S(A)$ for closed subsets $A \subset X_P$ (Example 2). In fact, Example 2 shows that there are closed subsets $A \subset X_P$ with $\psi(A) \subsetneqq S(A)$ whenever there are infinite fans in X_P. This proves one half of

CONJECTURE 1. *The following conditions are equivalent*:

(i) $s(X_P) = 1$.

(ii) $\mathrm{st}(T) < \infty$ *for all* $T \in X_P$, $\mathrm{st}(T) \geqq 2$ *for some* $T \in X_P$ (*i.e.*, X_P *contains nontrivial fans, but no infinite ones*).

This conjecture was a theorem in an earlier version of the paper. But Murray Marshall found a gap in the proof which I was unable to fill. The next couple of results are partial solutions of the conjecture.

THEOREM 16. *Let* X_P *be a space of orderings with* $\mathrm{st}(T) < \infty$ *for all* $T \in X_P$. *Then* $A = \phi\,\psi(A)$ *implies* $A = S(A)$.

Proof. Using the Zariski spectrum of the real holomorphy ring A_k of k ([**20**]) one sees that there is no infinite sequence $v_0, v_1, \ldots$ of real valuations such that v_{i+1} is finer than v_i and

$$[\Gamma_{v_{i+1}}{:}v_{i+1}(P^*)] > [\Gamma_{v_i}{:}v_i(P^*)].$$

For, assume that there exists such a sequence. Let $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \ldots$ be the corresponding sequence of prime ideals of A_k. The union $\mathfrak{P}$ of this chain is itself a prime ideal. The corresponding valuation v is finer than all v_i. Hence,

$$[\Gamma_v{:}v(P^*)] = \infty$$

which shows that there is an infinite fan in X_P, a contradiction.

Assume by way of contradiction that there is some $A \subset X_P$ with $A = \phi\,\psi(A)$ and $A \neq S(A)$. Pick $T \in S(A) \setminus A$. Thus $A \cap F(T) \neq \emptyset$ and this is saturated in the fan $F(T)$. So there is some $z \in k^*$ such that

$$T \in H(z) \quad \text{and} \quad A \cap F(T) \cap H(z) = \emptyset.$$

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

Let z_0 be one these z's such that $v_0 = v_{z_0}$ is the coarsest of the valuations v_z arising in this way. k_0 is the residue field of v_0, X_0 the space of orderings compatible with v_0, $\overline{X}_0$ the space of orderings k_0 induced by X_0, $p_0:X_0 \to \overline{X}_0$ the canonical mapping. Note that $A \cap X_0 = \phi\ \psi(A \cap X_0)$. The following argument shows that $T \in S(A \cap X_0)$: Suppose that $T \notin S(A \cap X_0)$. Then there is some $t \in k^*$ with

$$T \in H(t) \quad \text{and} \quad A \cap X_0 \cap H(t) = \emptyset.$$

In particular, $A \cap F(T) \cap H(t) = \emptyset$. By the choice of z_0, v_t is finer than v_0, i.e., $X_{v_t} \subset X_0$. But then

$$T \in H(t) \cap X_{v_t} \quad \text{and} \quad A \cap X_{v_t} \cap H(t) = \emptyset.$$

Corollary 13 shows that $T \notin S(A)$, a contradiction. Thus, $T \in S(A \cap X_0)$ and we may assume that $A \subset X_0$.

It is easy to see that

$$p_0(A) = \phi\ \psi\ p_0(A) \quad \text{and} \quad p_0(S(A)) \subset S(p_0(A)).$$

First suppose that $p_0(A) = Sp_0(A)$. Assume that there are $S_1 \in A \cap H(z_0)$ and $S_2 \in A \cap H(-z_0)$ with $p_0(S_1) = p_0(S_2)$. By the choice of z_0 there is some $S \in A \cap p_0{}^{-1}p_0(T) \cap H(-z_0)$. But then

$$S_1S_2S \in A \cap F(T) \cap H(z_0) = \emptyset:$$

a contradiction. Thus,

$$Sp_0(A) = p_0(A) = p_0(A \cap H(z_0)) \cup p_0(A \cap H(-z_0))$$

is a disjoint union of closed and open subsets. Now define α and a just as in the proof of Theorem 15. This leads to $T \in H(az_0)$, $A \cap H(az_0) = \emptyset$, which shows that $T \notin S(A)$. But this contradicts the choice of T. Therefore,

$$p_0(A) = \phi\ \psi\ p_0(A) \subsetneq Sp_0(A).$$

By induction we obtain a sequence $v_0, v_1, \ldots$ of valuations as described at the beginning of the proof, a contradiction.

Theorem 16 shows that in order to prove the conjecture one only has to show that $\psi(A)$ is closed for $A \subset X_P$ closed. The next theorem says that the conjecture is true under the additional hypothesis that the global stability index is finite.

THEOREM 17. *Let X_P be a space of orderings with* $\mathrm{st}(X_P) < \infty$. *Then* $s(X_P) \leqq 1$.

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

Proof. Let $A \subset X_P$ be closed and assume that $T \in S(A) \setminus \psi(A)$. Set

$$A' = A \cup (F(T) \cap \psi(A)).$$

Then A' is closed and $T \in S(A') \setminus \psi(A')$. Hence we may assume that $A = A'$. Since $F(T)$ is finite there is some z_0 as in the proof of Theorem 16. Define v_0, k_0, X_0, $\overline{X}_0$, p_0 as in the proof of Theorem 16. Again we may assume that $A \subset X_0$. If $p_0(\psi(A)) = \psi\, p_0(A) \subsetneq Sp_0(A)$, continue the construction of valuations. By the first step in the proof of Theorem 16 this process must break off after a finite number of steps. Therefore, it only remains to consider the case that $\psi\, p_0(A) = Sp_0(A)$.

As in the proof of Theorem 16,

$$Sp_0(A) = p_0(\psi(A)) = p_0(\psi(A) \cap H(z_0)) \cup p_0(\psi(A) \cap H(-z_0))$$

is a disjoint union. However, it is not clear whether $p_0(\psi(A) \cap H(z_0))$ and $p_0(\psi(A) \cap H(-z_0))$ are closed. To prove this set

$$B_0 = p_0(A \cap H(z_0)), \quad C_0 = p_0(A \cap H(-z_0)), \quad B_{n+1} = \psi(B_n C_n^2),$$

$$C_{n+1} = \psi(C_n B_n^2), \quad B = \cup B_n, \quad C = \cup C_n.$$

Then

$$B = p_0(\psi(A) \cap H(z_0)), \quad C = p_0(\psi(A) \cap H(-z_0)).$$

Since B_n and C_n are closed for all n, it suffices to show that $B = B_n$, $C = C_n$ for some n. In fact, it will turn out that this is the case for $n = 2s$, where $s = \mathrm{st}(\overline{X_0})$. This proof is done by induction:

If $s \leqq 1$, $Sp_0(A) = p_0(A) = B_0 \cup C_0$. Now suppose that $s > 1$. Assume that $B \neq B_{n+2}$ for some n. It will be shown that $n < 2(s - 1)$. Pick $T \in B \setminus B_{n+2}$. Since

$$\psi(F(T) \cap (B_{n+1} \cup C_{n+1})) \subset B_{n+2} \cup C_{n+2},$$

there is some $t \in k_0$ such that

$$T \in H(t) \quad \text{and} \quad F(T) \cap (B_{n+1} \cup C_{n+1}) \cap H(t) = \emptyset.$$

Choose t such that $w = v_t$ is the coarsest valuation arising in this way. k_w is the residue field of w, X_w the space of orderings of k_0 compatible with w, $\overline{X_w}$ the space of orderings of k_w induced by X_w, $p_w: X_w \to \overline{X_w}$ the natural mapping. As before we may assume that $B_0 \cup C_0 \subset X_w$. Suppose that

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press

$$p_w(B_n \cup C_n) = Sp_w(B_0 \cup C_0) = \psi\, p_w(B_0 \cup C_0).$$

As before,

$$Sp_w(B_0 \cup C_0) = p_w((B_n \cup C_n) \cap H(t)) \cup p_w((B_n \cup C_n) \cap H(-t))$$

is a disjoint union of closed subsets. As in the proof of Theorem 15, this shows that $T \notin S(B_0 \cup C_0) = B \cup C$, a contradiction. Therefore,

$$p_w(B_n \cup C_n) \neq Sp_w(B_0 \cup C_0),$$

and induction shows that $n < 2(s - 1)$ (since $\mathrm{st}(\overline{X_w}) \leqq s - 1$)).

Since the characterization of spaces X_P with $s(X_P) = 1$ is not complete one cannot expect a complete characterization for any higher values of $s(X_P)$. However, there are a few partial results:

An easy consequence of Example 3 is that $s(X_P) \leqq 2$ implies that for any valuation v with value group Γ_v such that $\Gamma_v \neq v(P^*)$ the space $\overline{X_v}$ (orderings of the residue field k_v induced by X_v) is connected or $\overline{X_v}$ has no infinite fans. My conjecture is that this property characterizes the spaces with $s(X_P) \leqq 2$.

CONJECTURE 2. *The following conditions are equivalent*:

(i) $s(X_P) \leqq 2$.

(ii) *For any valuation v with $\Gamma_v \neq v(P^*)$ either $\overline{X_v}$ is connected or $\overline{X_v}$ has no infinite fans.*

Suppose that condition (ii) of Conjecture 2 holds for the space X_P. If v is any valuation of k then condition (ii) is also true for $\overline{X_v}$.

It is useful to note the following:

LEMMA 18. *If condition* (ii) *of Conjecture* 2 *holds for the space X_P, if v is a valuation with $\Gamma_v \neq v(P^*)$ and if $\overline{X_v}$ has an infinite fan then X_v is a connected component of X_P.*

Proof. X_v is clearly connected since $\Gamma_v \neq v(P^*)$. Assume that there is some $T \in X_p \setminus X_v$ which is in the connected component of X_v. By [**4**, Theorem 2.7], there is a finest valuation w compatible with T and the orderings in X_v, and $\Gamma_w \neq w(P^*)$. Since w is coarser than v, $\overline{X_w}$ contains an infinite fan and is therefore connected (by condition (ii)). On the other hand, by the choice of w, the image of T in $\overline{X_w}$ cannot be related to any ordering in the image of X_v in $\overline{X_w}$. This is a contradiction.

Here, as in the case of Conjecture 1, there are only partial results supporting the nontrivial direction of the conjecture:

THEOREM 19. *Suppose that condition* (ii) *of Conjecture* 2 *holds in the space* X_P. *Then* $A = \phi\,\psi(A)$ *implies* $A = S(A)$.

Proof. Let $A = \phi\,\psi(A)$ and assume that there is some $T \in S(A) \setminus A$. Pick $z \in k^*$ such that

$$T \in H(z) \quad \text{and} \quad A \cap F(T) \cap H(z) = \emptyset.$$

Note that $A \cap F(T) \neq \emptyset$ by Corollary 3. Let $v = v_z$. By definition of v, $\Gamma_v \neq v(P^*)$. If $\overline{X_v}$ has no infinite fans,

$$p(A \cap X_v) = \phi\,\psi(p(A \cap X_v)) = Sp(A \cap X_v)$$

(Theorem 16). As in the proof of Theorem 16 one sees that

$$Sp(A \cap X_v) = p(A \cap X_v) = p(A \cap X_v \cap H(z)) \cup p(A \cap X_v \cap H(-z))$$

is a disjoint union of closed and open subsets. If $p(T) \in p(A \cap X_v)$, then, as in the proof of Theorem 15, one obtains $a \in A_v^*$ such that $T \in H(az)$ whereas $A \cap X_v \cap H(az) = \emptyset$. This shows that $T \notin S(A)$ (Corollary 13), a contradiction. Now suppose that $p(T) \notin p(A \cap X_v)$. As in the proof of Theorem 15, there is some $a \in A_v^*$ such that $A \cap X_v \subset H(-az)$. If $T \in H(az)$ then Corollary 13 leads to a contradiction. Otherwise, by $p(T) \notin p(A \cap X_v)$, one finds some $b \in A_v^*$ such that $T \in H(b)$ and $A \cap X_v \subset H(-b)$. Now, $T \in H(abz)$, $A \cap X_v \cap H(abz) = \emptyset$ gives the desired contradiction.

Now suppose that $\overline{X_v}$ has an infinite fan. X_v is a connected component (Lemma 18). As in the proof of Theorem 16, we may assume that $A \subset X_v$. Let w be the finest valuation compatible with X_v, $p:X_w \to \overline{X_w}$ the natural map. Then $p(A) = Sp(A)$. For, if there is some $S \in X_w$ such that $p(S) \in Sp(A) \setminus p(A)$, then there is some $t \in k_w^*$ such that

$$p(S) \in H(t) \quad \text{and} \quad p(A) \cap F(p(S)) \cap H(t) = \emptyset.$$

Let $u = v_t$. If $((\overline{X_w})_u)^-$ has an infinite fan then the remark after Conjecture 2 and Lemma 18 imply that $(\overline{X_w})_u = \overline{X_w}$. But then the composition of the valuations w and u is compatible with X_w and properly finer than w. This contradicts the choice of w. Thus, $((\overline{X_w})_u)^-$ does not have an infinite fan. Now the first part of the proof shows that also in this case there is a contradiction. Altogether, this proves $p(A) = Sp(A)$. Finally, the theorem follows by another application of the techniques of the proof of Theorem 16.

The proof of the next result requires the same techniques which have been used repeatedly in the proofs of the preceding results. Therefore it will be omitted.

THEOREM 20. *Suppose that* $n \in \mathbf{N}$ *and that* X_P *has the following property: For any valuation* v *with* $\Gamma_v \neq v(P^*)$ *either there is a valuation* w *properly finer than* v *with* $X_w = X_v$ *or* $s(\overline{X_v}) \leqq n$ *and* $A = \phi\,\psi(a)$ *implies* $A = S(A)$ *in* $\overline{X_v}$. *Then* $s(X_P) \leqq n + 3$ *if* n *is odd,* $s(X_P) \leqq n + 2$ *if* n *is even.*

An immediate corollary of this and Theorem 17 is

COROLLARY 21. *Suppose that for all valuations* v *with* $\Gamma_v \neq v(P^*)$ *either there is some valuation* w *properly finer than* v *with* $X_w = X_v$ *or* $\mathrm{st}(\overline{X_v})$ *is finite. Then* $s(X_P) \leqq 4$.

The paper concludes with an example showing that spaces X_P with $s(X_P) > 3$ do occur.

Example 4. Let $K = \mathbf{Q}(Y_1, Y_2, \dots)$, $K_i = \mathbf{R}((Y_1 - 1/2^i))((Y_2 - 1/2^i))\dots$, $i \in \mathbf{N}_0$. Each K_i carries a natural valuation which has a natural sequence of coarsenings. Let $f_i:K \rightarrow K_i$ be the natural embedding and u_i the valuation of K induced by the i-th coarsening of the natural valuation of K_i. The residue field of u_i is canonically isomorphic to $K^{(i)} = \mathbf{Q}(Y_1, \dots, Y_i)$. Let $\{s_1, s_2, \dots, t_1, t_2, \dots\} \subset \mathbf{R}$ be a subset which is algebraically independent over $\mathbf{Q}$. For each $i \in \mathbf{N}_0$, define a place $\mu_i:K \rightarrow \mathbf{R}$ by $\mu_i(\mathrm{Y}_j) = s_j$ for $j = 1, \dots, i$. Next, let $k = K(X_1, X_2, \dots)$. Embed k naturally into $L = \mathbf{R}((X_1))((X_2))\dots$. For $i \in \mathbf{N}_0$, let v_i be the valuation of k induced by the i-th coarsening of the natural valuation of L. The residue field of v_i is $k^{(i)} = K(X_1, \dots, X_i)$. $v_i:k \rightarrow k^{(i)}$ is the place belonging to v_i, and let $\rho_i:k^{(i)} \rightarrow \mathbf{R}$ be the placed defined by $\rho_i|K = \mu_i$, $\rho_i(X_j) = t_j$ for $j = 1, \dots, i$. Finally, $\varphi_i:k \rightarrow \mathbf{R}$ is the composition of the places v_i and ρ_i. Moreover, define $\varphi:k \rightarrow \mathbf{R}$ by $\varphi(Y_i) = s_i$, $\varphi(X_i) = t_i$ for all $i \in \mathbf{N}$. By using [**20**, Kap. 1], it is easy to verify that the subset $P = \{\varphi, \varphi_0, \varphi_1, \dots\} \subset \widetilde{X}_k$ is closed and has φ as its only accumulation point. Thus, $\lambda^{-1}(P) \subset X_k$ is a saturated subset (Corollary 3). Let $X_P = \lambda^{-1}(P)$. Let $F = \lambda^{-1}(\varphi)$, $F_i = \lambda^{-1}(\varphi_i)$ for all i. Then X_P has two connected components, namely F, which consists of the archimedean ordering T defined by φ, and $\cup F_i$. Let w_i be the valuation determined by φ_i, $W_i = \Gamma_{w_i} / w_i(P^*)$ and $V_i = \Gamma_{vi} / v_i(P^*)$. The character spaces $\chi(W_i)$ and $\chi(V_i)$ are subspaces of $\chi(k^* / P^*)$ with the following properties:

$$\chi(V_i) \subset \chi(W_i),\ \dim_{\mathbf{H}_2} \chi(V_i) = \infty,\ \dim_{\mathbf{H}_2} \chi(W_i) / \chi(V_i) = \infty,$$

$$\chi(V_{i+1}) \subsetneqq \chi(V_i),\ \chi(V_{i+1}) = \chi(W_{i+1}) \cap \chi(W_i).$$

For each $i \in \mathbf{N}_0$, fix a section $V_i \rightarrow W_i$ which in turn gives an epimorphism $h_i:\chi(W_i) \rightarrow \chi(V_i)$. Moreover, let

$$g_i : \chi(W_i) \to \chi(W_i) / \chi(V_i)$$

be natural. Now, for each $i \in \mathbf{N}_0$, fix some closed subset $A_i = \{T_{i0}, T_{i1}, \dots\} \subseteq \chi(W_i)$ such that $g_i(T_{i0}), g_i(T_{i1}), \dots$ are all different and linearly independent and the closed linear hull of $g_i(T_{i1}), g_i(T_{i2}), \dots$ is $\chi(W_i) / \chi(V_i)$ and such that

$$h_i(T_{i1}) = h_i(T_{i2}) = \dots = 1 \quad \text{and} \quad h_i(T_{i0}) \in \chi(V_i) / \chi(V_{i+1}).$$

For each i, fix some $T_i \in F_i$. Then $F_i = T_i\, \chi(W_i)$. Set

$$A = F \cup T_0 A_0 \cup T_1 A_1 \cup \dots \subset X_P.$$

By the topological structure of X_P this is a closed subset. Now,

$$A \subsetneqq \psi(A) \subsetneqq \phi\, \psi(A) \subsetneqq \psi\, \phi\, \psi(A) \subsetneqq \phi\, \psi\, \phi\, \psi(A).$$

For,

$$\psi(A) = F \cup \psi(T_0 A_0) \cup \psi(T_1 A_1) \cup \dots,$$

$$\phi\, \psi(A) = F \cup \phi\, \psi(T_0 A_0) \cup \dots,$$

$$T_0(\psi\, \phi\, \psi\, (A) \cap F_0) \cap \chi(V_0) = L(h_0(T_{00}), h_1(T_{11}), \dots),$$

which is not a closed subset of $\chi(V_0)$. This shows that the last of the above inequalities is in fact strict.

A construction similar to the one used in Example 3 can now be used to construct a space X of orderings with $s(X) \geqq 5$.

References

1. E. Becker and L. Bröcker, *On the description of the reduced Witt ring*, J. Alg. *52* (1978), 328-346.
2. E. Becker and E. Köing, *Reduzierte quadratische Formen*, Abh. Math. Sem. Univ. Hamb. *46* (1977), 143-177.
3. L. Bröcker, *Zur Theorie der quadratischen Formen über formal reellen Körpern*, Math. Ann. *210* (1974), 233-256.
4. ——— *Characterization of fans and hereditarily pythagorean fields*, Math. Z. *151* (1976), 149-163.
5. ——— *Über die Anzahl der Anordnungen eines kommutativen Körpers*, Arch. Math. *29* (1977), 458-464.
6. M. Coste and M.-F. Coste-Roy, *La topologie du spectre reel*, Preprint.
7. T. Craven, *Stability in Witt rings*, Transactions AMS *225* (1977), 227-242.
8. D. W. Dubois, *Infinite primes and ordered fields*, Dissertationes Math. *69* (1970).
9. R. Elman and T. Y. Lam, *Quadratic forms over formally real fields and pythagorean fields*, Amer. J. M. *94* (1972), 1155-1194.
10. R. Elman, T. Y. Lam and A. Wadsworth, *Pfister ideals in Witt-rings*, Math. Ann. *245* (1979), 219-245.
11. J. Kleinstein, *Abstract Witt rings*, Thesis.

12. ——— *Abstract Witt rings*, In: Conference on quadratic forms – 1976, Kingston, Ontario (1977).

13. J. Kleinstein and A. Rosenberg, *Signatures and semisignatures of abstract Witt rings and Witt rings of semilocal rings*, Can. J. Math. *30* (1978), 872-895.

14. M. Knebusch, A. Rosenberg and R. Ware, *Signatures on semilocal rings*, J. Alg. *26* (1973), 208-250.

15. T. Y. Lam, *The algebraic theory of quadratic forms* (Benjamin, Reading, Mass., 1973).

16. M. Marshall, *Classification of finite spaces of orderings*, Can. J. Math. *31* (1979), 320-330.

17. ——— *The Witt ring of a space of orderings*, Transactions AMS *258* (1980), 505-521.

18. ——— *Spaces of orderings IV*, Preprint.

19. A. Prestel, *Lectures on formally real fields*, IMPA, Rio de Janeiro (1975).

20. H.-W. Schülting, *Über reelle Stellen eines Körpers und ihren Holomorphiering*, Dissertation, Dortmund (1979).

Universität München,
München, W. Germany

https://doi.org/10.4153/CJM-1983-025-x Published online by Cambridge University Press