THE STRENGTH OF AN AXIOM OF FINITE CHOICE
FOR BRANCHES IN TREES

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Abstract. In their logical analysis of theorems about disjoint rays in graphs, Barnes, Shore, and the author (hereafter BGS) introduced a weak choice scheme in second-order arithmetic, called the Σ_1^1 axiom of finite choice (hereafter finite choice). This is a special case of the Σ_1^1 axiom of choice (Σ_1^1-AC$_0$) introduced by Kreisel. BGS showed that Σ_1^1-AC$_0$ suffices for proving many of the aforementioned theorems in graph theory. While it is not known if these implications reverse, BGS also showed that those theorems imply finite choice (in some cases, with additional induction assumptions). This motivated us to study the proof-theoretic strength of finite choice. Using a variant of Steel forcing with tagged trees, we show that finite choice is not provable from the Δ_1^1-comprehension scheme (even over ω-models). We also show that finite choice is a consequence of the arithmetic Bolzano–Weierstrass theorem (introduced by Friedman and studied by Conidis), assuming Σ_1^1-induction. Our results were used by BGS to show that several theorems in graph theory cannot be proved using Δ_1^1-comprehension. Our results also strengthen results of Conidis.

§1. Introduction. Reverse mathematics, as initiated by Friedman [5] in the 1970s, is a program in the foundations of mathematics which aims to find the axioms or formal systems needed in order to carry out proofs of particular mathematical theorems. Early results by Friedman, Simpson, and others showed that several basic theorems of analysis, algebra, and combinatorics, when formalized in second-order arithmetic, are equivalent to one of five sets of axioms, now known as the "Big Five." (The standard reference for subsystems of second-order arithmetic is [15].) These equivalences were proved over the base theory RCA$_0$, which formalizes computable mathematics and is the weakest of the Big Five. Later work revealed several exceptions to the Big Five phenomenon, including theorems that lie strictly between consecutive levels of the Big Five, and theorems that are incomparable to some level of the Big Five (for instance, Ramsey's theorem for pairs). In this paper, we study a weak form of the axiom of choice which lies strictly between consecutive levels of the Big Five, namely ACA$_0$ and ATR$_0$.

The study of choice schemes in second-order arithmetic predates reverse mathematics. In 1962, Kreisel [8] introduced the Σ_1^1 axiom of choice (Σ_1^1-AC), which asserts that

$$\forall n \exists X \varphi(n, X) \to \exists (X_n)_n \forall n \varphi(n, X_n)$$

for each formula $\varphi(n, X)$ in the language of second-order arithmetic which is arithmetical (i.e., has no set quantifiers). Equivalently,[1] it asserts that for any sequence of trees $(T_n)_n$, each of which has a branch (i.e., a ray in the sense of graph theory), one can choose a branch from each T_n.

Kreisel's interest in Σ_1^1-AC stemmed at least partially from its intimate connection with the hyperarithmetic sets. (For background on hyperarithmetic theory, see [13].) This connection arises from studying the ω-models (defined below) of $RCA_0 + \Sigma_1^1$-AC, or Σ_1^1-AC_0 for short.

A structure in the language of second-order arithmetic has the form

$$(N, S, +, \cdot, <, 0, 1, \in),$$

where $(N, +, \cdot, <, 0, 1)$ is a structure in the language of first-order arithmetic (called the *first-order part*), S is a collection of subsets of N (called the *second-order part*), and $\in$ is the membership relation between elements of N and elements of S. Such a structure is said to be an *ω-model* if its first-order part is the standard natural numbers $\mathbb{N}$ and the symbols $+, \cdot$, etc. have their standard interpretations. We identify an ω-model with its second-order part.

Kreisel [8] showed that the ω-model HYP consisting of all hyperarithmetical subsets of $\mathbb{N}$ satisfies Σ_1^1-AC_0, and furthermore, HYP is the minimum ω-model of Σ_1^1-AC_0. This is analogous to the well-known facts that ACA_0 has a minimum ω-model ARITH consisting of all arithmetical subsets of $\mathbb{N}$, and RCA_0 has a minimum ω-model REC consisting of all recursive (a.k.a. computable) subsets of $\mathbb{N}$. In fact, Friedman [5] observed that a subset of $\mathcal{P}(\mathbb{N})$ is a model of RCA_0 (ACA_0, respectively) *if and only if* it is closed under (effective) join $\oplus$ and Turing reduction $\leq_T$ (and the Turing jump operator, respectively). In summary:

Theory	Minimum ω-model	Closure of ω-models
RCA_0	REC	$\oplus, \leq_T$
ACA_0	ARITH	$\oplus, \leq_T$, Turing jump
Σ_1^1-AC_0	HYP	

One might hope to fill in the final entry of the table with "$\oplus$, hyperarithmetical reduction." Every ω-model of Σ_1^1-AC_0 is closed under join and hyperarithmetical reduction [8]. However, not every subset of $\mathcal{P}(\mathbb{N})$ which is closed under join and hyperarithmetical reduction satisfies Σ_1^1-AC_0. The issue is not that Σ_1^1-AC_0 is too strong; rather, there is *no theory* T such that the ω-models of T are exactly those which are closed under join and hyperarithmetical reduction [17, 2.2.2]. This motivated the following definition:

DEFINITION 1.1 (Steel [16], Montalbán [9]). A theory T is a *theory of hyperarithmetic analysis* (*THA*) if:

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[1]To prove this, use [15, V.5.4] and observe that Σ_1^1-AC_0 and the statement about trees each imply ACA_0 over RCA_0.

– every ω-model of T is closed under join and hyperarithmetic reduction, and
– T holds in $\mathrm{HYP}(Y)$ for every $Y \subseteq \mathbb{N}$ (here $\mathrm{HYP}(Y)$ consists of all subsets of $\mathbb{N}$ which are hyperarithmetically reducible to Y).

Before we discuss examples of THAs, note that each THA T characterizes the notion of hyperarithmetic reduction in the following sense: A set $X \subseteq \mathbb{N}$ is hyperarithmetically reducible to a set $Y \subseteq \mathbb{N}$ if and only if every ω-model of T which contains Y also contains X.

The aforementioned results of Kreisel [8] relativize to show that $\Sigma_1^1\text{-AC}_0$ is a THA. In the 60s and 70s a number of THAs were identified and proved to be nonequivalent [4, 16, 17]. These THAs are natural fragments of well-studied axiom schemes such as choice schemes or comprehension schemes. In 2006, Montalbán [9] was the first to identify a THA which is a published theorem not stated using concepts from logic (such as first-order formulas). This THA, known as INDEC, is a result of Jullien (see [9]) about indecomposability of linear orderings. Other works on THAs include [2, 10–12].

Recently Barnes, Goh, and Shore [1] (hereafter BGS) identified several theorems of graph theory which are THAs. These theorems are variations of classical results of Halin [6] on disjoint rays in graphs. For simplicity we only discuss the oldest of these results (see [3, Theorem 8.2.5(i)] for its statement), which we call Halin's theorem. (Much of the discussion below applies to variations of Halin's theorem as well; see [1].) In their proof that Halin's theorem is a THA, BGS [1] showed that Halin's theorem is provable using $\Sigma_1^1\text{-AC}_0$, and conversely, Halin's theorem implies the following weakening of $\Sigma_1^1\text{-AC}_0$ (assuming additional induction axioms on top of what is available in RCA_0):

DEFINITION 1.2 (BGS [1]).   *Finite-$\Sigma_1^1\text{-AC}_0$* consists of RCA_0 and

$$(\forall n)(\exists \text{ nonzero finitely many } X)\varphi(n, X) \to (\exists (X_n)_n)(\forall n)\varphi(n, X_n)$$

for each arithmetical formula $\varphi(n, X)$. Formally, "($\exists$ nonzero finitely many $X)\varphi(n, X)$" means that there is a nonempty sequence $(X_i)_{i<j}$ such that for each X, $\varphi(n, X)$ holds if and only if $X = X_i$ for some $i < j$.

Henceforth we will refer to finite-$\Sigma_1^1\text{-AC}_0$ as *finite choice*.

In this paper we present implications and nonimplications between finite choice and other known THAs (Theorems 1.3 and 1.5). First we ought to discuss why finite choice is a THA. To show that a theory is a THA, it suffices to show that it is sandwiched between two THAs. Of course $\Sigma_1^1\text{-AC}_0$ implies finite choice. On the other hand, finite choice implies another known THA: *unique-$\Sigma_1^1\text{-AC}_0$*, which asserts (in addition to RCA_0) that

$$\forall n \exists ! X\varphi(n, X) \to \exists (X_n)_n \forall n \varphi(n, X_n)$$

for each arithmetical formula $\varphi(n, X)$. Unique-$\Sigma_1^1\text{-AC}_0$ has appeared in the literature with names such as arithmetical replacement $\Pi_{(\omega)}^0\text{-RA}$ [17, p. 6], Π_0^1-replacement [16, p. 74], and most commonly weak-$\Sigma_1^1\text{-AC}_0$ (e.g., [15, VIII.4.12]). A proof that unique-$\Sigma_1^1\text{-AC}_0$ is a THA can be found in [15, VIII.4.15–16].

Since finite choice is sandwiched between $\Sigma_1^1\text{-AC}_0$ and unique-$\Sigma_1^1\text{-AC}_0$, we can calibrate its proof-theoretic complexity by comparing it with other THAs in this

position. One such THA is the Δ_1^1-comprehension scheme (Δ_1^1-CA$_0$), which asserts (in addition to RCA$_0$) that

$$\forall n(\varphi(n) \leftrightarrow \neg\psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

for any $\varphi(n)$ and $\psi(n)$ which are Σ_1^1. It is easy to see that Σ_1^1-AC$_0 \rightarrow \Delta_1^1$-CA$_0 \rightarrow$ unique-Σ_1^1-AC$_0$. In order to separate Δ_1^1-CA$_0$ and Σ_1^1-AC$_0$, Steel [16] introduced a new forcing notion and used it to construct an ω-model which satisfies Δ_1^1-CA$_0$ but not Σ_1^1-AC$_0$. Van Wesep [17] modified Steel's methods to construct an ω-model which satisfies unique-Σ_1^1-AC$_0$ but not Δ_1^1-CA$_0$. Subsequently various modifications of Steel forcing have been used to separate THAs [2, 9–12]. Using yet another variant of Steel forcing (Section 3.3), we shall prove the following:

THEOREM 1.3. *There is an ω-model which satisfies Δ_1^1-CA$_0$ but not finite choice. Therefore Δ_1^1-CA$_0$ does not imply finite choice, even if additional induction axioms are assumed.*

In particular, finite choice is strictly stronger than unique-Σ_1^1-AC$_0$. Theorem 1.3 was used by BGS [1] to show that Halin's theorem and many of its variants cannot be proved using Δ_1^1-CA$_0$ (even if additional induction axioms are assumed).

To show that finite choice is strictly weaker than Σ_1^1-AC$_0$, we will establish a connection between finite choice and a known THA: the arithmetic Bolzano–Weierstrass theorem ABW introduced by Friedman [5]. Our statement of ABW below follows Conidis [2, p. 4470].

DEFINITION 1.4. The *arithmetic Bolzano–Weierstrass theorem* (ABW) states that if $A(X)$ is an arithmetic predicate on 2^N, then either $A(X)$ has finitely many solutions, or the set $\{X \in 2^N : A(X)\}$ of A-solutions has an accumulation point.

Friedman asserted that ABW follows from Σ_1^1-AC$_0$. Conidis [2, Theorem 2.1(2)] furnished a proof of that statement. In addition, [2, Theorem 2.1(4)] showed that ABW implies unique-Σ_1^1-AC$_0$ over the base theory consisting of RCA$_0$ and the induction scheme for Σ_1^1 formulas (denoted IΣ_1^1). (To the best of our knowledge it is not known if ABW implies unique-Σ_1^1-AC$_0$ over just RCA$_0$. Nevertheless, as every ω-model satisfies IΣ_1^1, Conidis's result shows that every ω-model of RCA$_0$ + ABW satisfies unique-Σ_1^1-AC$_0$.) We will strengthen Conidis's result by proving the following (Section 2):

THEOREM 1.5. ABW *implies finite choice over* RCA$_0$ + IΣ_1^1.

It follows that finite choice is strictly weaker than Σ_1^1-AC$_0$, as RCA$_0$ + IΣ_1^1 + ABW does not imply Σ_1^1-AC$_0$ or even Δ_1^1-CA$_0$ (as witnessed by Van Wesep's model, see [17, Lemma 1.4] and [2, Theorem 4.7]).

Having established a connection between ABW and finite choice (Theorem 1.5), our Theorem 1.3 gives an alternate, arguably simpler, proof of Conidis's [2, Theorem 3.1] result that Δ_1^1-CA$_0$ does not imply ABW. (See Remark 3.1.)

§2. Arithmetic Bolzano–Weierstrass. We adapt a proof of Conidis [2, Theorem 2.1(4)] to prove Theorem 1.5, which asserts that RCA$_0$ + IΣ_1^1 + ABW $\vdash$ finite choice.

PROOF OF THEOREM 1.5.   Suppose that $\varphi(n, Y)$ is an arithmetical predicate which is an instance of finite choice, i.e., for each n, $\varphi(n, Y)$ has finitely many solutions. Without loss of generality, we may assume that $\varphi(n, \emptyset)$ always fails.

Define an arithmetical predicate A on 2^N as follows: $A(\langle X_n \rangle_n)$ holds if

$$\exists n_0((\forall n \leq n_0)[\varphi(n, X_n)] \wedge (\forall n > n_0)[X_n = \emptyset]).$$

Using $\mathsf{I}\Sigma_1^1$ and the assumption that for each n, $\varphi(n, Y)$ has a solution, we can show that for each n_0, $A(\langle X_n \rangle_n)$ has at least n_0 distinct solutions. (Note if X_n is $\emptyset$, then it does not code a solution to $\varphi(n, \cdot)$.) Hence $A(\langle X_n \rangle_n)$ is an instance of ABW.

By ABW, the set $\{\langle X_n \rangle_n : A(\langle X_n \rangle_n)\}$ has an accumulation point Y. Interpret Y as a sequence $\langle Y_n \rangle_n$. We claim that for all n, $\varphi(n, Y_n)$ holds.

Suppose towards a contradiction that $\varphi(k, Y_k)$ fails. Since $\varphi(k, \cdot)$ has only finitely many solutions, there is some m sufficiently large such that $Y_k \restriction m \neq Y \restriction m$ for every Y such that $\varphi(k, Y)$ holds.

Now, by our choice of $\langle Y_n \rangle_n$, there are infinitely many $\langle X_n \rangle_n$ satisfying A such that X_k extends $Y_k \restriction m$. For any such $\langle X_n \rangle_n$, $\varphi(k, X_k)$ fails, so by definition of A, $X_n = \emptyset$ for all $n \geq k$.

But for each $n < k$, $\varphi(n, \cdot)$ has only finitely many solutions, so there cannot be infinitely many $\langle X_n \rangle_n$ satisfying the above conditions. Contradiction. We have showed that $\langle Y_n \rangle_n$ is a finite choice solution to $\varphi(n, Y)$.   ⊣

We do not know if finite choice implies ABW.

§3. Δ_1^1-comprehension does not imply finite choice.

The rest of the paper is devoted to proving Theorem 1.3 using a new variant of Steel's [16] tagged tree forcing. We assume familiarity with forcing in arithmetic (see, e.g., [14, Section 3]). A helpful reference for forcing in arithmetic over ramified languages is [13, III.4 and IV.3].

We begin by fixing some notation regarding trees. By a *tree* we mean a subset of $\mathbb{N}^{<\mathbb{N}}$ which is closed under prefix. An element f of $\mathbb{N}^{\mathbb{N}}$ is a *branch through a tree* T, written $f \in [T]$, if every prefix of f lies in T. If some σ in T is a prefix of a branch through T, we say that σ is *extendible* in T. The empty string is denoted by $\emptyset$. The length of a string σ is denoted by $|\sigma|$. If σ is a prefix of τ (or a branch f), we write $\sigma \subseteq \tau$ ($\sigma \subseteq f$, respectively). If σ is a nonempty string, σ^- denotes the prefix of σ of length $|\sigma| - 1$. If σ is a string and T is a tree, $\sigma \cap T$ denotes the longest prefix of σ which lies in T.

3.1. The model.

We shall construct an ω-model $M_\infty \subset \mathcal{P}(\mathbb{N})$ which satisfies Δ_1^1-CA$_0$ but not finite choice. The basic setup follows [10].

To define M_∞ we will construct a generic object

$$\langle T^G, \{f_i^G : i \in \mathbb{N}\}, h^G \rangle,$$

where T^G is a tree, each f_i^G is a distinct branch through T^G, and $h^G : T^G \to \omega_1^{CK} \cup \{\infty\}$ is the well-founded rank function on T^G, i.e., for all σ in the well-founded part of T^G, we have $h^G(\sigma) = \sup\{h^G(\tau) + 1 : \tau \in T^G, \tau \supseteq \sigma\}$, while for all σ which is extendible in T^G, we have $h^G(\sigma) = \infty$. Our convention is that $\infty = \infty + 1$ and $\infty > \infty > \alpha$, so $h^G(\sigma) = \sup\{h^G(\tau) + 1 : \tau \in T^G, \tau \supseteq \sigma\}$ for all σ in T^G.

Then, for each finite $F \subset \mathbb{N}$ (written $F \subset_f \mathbb{N}$), we define

$$M_F = \{X \subseteq \mathbb{N} : \exists \mu < \omega_1^{CK}(X \leq_T (T^G \oplus \langle f_i^G \rangle_{i \in F})^{(\mu)})\},$$

where $Y^{(\mu)}$ denotes the transfinite iteration of the Turing jump of Y up to level μ. (For background on transfinite iterations of the Turing jump, see [13]. Later, we will express M_F as the sets which are computable in $H_{\mu,F}$ for some $\mu < \omega_1^{CK}$, defined below.) Notice that μ ranges over the computable ordinals, rather than the ordinals which are computable in $\omega_1^{T^G \oplus \langle f_i^G \rangle_{i \in F}}$. Nonetheless we will show in Corollary 3.14 that $M_F = \mathrm{HYP}(T^G \oplus \langle f_i^G \rangle_{i \in F})$.

Finally, our desired ω-model is defined by

$$M_\infty = \bigcup_{F \subset_{\mathrm{f}} \mathbb{N}} M_F.$$

Notice that h^G does not appear in the definitions of M_F or M_∞. Nonetheless it will play a crucial role in showing that M_∞ has the properties we desire.

Let us sketch why finite choice fails in M_∞. We will show (Lemma 3.9) that for each $F \subset_{\mathrm{f}} \mathbb{N}$, $M_F \cap [T^G] = \{f_i^G : i \in F\}$. This implies that the branches through T^G in M_∞ are exactly the branches f_i^G, for $i \in \mathbb{N}$. This also implies that M_∞ does not contain any infinite sequence of distinct branches f_i^G, for such a sequence has to lie in some M_F, contradicting Lemma 3.9.

This allows us to construct a failure of finite choice: For each n, let T_n be the subtree of T^G passing through $\langle n \rangle$. Our forcing will ensure that each $[T_n]$ contains (nonzero) finitely many f_i^G. Hence M_∞ thinks that $\langle T_n \rangle_n$ is an instance of finite choice. Any finite choice solution to $\langle T_n \rangle_n$ would be an infinite sequence of distinct branches f_i^G (they are distinct because the nth branch has first entry n), and hence would not lie in M_∞, by the previous paragraph.

In preparation for describing the forcing, we shall give every element of M_∞ a name. Define by recursion on $\mu < \omega_1^{CK}$:

$$H_{1,F} = T^G \oplus \langle f_i^G \rangle_{i \in F}, \quad S_{\mu,F,e} = W_e^{H_{\mu,F}}, \quad H_{\mu,F} = \bigoplus_{v < \mu, e \in \mathbb{N}} S_{v,F,e}$$

for $\mu < \omega_1^{CK}$, $F \subset_{\mathrm{f}} \mathbb{N}$, $e \in \mathbb{N}$. (Here W_e^Y is the eth computably enumerable set relative to the oracle Y.)

3.2. The forcing language. We consider a ramified language $\mathcal{L}_\infty$ which extends the language of second-order arithmetic with constants for each element of M_∞, and various types of restricted set variables.

In order to define $\mathcal{L}_\infty$, we first define a language $\mathcal{L}_F$ for each $F \subset_{\mathrm{f}} \mathbb{N}$. $\mathcal{L}_F$ consists of first-order formulas in the language of second-order arithmetic generated by the following set variables and constants: for each $D \subseteq F$, there are unranked set variables of the form X_D and ranked set variables of the form X_D^λ for each $\lambda < \omega_1^{CK}$, while the constants are intended to name every element of M_F:

  – $\mathbf{T}, \mathbf{f}_i$ for $i \in F$;
  – for each $v < \omega_1^{CK}$, $\mathbf{H}_{v,F}$ and $\mathbf{S}_{v,F,e}$ for each $e \in \mathbb{N}$.

Let C_F denote the above set of constants. If $\mathbf{S}$ is of the form $\mathbf{S}_{v,F,e}$, we define $\mathrm{dom}(\mathbf{S})$ to be F. We define C^μ to be the set of all constants of the form $\mathbf{H}_{v,F}$ or $\mathbf{S}_{v,F,e}$ for some $v < \mu$.

The language $\mathcal{L}_\infty$ consists of $\bigcup_{F \subset_{\mathrm{f}} \mathbb{N}} \mathcal{L}_F$, unranked set variables of the form X, and ranked set variables of the form X^λ for each $\lambda < \omega_1^{CK}$.

Each formula ψ of $\mathcal{L}_\infty$ can be assigned an ordinal rank as follows:

$$\mathrm{rk}(\psi) = \omega_1^{CK} \cdot u(\psi) + \omega^2 \cdot o(\psi) + \omega \cdot r(\psi) + n(\psi),$$

where:

- $u(\psi) < \omega$ denotes the number of unranked quantifiers in ψ;
- $o(\psi) < \omega_1^{CK}$ denotes the least upper bound of

$$\{v : v \text{ is the superscript of a bound variable in } \psi\}$$
$$\cup \{v + 1 : \text{ some constant of the form } \mathbf{S}_{v,F,e} \text{ or } \mathbf{H}_{v,F} \text{ occurs in } \psi\};$$

- $r(\psi) < \omega$ denotes the number of ranked set quantifiers in ψ;
- $n(\psi) < \omega$ denotes the number of connectives.

We say that a formula ψ of $\mathcal{L}_\infty$ is *ranked* if all of its bound variables are ranked, or equivalently, if $\mathrm{rk}(\psi) < \omega_1^{CK}$.

A variable of the form X_D^λ or X_D is *F-restricted* if $D \subseteq F$. A constant of the form $H_{\mu,D}$ or $S_{\mu,D,e}$ is *F-restricted* if $D \subseteq F$. A formula of $\mathcal{L}_\infty$ is *F-restricted* if all of its bound variables and constants are *F*-restricted.

A formula is Σ-*over*-$\mathcal{L}_F$ if it is built up from ranked *F*-restricted formulas using $\wedge$, $\forall n$, and $\exists X$.

For any formula ψ and any $\mu < \omega_1^{CK}$, we define ψ^μ by replacing every unranked set variable in ψ with its ranked counterpart, i.e., X is replaced by X^μ and X_F is replaced by X_F^μ.

3.3. The forcing notion. The forcing $\mathbb{P}$ consists of tuples

$$p = \langle T^p, f^p, h^p \rangle$$

such that:

(C1) $T^p \subseteq \mathbb{N}^{<\mathbb{N}}$ is a finite tree;

(C2) f^p is a finite partial function from $\mathbb{N}$ to $T^p \backslash \{\emptyset\}$ such that each $\sigma \in T^p$ of length 1 is an initial segment of some $f^p(i)$
(we view f^p as a finite collection of finite paths in T^p);

(C3) $h^p : T^p \rightarrow \omega_1^{CK} \cup \{\infty\}$ satisfies the following:
   (a) h^p is a *rank function*, i.e., if $\tau \subsetneq \sigma$, then $h^p(\tau) > h^p(\sigma)$
      (by fiat $\infty > \infty > \alpha$ for every α);
   (b) for all $i \in \mathrm{dom}(f^p)$, $h^p(f^p(i)) = \infty$;
   (c) $h^p(\emptyset) = \infty$.

We think of h^p as a *labeling* or *tagging* of nodes in T^p. Condition (C2) is novel; ordinary Steel forcing only requires that f^p is a finite partial function from $\mathbb{N}$ to T^p. We impose (C2) in order to ensure that every node of length 1 is extendible in the generic tree T^G. Note that by (C2) and (C3)(b), for each σ in T^p of length 1, we must have $h^p(\sigma) = \infty$. Therefore, if T^p contains any node of length 1, then (C3)(c) follows from (C2) and (C3)(b). There may be nodes σ (of length greater than 1) such that $h^p(\sigma) = \infty$ yet there is no i such that $\sigma \subseteq f^p(i)$.

We say that q *extends* p, written $q \leq p$, if:

(E1) $T^q \supseteq T^p$;

(E2) for all $i \in \mathrm{dom}(f^p)$, $f^q(i)$ is defined and $f^p(i) = f^q(i) \cap T^p$
(old paths cannot be extended in the old tree);

(E3) for all $i \in \mathrm{dom}(f^q) \backslash \mathrm{dom}(f^p)$, $f^q(i) \cap T^p = \emptyset$
(new paths can only intersect the old tree at the root);

(E4) $h^q \supseteq h^p$.

Conditions (E2) and (E3) are needed for us to control the complexity of the forcing relation. Condition (E2) is present in ordinary Steel forcing [16, p. 68]. One could weaken (E2) by considering only conditions p where each $f^p(i)$ is a leaf in T^p (such conditions are dense in the above forcing): In this case we could replace (E2) by "for all $i \in \mathrm{dom}(f^p)$, $f^q(i)$ is defined and $f^p(i) \subseteq f^q(i)$." Condition (E3) is not in [16] but needs to be added to it, as pointed out in [10, Section 2.2].

REMARK 3.1. Conidis [2, Section 3.2] used a more complicated variant of Steel forcing (with locks on certain nodes of T^p which can be toggled on and off) to prove that Δ_1^1-CA$_0$ does not imply ABW$_0$. Theorems 1.3 and 1.5 together provide an alternate proof of his result.

We will take G to be a sufficiently $\mathbb{P}$-generic filter (G must decide all Σ-over-$\mathcal{L}_F$ formulas for all $F \subset_\mathrm{f} \mathbb{N}$, as well as all formulas of the form $\forall n(\psi(n) \leftrightarrow \neg\varphi(n))$, where φ and ψ are Σ-over-$\mathcal{L}_F$ for some $F \subset_\mathrm{f} \mathbb{N}$). Then we may define $T^G = \bigcup_{p \in G} T^p$, $f_i^G = \bigcup_{p \in G} f^p(i)$ for $i \in \mathbb{N}$, and $h^G = \bigcup_{p \in G} h^p$. By genericity, each f_i^G is an infinite branch in T^G, the branches f_i^G are distinct, and h^G is the well-founded rank function on T^G.

We encode each condition as a number as follows. Fix a computable linear ordering with well-founded part ω_1^{CK}. (Such orderings are known as pseudo-well-orderings; see [7].) We identify each $\alpha < \omega_1^{CK}$ with its corresponding element in the well-founded part. When we write $\alpha < \beta$, we always refer to their order as ordinals rather than the natural number ordering. Using the above identification and standard pairing functions, we may encode $\mathbb{P}$ as a Π_1^1 subset of $\mathbb{N}$. For each $\alpha < \omega_1^{CK}$, define $\mathbb{P}_\alpha$ to be the set of all conditions p such that the range of h^p is contained in $\alpha \cup \{\infty\}$. We have $\mathbb{P} = \bigcup_{\alpha < \omega_1^{CK}} \mathbb{P}_\alpha$. Each $\mathbb{P}_\alpha$ is computable, uniformly in α.

3.4. The forcing relation. The forcing relation for formulas in $\mathcal{L}_\infty$ is defined by recursion as follows:

(1) for quantifier-free formulas of arithmetic ψ, $p \Vdash \psi$ if and only if ψ is true;
(2) $p \Vdash \sigma \in \mathbf{T}$ if either $|\sigma| < 2$ and $\sigma \in T^p$, or $\sigma^- \in T^p$ and $h^p(\sigma^-) \geq 1$;
(3) $p \Vdash \langle \mathbf{n}, \mathbf{m} \rangle \in \mathbf{f}_i$ if $i \in \mathrm{dom}(f^p)$ and $f^p(i)(n) = m$;
(4) $p \Vdash \langle 0, \sigma \rangle \in \mathbf{H}_{1,F}$ if $p \Vdash \sigma \in \mathbf{T}$;
(5) $p \Vdash \langle 1, \langle \mathbf{i}, \langle \mathbf{n}, \mathbf{m} \rangle \rangle \rangle \in \mathbf{H}_{1,F}$ if $i \in F$ and $p \Vdash \langle \mathbf{n}, \mathbf{m} \rangle \in \mathbf{f}_i$;
(6) for $v > 1$, $p \Vdash \mathbf{n} \in \mathbf{S}_{v,F,e}$ if $p \Vdash \exists s R(\mathbf{H}_{v,F}; \mathbf{e}, s, \mathbf{n})$ where R codes a universal Turing machine;
(7) for $v > 1$, $p \Vdash \langle \mathbf{e}, \mathbf{n}, \mu \rangle \in \mathbf{H}_{v,F}$ if $\mu < v$ and $p \Vdash \mathbf{n} \in \mathbf{S}_{\mu,F,e}$;
(8) $p \Vdash \forall x \psi(x)$ if for all $n \in \mathbb{N}$, $p \Vdash \psi(\mathbf{n})$;
(9) $p \Vdash \forall X_F^\lambda \psi(X_F^\lambda)$ if for all $v < \lambda$, $e \in \mathbb{N}$, $p \Vdash \psi(\mathbf{S}_{v,F,e})$;
(10) $p \Vdash \forall X^\lambda \psi(X^\lambda)$ if for all $v < \lambda$, $e \in \mathbb{N}$, $F \subset_\mathrm{f} \mathbb{N}$, $p \Vdash \psi(\mathbf{S}_{v,F,e})$;
(11) $p \Vdash \forall X_F \psi(X_F)$ if for all $v < \omega_1^{CK}$, $e \in \mathbb{N}$, $p \Vdash \psi(\mathbf{S}_{v,F,e})$;
(12) $p \Vdash \forall X \psi(X)$ if for all $v < \omega_1^{CK}$, $e \in \mathbb{N}$, $F \subset_\mathrm{f} \mathbb{N}$, $p \Vdash \psi(\mathbf{S}_{v,F,e})$;
(13) $p \Vdash \varphi \wedge \psi$ if $p \Vdash \varphi$ and $p \Vdash \psi$;
(14) $p \Vdash \neg\psi$ if for every $q \leq p$, $q \nVdash \psi$.

LEMMA 3.2. *The following facts hold of* $\Vdash$:

(1) *If* $p \Vdash \psi$, *then for every sufficiently generic filter* G *which contains* p, M_∞ *satisfies* ψ.
(2) *If* G *is sufficiently generic and* M_∞ *satisfies* ψ, *then there is some* $p \in G$ *such that* $p \Vdash \psi$.

Analogous facts hold for $\psi \in \mathcal{L}_F$ *and each model* M_F.

Just as we encoded each forcing condition as a number, we may also encode each formula in $\mathcal{L}_\infty$ as a number. We will show that the forcing relation for certain classes of formulas (encoded as a relation on numbers) can be computed within M_∞ (Lemma 3.12).

3.5. Analyzing the forcing relation for ranked formulas. In order to control the complexity of the forcing relation, we will show that conditions that are sufficiently similar to each other force the same formulas of sufficiently low complexity. The basic notion of similarity in Steel forcing is known as *retagging*:

DEFINITION 3.3 [16, Definition 4]. Suppose $\mu < \omega_1^{CK}$ and $F \subset_f \mathbb{N}$. If p and p^* are conditions, we say that they are *μ-F-retaggings* if:

– $T^p = T^{p^*}$ and $f^p \upharpoonright F = f^{p^*} \upharpoonright F$;
– h^p and h^{p^*} agree on labels strictly smaller than μ;
– if $h^p(\sigma) \geq \mu$, then $h^{p^*}(\sigma) \geq \mu$.

We make some observations:

– μ-F-retagging is an equivalence relation on conditions.
– If p and p^* are μ-F-retaggings, then for any $\mu' \leq \mu$ and any $F' \subseteq F$, p and p^* are μ'-F'-retaggings as well.

Central to our analysis will be the ability to replace a quantifier over $\mathbb{P}$ with a quantifier over some subset $\mathbb{P}_\beta$ (recall that $\mathbb{P}$ is Π_1^1 while $\mathbb{P}_\beta$ is computable). Once we have established connections between retagging and the forcing relation, the following lemma will be useful for achieving the above:

LEMMA 3.4. *For any* $q \in \mathbb{P}$ *and any* $\beta < \omega_1^{CK}$, *there is some* $q^* \in \mathbb{P}_\beta$ *such that*:

– q *and* q^* *are* β-F-retaggings *for every* $F \subset_f \mathbb{N}$;
– *for every* $p \in \mathbb{P}_\beta$ *such that* $q \leq p$, *we have* $q^* \leq p$.

PROOF. Define $T^{q^*} = T^q$, $f^{q^*} = f^q$, and

$$h^{q^*}(\tau) = \begin{cases} \infty, & h^q(\tau) \geq \beta, \\ h^q(\tau), & h^q(\tau) < \beta. \end{cases}$$

It is easy to see that q^* satisfies the desired properties. $\dashv$

A cornerstone of the method of Steel forcing is the following basic retagging lemma.

LEMMA 3.5. *Let* p *and* p^* *be* $\omega\beta$-F-retaggings. *Then for all* $q \leq p$ *and all* $\gamma < \beta$, *there exists* $q^* \leq p^*$ *such that* q *and* q^* *are* $\omega\gamma$-F-retaggings. *Furthermore, if* $\mathrm{dom}(f^{p^*}) \subseteq F$ (*and hence* $f^{p^*} \subseteq f^p$), *then we can choose* q^* *such that* $f^{q^*} \subseteq f^q$.

The first part of the lemma is identical in form to [10, Lemma 2.5]. The second part has no analog in [10], but its analog would follow immediately from the proof of [10, Lemma 2.5]. For our forcing we will construct q^* (specifically f^{q^*}) slightly differently in order to prove the second part. The second part will be used in the proof of Lemma 3.11.

Proof. Define q^* as follows:

- $T^{q^*} = T^q$;
- in general, we define f^{q^*} to be the disjoint union of:
  - $f^q \restriction F$,
  - $f^{p^*} \restriction (\mathrm{dom}(f^{p^*}) \backslash (\mathrm{dom}(f^q) \cap F))$, and
  - a function g such that $\mathrm{dom}(g)$ is disjoint from $F \cup \mathrm{dom}(f^{p^*})$ and $\mathrm{range}(g) = \{\sigma \in T^{q^*} \backslash T^{p^*} : |\sigma| = 1\}$;
- if $\mathrm{dom}(f^{p^*}) \subseteq F$ and we want to satisfy the second part of the lemma, we define $f^{q^*} = f^q \restriction (F \cup \{i : f^q(i) \cap T^p = \emptyset\})$;
- h^{q^*} is defined by cases:

$$h^{q^*}(\tau) = \begin{cases} h^{p^*}(\tau), & \text{if } \tau \in T^p, \\ \infty, & \text{if } \exists i (\tau \subseteq f^{q^*}(i)), \\ h^q(\tau), & \text{if } h^q(\tau) < \omega\gamma, \\ \omega\gamma + |\tau|_Q, & \text{otherwise}, \end{cases}$$

where $Q = \{\sigma \in T^q : h^q(\sigma) \geq \omega\gamma\}$ and $|\tau|_Q \in \mathbb{N}$ denotes the rank of τ in the finite tree Q.

h^{q^*} *is well-defined.* First, we show that the second and third cases in the definition of h^{q^*} are mutually exclusive. Suppose $\tau \subseteq f^{q^*}(i)$ for some i.

Case 1. If $f^{q^*}(i) = f^q(i)$, then $h^q(\tau) = \infty > \omega\gamma$.

Case 2. If $f^{q^*}(i) = f^{p^*}(i)$, then $\tau \in T^p$ and $h^{p^*}(\tau) = \infty$. Since p and p^* are $\omega\beta$-F-retaggings, we have $h^p(\tau) \geq \omega\beta$. It follows that $h^q(\tau) = h^p(\tau) \geq \omega\beta > \omega\gamma$.

Case 3. Otherwise, $|f^{q^*}(i)| = 1$. Then $h^q(\tau) = \infty > \omega\gamma$.

Second, in order to show that the first and second cases in the definition of h^{q^*} do not conflict, we shall show that if $\tau \in T^p$ and $\tau \subseteq f^{q^*}(i)$ for some i, then $h^{p^*}(\tau) = \infty$.

Case 1. Suppose $f^{q^*}(i) = f^q(i)$ and $i \in F$. *1a.* If $f^q(i) \cap T^p = \emptyset$, then $\tau = \emptyset$ so $h^{p^*}(\tau) = \infty$. *1b.* Otherwise, by (E2) and (E3), we have $f^q(i) \cap T^p = f^p(i)$. Since p and p^* are $\omega\beta$-F-retaggings, we have $f^{p^*}(i) = f^p(i)$. It follows that $\tau \subseteq f^q(i) \cap T^p = f^p(i) = f^{p^*}(i)$, implying $h^{p^*}(\tau) = \infty$.

Case 2. If $f^{q^*}(i) = f^{p^*}(i)$, then $h^{p^*}(\tau) \geq h^{p^*}(f^{p^*}(i)) = \infty$.

Case 3. Otherwise, $|f^{q^*}(i)| = 1$. Then $h^{p^*}(\tau) = \infty$.

To see that the first and third cases in the definition of h^{q^*} do not conflict, suppose $\tau \in T^p$ and $h^q(\tau) < \omega\gamma$. We have $h^p(\tau) = h^q(\tau) < \omega\gamma$, so by retagging, $h^{p^*}(\tau) = h^p(\tau) = h^q(\tau)$ as desired. We have shown that h^{q^*} is well-defined.

q^* *satisfies* (C2). We address the two definitions of f^{q^*} separately. Suppose $\sigma \in T^{q^*}$ has length 1. We want to show that there is some i such that $\sigma \subseteq f^{q^*}(i)$.

We begin by considering the first definition of f^{q^*}. First, consider the case $\sigma \in T^{p^*}$. If there is some $i \in \mathrm{dom}(f^{p^*}) \backslash (\mathrm{dom}(f^q) \cap F)$ such that $\sigma \subseteq f^{p^*}(i)$,

then we are done. Otherwise, since p^* satisfies (C2), there is some $i \in \mathrm{dom}(f^{p^*}) \cap \mathrm{dom}(f^q) \cap F$ such that $\sigma \subseteq f^{p^*}(i)$. Then $f^{q^*}(i) = f^q(i) \supseteq f^p(i) = f^{p^*}(i) \supseteq \sigma$ as desired.

Second, if $\sigma \in T^q \backslash T^{p^*}$, then there is some $i \in \mathrm{dom}(g)$ such that $g(i) = \sigma$. We have $f^{q^*}(i) = g(i) = \sigma$ as desired.

Next we consider the second definition of f^{q^*}. Assume that $\mathrm{dom}(f^{p^*}) \subseteq F$. If $\sigma \in T^{p^*}$, then since p^* satisfies (C2), there is some $i \in \mathrm{dom}(f^{p^*})$ such that $\sigma \subseteq f^{p^*}(i)$. Such i lies in F by assumption, so $f^{q^*}(i) = f^q(i) \supseteq f^p(i) = f^{p^*}(i) \supseteq \sigma$ as desired.

If $\sigma \in T^{q^*} \backslash T^{p^*}$, then since q satisfies (C2), there is some $i \in \mathrm{dom}(f^q)$ such that $\sigma \subseteq f^q(i)$. Since $\sigma \notin T^p$ and $|\sigma| = 1$, it follows that $f^q(i) \cap T^p = \emptyset$. So i lies in $\mathrm{dom}(f^{q^*})$. Hence $f^{q^*}(i) = f^q(i) \supseteq \sigma$ as desired. This completes the proof that q^* satisfies (C2).

To show that q^* is a condition, it remains to check (C3). We omit the verification as it is exactly the same as that for ordinary Steel forcing.

q^* *extends* p^*. (E1) and (E4) are immediate. To check (E2), consider any $i \in \mathrm{dom}(f^{p^*})$.

CASE 1. Suppose $i \in F$. By retagging, we have $i \in \mathrm{dom}(f^p) \subseteq \mathrm{dom}(f^q)$ and $f^{p^*}(i) = f^p(i)$. In both definitions of f^{q^*} we have $f^{q^*}(i) = f^q(i)$, so $f^{q^*}(i) \cap T^{p^*} = f^q(i) \cap T^p = f^p(i) = f^{p^*}(i)$ (the second equality holds by (E2)).

CASE 2. If $i \notin F$, then we have $\mathrm{dom}(f^{p^*}) \not\subseteq F$. So we are only concerned with the first definition of f^{q^*}. In this case $f^{q^*}(i)$ is defined to be $f^{p^*}(i)$, which lies in T^{p^*}, so $f^{q^*}(i) \cap T^{p^*} = f^{p^*}(i)$.

To check (E3), consider any $i \in \mathrm{dom}(f^{q^*}) \backslash \mathrm{dom}(f^{p^*})$. We analyze the two definitions of f^{q^*} separately.

We begin by considering the first definition of f^{q^*}.

CASE 1. $i \in \mathrm{dom}(f^q) \cap F$ and $f^{q^*}(i) = f^q(i)$. Since $i \in F \backslash \mathrm{dom}(f^{p^*})$, we have $i \notin \mathrm{dom}(f^p)$ by retagging. Hence by (E3) for $q \leq p$, we have $f^q(i) \cap T^p = \emptyset$ as desired.

CASE 2. Otherwise, $i \in \mathrm{dom}(g)$ and $f^{q^*}(i) = g(i)$. In this case, $g(i) \notin T^{p^*}$ and $|g(i)| = 1$, so $g(i) \cap T^{p^*} = \emptyset$ as desired.

Next we consider the second definition of f^{q^*}. If $i \in F$, then the reasoning is identical to Case 1 above. Otherwise, by the definition of f^{q^*}, we have $f^{q^*}(i) \cap T^p = f^q(i) \cap T^p = \emptyset$ as desired. This completes the proof that q^* extends p^*.

Finally, it is easy to check that q and q^* are $\omega\gamma$-F-retaggings. $\dashv$

The following consequences of the basic retagging lemma are routine (see [10, Lemma 2.4 and Corollary 2.6]). We include their proofs for completeness.

LEMMA 3.6. *Let ψ be a ranked formula in $\mathcal{L}_F$. Suppose that p and p^* are $(\omega \cdot \mathrm{rk}(\psi))$-$F$-retaggings. Then $p \Vdash \psi$ if and only if $p^* \Vdash \psi$.*

PROOF. We proceed by induction on the rank of ψ. The only nontrivial case is when ψ is $\neg\varphi$. Assuming that p^* and p are $(\omega \cdot \mathrm{rk}(\neg\varphi))$-$F$-retaggings and that $p^* \Vdash \neg\varphi$, we want to show that $p \Vdash \neg\varphi$, i.e., for all $q \leq p$, we have $q \not\Vdash \varphi$. By Lemma 3.5, there is some $q^* \leq p^*$ such that q^* and q are $(\omega \cdot \mathrm{rk}(\varphi))$-$F$-retaggings. Since

$p^* \Vdash \neg\varphi$, we have $q^* \nVdash \varphi$. Applying the inductive hypothesis to q and q^* shows that $q \nVdash \varphi$ as desired.   ⊣

COROLLARY 3.7. *Suppose $p \in \mathbb{P}_\mu$ and $\psi \in \mathcal{L}_F$. If there is some $q \leq p$ such that $q \Vdash \psi$, then there is some $q' \leq p$ in $\mathbb{P}_{\max\{\mu, \omega \cdot \mathrm{rk}(\psi)\}}$ such that $q' \Vdash \psi$. Therefore, $p \Vdash \neg\psi$ if and only if for all $q \leq p$ in $\mathbb{P}_{\max\{\mu, \omega \cdot \mathrm{rk}(\psi)\}}$, $q \nVdash \psi$.*

PROOF. Apply Lemma 3.4 to obtain some $q' \leq p$ in $\mathbb{P}_{\max\{\mu, \omega \cdot \mathrm{rk}(\psi)\}}$ such that q and q' are $(\max\{\mu, \omega \cdot \mathrm{rk}(\psi)\})$-$F$-retaggings. By Lemma 3.6, $q' \Vdash \psi$.   ⊣

COROLLARY 3.8. *For each condition p and each formula ψ in $\mathcal{L}_F$, $\emptyset^{(\mathrm{rk}(\psi))}$ can decide whether $p \Vdash \psi$ uniformly in p and ψ.*

PROOF. Induction on $\mathrm{rk}(\psi)$ using Corollary 3.7.   ⊣

We may now analyze which branches on T^G lie in each M_F. The proof below follows [10, Lemma 2.7], with modifications in order to account for (C2).

LEMMA 3.9. *For each $F \subset_{\mathrm{f}} \mathbb{N}$, $M_F \cap [T^G] = \{f_i^G : i \in F\}$. Hence $M_\infty \cap [T^G] = \{f_i^G : i \in \mathbb{N}\}$, but no infinite sequence of distinct f_i^G lies in M_∞.*

PROOF. Suppose towards a contradiction that $S = S_{v,F,e} \in [T^G]$ is not f_i^G for any $i \in F$. Then there is some prefix $\sigma \subset S$ such that $\sigma \not\subset f_i^G$ for any $i \in F$. For later purposes, fix such σ with length greater than 1. By Lemma 3.2, fix $p \in G$ such that $p \Vdash \varphi(\mathbf{S})$, where $\varphi(\mathbf{S})$ is

$$\mathbf{S} \in [\mathbf{T}] \ \wedge \ \boldsymbol{\sigma} \subset \mathbf{S} \ \wedge \ (\forall i \in F)(\boldsymbol{\sigma} \not\subset \mathbf{f}_i).$$

Note $\varphi(\mathbf{S})$ is a ranked formula in $\mathcal{L}_F$.

By genericity, we may assume that $\sigma \in T^p$. Next, fix $\beta < \omega_1^{CK}$ large enough so that $\beta > \omega \cdot \mathrm{rk}(\varphi(\mathbf{S}))$ and $p \in \mathbb{P}_\beta$.

Since p forces that σ is extendible in T^G, we have $h^p(\sigma) = \infty$ (by Lemma 3.2). We now define p^* which is a β-F-retagging of p, such that $h^{p^*}(\sigma) \in [\beta, \omega_1^{CK})$. Define $T^{p^*} = T^p$. Define f^{p^*} to be the union of $f^p \restriction F$ and a function g such that $\mathrm{dom}(g)$ is disjoint from F and $\mathrm{range}(g) = \{\sigma \in T^p : |\sigma| = 1\}$. Finally, define

$$h^{p^*}(\tau) = \begin{cases} \beta + |\tau|_Q, & \text{if } \tau \supseteq \sigma \wedge h^p(\tau) = \infty, \\ h^p(\tau), & \text{otherwise,} \end{cases}$$

where $Q = \{\tau : \tau \subseteq \sigma \vee (\tau \supseteq \sigma \wedge h^p(\tau) = \infty)\}$.

Since $h^p(\sigma) = \infty$ and $|\sigma| > 1$, it is straightforward to check that h^{p^*} is a rank function and $h^{p^*}(\emptyset) = \infty$. In order to show that p^* is a condition, it suffices to check that $h^{p^*}(f^{p^*}(i)) = \infty$ for all $i \in \mathrm{dom}(f^{p^*})$. If $i \in F$, we have $\sigma \not\subseteq f^p(i) = f^{p^*}(i)$, so $h^{p^*}(f^{p^*}(i)) = h^p(f^{p^*}(i)) = h^p(f^p(i)) = \infty$. If $i \in \mathrm{dom}(f^{p^*}) \backslash F$ (that is, $\mathrm{dom}(g)$), then $f^{p^*}(i)$ has length 1, so $h^{p^*}(f^{p^*}(i)) = h^p(f^{p^*}(i)) = \infty$.

Finally, it is clear that p^* is a β-F-retagging of p, and hence an $(\omega \cdot \mathrm{rk}(\varphi(\mathbf{S})))$-$F$-retagging of p. By Lemma 3.6 we have

$$p^* \Vdash \mathbf{S} \in [\mathbf{T}] \wedge \boldsymbol{\sigma} \subset \mathbf{S},$$

which is impossible because $h^{p^*}(\sigma) \neq \infty$.   ⊣

The following lemma is new; (C2) was designed in order to prove it.

LEMMA 3.10. M_∞ *does not satisfy finite choice.*

PROOF. Consider the following arithmetical predicate $\varphi(n, X)$ (with parameter T^G):

$$X \in [T^G] \ \wedge \ X(0) = n.$$

First, we claim that for each n, there is some $X \in M_\infty$ such that $\varphi(n, X)$ holds. By genericity, fix some $p \in G$ such that $\langle n \rangle \in T^p$. By (C2), there is some i such that $\langle n \rangle \subseteq f^p(i)$. Then $\varphi(n, f_i^G)$ holds.

Second, we claim that for each n, there are at most finitely many $X \in M_\infty$ such that $\varphi(n, X)$ holds. By Lemma 3.9, it suffices to show that there are only finitely many i such that $\varphi(n, f_i^G)$ holds. As above, fix $p \in G$ such that $\langle n \rangle \in T^p$. For each i, we claim that if $i \notin \text{dom}(f^p)$, then $\langle n \rangle \nsubseteq f_i^G$. Suppose to the contrary that there is some $i \notin \text{dom}(f^p)$ such that $\langle n \rangle \subseteq f_i^G$. Fix some $q \in G$ such that $q \Vdash \langle \mathbf{n} \rangle \subseteq \mathbf{f}_i$. Let $r \in G$ be a common refinement of p and q. Then $\langle n \rangle \subseteq f^r(i)$. But then $f^r(i) \cap T^p \neq \emptyset$, violating (E3) for $r \leq p$.

This shows that $\varphi(n, X)$ is an instance of finite choice in M_∞. By Lemma 3.9, M_∞ does not contain any infinite sequence of distinct branches through T^G, so M_∞ does not contain any finite choice solution to $\varphi(n, X)$.   ⊣

3.6. Analyzing the forcing relation for Σ-over-$\mathcal{L}_F$ formulas. In order to show that $M_\infty \models \Delta_1^1\text{-CA}_0$ (Section 3.7), we need to analyze the forcing relation for Σ-over-$\mathcal{L}_F$ formulas ψ. We begin by considering formulas ψ^ρ where $\rho < \omega_1^{CK}$. Note that such formulas may not lie in any $\mathcal{L}_{F'}$ because they may contain (existential) quantifiers over set variables of the form X^ρ.

To this end we exploit automorphisms of $\mathbb{P}$, as was done in [16, Lemma 10]. Each permutation $\pi : \mathbb{N} \to \mathbb{N}$ induces an automorphism $\hat{\pi}$ of $\mathbb{P}$ by permuting the finite paths in each condition: $T^{\hat{\pi}(p)} = T^p$, $h^{\hat{\pi}(p)} = h^p$, and $f^{\hat{\pi}(p)}(\pi(i)) = f^p(i)$. Each permutation π also induces an automorphism of $\mathcal{L}_\infty$: For each formula ψ, let $\pi\psi$ denote the formula obtained from ψ by replacing each $\mathbf{f}_i$ by $\mathbf{f}_{\pi(i)}$ and each $\mathbf{S}_{v,F,e}$ by $\mathbf{S}_{v,\pi``F,e}$. (In the proof below, we will denote $\mathbf{S}_{v,\pi``F,e}$ by $\pi\mathbf{S}_{v,F,e}$.) By induction on $\text{rk}(\psi)$, one can show that $p \Vdash \psi$ if and only if $\hat{\pi}(p) \Vdash \pi\psi$.

LEMMA 3.11. *Suppose ψ is Σ-over-$\mathcal{L}_F$, $p \Vdash \psi^\mu$, and $\text{dom}(f^p) \subseteq F$. If p and p^* are $(\omega \cdot \text{rk}(\psi^\mu) + \omega^2)$-F-retaggings, then $p^* \Vdash \psi^\mu$ as well.*

PROOF. We prove by induction on k that if ψ is built from ranked, F-restricted formulas using k steps and p is a condition such that $p \Vdash \psi^\mu$, $\text{dom}(f^p) \subseteq F$, and p and p^* are $(\omega \cdot \text{rk}(\psi^\mu) + \omega \cdot 2k)$-F-retaggings, then $p^* \Vdash \psi^\mu$. The base case holds by Lemma 3.6.

The only nontrivial inductive step is if ψ has the form $\exists X \varphi(X)$. We want to prove that for all $q^* \leq p^*$, there is some $r^* \leq q^*$ and some $\mathbf{S} \in C^\mu$ such that $r^* \Vdash \varphi^\mu(\mathbf{S})$. Our plan for doing so is illustrated in Figure 1.

Given $q^* \leq p^*$, there is some $q \leq p$ such that q and q^* are $(\omega \cdot \text{rk}(\psi^\mu) + \omega \cdot (2k+1))$-F-retaggings. Furthermore, since $\text{dom}(f^p) \subseteq F$, we can choose such q with $f^q \subseteq f^{q^*}$ (by the second part of Lemma 3.5).
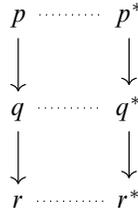
FIGURE 1. Illustration of the proof of Lemma 3.11. Arrows correspond to extension in the forcing. Dotted lines correspond to retaggings.

Since $p \Vdash \psi^\mu$ and $q \leq p$, there is some $r \leq q$ and some $\mathbf{S} \in C^\mu$ such that $r \Vdash \varphi^\mu(\mathbf{S})$. By extending r, we may assume that $F \cup \mathrm{dom}(\mathbf{S}) \subseteq \mathrm{dom}(f^r)$. To complete the inductive step, we want to define $r^* \leq q^*$ such that r and r^* are $(\omega \cdot \mathrm{rk}(\psi^\mu) + \omega \cdot 2k)$-$\mathrm{dom}(f^r)$-retaggings. By inductive hypothesis, it would follow that $r^* \Vdash \varphi^\mu(\mathbf{S})$ as desired.

The naive definition of r^* fails because there may be conflict between f^r and f^{q^*}. In order to avoid this conflict, we use an automorphism of $\mathbb{P}$ to permute r and $\mathbf{S}$ as follows. Consider a permutation π which fixes $F \cup \mathrm{dom}(f^q)$ and moves $\mathrm{dom}(f^r) \backslash (F \cup \mathrm{dom}(f^q))$ to some set which is disjoint with $\mathrm{dom}(f^{q^*})$. Since π fixes $\mathrm{dom}(f^q)$, we have $\hat\pi r \leq q$. Since π fixes F and φ is Σ-over-$\mathcal{L}_F$, we have $\pi\varphi^\mu = \varphi^\mu$. So $\hat\pi r \Vdash \varphi^\mu(\pi\mathbf{S})$. Therefore, by replacing r and $\mathbf{S}$ with $\hat\pi r$ and $\pi\mathbf{S}$, we may assume that $\mathrm{dom}(f^r) \backslash (F \cup \mathrm{dom}(f^q))$ and $\mathrm{dom}(f^{q^*})$ are disjoint.

We claim that q and q^* are $(\omega \cdot \mathrm{rk}(\psi^\mu) + \omega \cdot (2k+1))$-$\mathrm{dom}(f^r)$-retaggings. Since $f^q \subseteq f^{q^*}$, this reduces to showing that $\mathrm{dom}(f^{q^*}) \cap \mathrm{dom}(f^r) \subseteq \mathrm{dom}(f^q)$. Suppose $i \in \mathrm{dom}(f^{q^*}) \cap \mathrm{dom}(f^r)$. By assumption on r, it follows that $i \in F \cup \mathrm{dom}(f^q)$. If $i \in \mathrm{dom}(f^q)$, then we are done. Otherwise $i \in F$, but since $i \in \mathrm{dom}(f^{q^*})$ and q and q^* are $(\omega \cdot \mathrm{rk}(\psi^\mu) + \omega \cdot (2k+1))$-$F$-retaggings, it follows that $i \in \mathrm{dom}(f^q)$ as well. This proves the claim.

By Lemma 3.5, there is some $r^* \leq q^*$ such that r and r^* are $(\omega \cdot \mathrm{rk}(\psi^\mu) + \omega \cdot 2k)$-$\mathrm{dom}(f^r)$-retaggings. Since $F \cup \mathrm{dom}(\mathbf{S}) \subseteq \mathrm{dom}(f^r)$, we may apply the inductive hypothesis and the fact that $r \Vdash \varphi^\mu(\mathbf{S})$ to conclude that $r^* \Vdash \varphi^\mu(\mathbf{S})$ as desired. $\dashv$

The following lemma is not stated in [10] but some form of it is used in the proof of [10, Lemma 2.9(1)].

LEMMA 3.12. *Given a condition p, a formula ψ which is Σ-over-$\mathcal{L}_F$, and some $\rho < \omega_1^{CK}$, $\emptyset^{(\mathrm{rk}(\psi^\rho))}$ can decide whether $p \Vdash \psi^\rho$ uniformly in p, ψ, and ρ.*

PROOF. We proceed by induction on the number of steps it takes to construct ψ from ranked F-restricted formulas. The base case holds by Corollary 3.8.

For the inductive step, the only nontrivial case is when ψ has the form $\exists X\varphi(X)$. Fix any $\mu \geq \omega \cdot \mathrm{rk}(\varphi^\rho(\mathbf{S})) + \omega^2 + \omega$ such that $p \in \mathbb{P}_\mu$. We claim that $p \Vdash \exists X^\rho \psi^\rho(X^\rho)$ if and only if for every $q \leq p$ in $\mathbb{P}_\mu$, there is some $r \leq q$ in $\mathbb{P}_\mu$ and some $\mathbf{S} \in C^\rho$ such that $r \Vdash \varphi^\rho(\mathbf{S})$. This claim suffices for proving this case of the inductive step.

To prove the forward direction, suppose we are given $q \leq p$ in $\mathbb{P}_\mu$. By assumption, there is some $r \leq q$ and some $\mathbf{S} \in C^\rho$ such that $r \Vdash \varphi^\rho(\mathbf{S})$. We can retag r to some
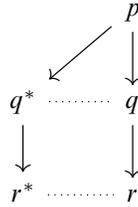
$$p$$

FIGURE 2. Illustration of the proofs of Lemmas 3.12 and 3.13. Arrows correspond to extension in the forcing. Dotted lines correspond to retaggings.

$r^* \leq q$ in $\mathbb{P}_\mu$ such that r and r^* are μ-$(F \cup \mathrm{dom}(f^r))$-retaggings (Lemma 3.4). Since $\mu \geq \omega \cdot \mathrm{rk}(\varphi^\rho(\mathbf{S})) + \omega^2$, it follows from Lemma 3.11 that $r^* \Vdash \varphi^\rho(\mathbf{S})$ as desired.

To prove the backward direction, suppose we are given $q \leq p$. In order to construct some $r \leq q$ and some $\mathbf{S} \in C^\rho$ such that $r \Vdash \varphi^\rho(\mathbf{S})$, we follow Figure 2. We begin by retagging q to some $q^* \leq p$ in $\mathbb{P}_\mu$ such that q and q^* are μ-F'-retaggings for every $F' \subset_f \mathbb{N}$ (Lemma 3.4). By assumption, there is some $r^* \leq q^*$ in $\mathbb{P}_\mu$ and some $\mathbf{S} \in C^\rho$ such that $r^* \Vdash \varphi^\rho(\mathbf{S})$.

To complete the proof of the backward direction, we want to produce some $r \leq q$ which is a retagging of r^*. Define F' to be the union of F, $\mathrm{dom}(f^{r^*})$ and $\mathrm{dom}(\mathbf{S})$ (recall that $\mathrm{dom}(\mathbf{S}_{v,D,e})$ is defined to be D.) Then $\varphi^\rho(\mathbf{S})$ is Σ-over-$\mathcal{L}_{F'}$. Since $\mu \geq \omega \cdot \mathrm{rk}(\varphi^\rho(\mathbf{S})) + \omega^2 + \omega$, there is some $r \leq q$ such that r and r^* are $(\omega \cdot \mathrm{rk}(\varphi^\rho(\mathbf{S})) + \omega^2)$-$F'$-retaggings (Lemma 3.5). It follows from Lemma 3.11 that $r \Vdash \varphi^\rho(\mathbf{S})$ as desired. $\dashv$

Next, we shall show that in order to understand the forcing relation for Σ-over-$\mathcal{L}_F$ formulas φ, it suffices to consider formulas φ^ρ where $\rho < \omega_1^{CK}$. This will be useful for showing that $M_\infty \models \Delta_1^1\text{-CA}_0$.

LEMMA 3.13. *Suppose ψ is Σ-over-$\mathcal{L}_F$. If $p \Vdash \psi$, then there is some $\mu < \omega_1^{CK}$ such that for all $\rho \in [\mu, \omega_1^{CK})$, $p \Vdash \psi^\rho$.*

PROOF. We proceed by induction on the number of steps it takes to construct ψ from ranked F-restricted formulas. The base case is trivial. For the inductive step, the only nontrivial case is when ψ has the form $\exists X \varphi(X)$. Since $p \Vdash \exists X \varphi(X)$, for every $q \leq p$, there exist $r \leq q$ and $\mathbf{S}$ such that $r \Vdash \varphi(\mathbf{S})$. By the inductive hypothesis, $r \Vdash \varphi^\rho(\mathbf{S})$ for all sufficiently large $\rho < \omega_1^{CK}$.

Hence, for each $q \leq p$, there exist $\gamma_q < \omega_1^{CK}$, a condition $r \leq q$ in $\mathbb{P}_{\gamma_q}$, and $\mathbf{S} \in C^{\gamma_q}$ such that $r \Vdash \varphi^{\gamma_q}(\mathbf{S})$. By Lemma 3.12, we can hyperarithmetically search for the least such γ_q, r and $\mathbf{S}$.

By boundedness, for each $\beta < \omega_1^{CK}$, there is some $\gamma < \omega_1^{CK}$ such that for every $q \leq p$ in $\mathbb{P}_\beta$, there exists $\gamma_q < \gamma$ as above. For later purposes, we will choose $\gamma < \omega_1^{CK}$ sufficiently large such that $\omega \cdot \mathrm{rk}(\varphi^{\gamma_q}(\mathbf{S})) + \omega^2 + \omega \leq \gamma$ for every $q \leq p$ in $\mathbb{P}_\beta$.

Since we can find γ from β hyperarithmetically, by boundedness, there is some limit $\mu < \omega_1^{CK}$ such that (1) $p \in \mathbb{P}_\mu$; (2) for every $\beta < \mu$, there is some $\gamma < \mu$ satisfying the previous paragraph.

We shall show that $p \Vdash \exists X^\mu \varphi^\mu(X^\mu)$. Given $q \leq p$, we want to construct $r \leq q$ and $\mathbf{S} \in C^\mu$ such that $r \Vdash \varphi^\mu(\mathbf{S})$. Our plan for doing so is illustrated in Figure 2.

We begin by retagging q to some $q^* \leq p$ in $\mathbb{P}_\mu$ such that q and q^* are μ-F'-retaggings for every $F' \subset_f \mathbb{N}$ (Lemma 3.4). Since μ is a limit and $\mathrm{dom}(h^{q^*}) = T^{q^*}$ is finite, we have $q^* \in \mathbb{P}_\beta$ for some $\beta < \mu$. By construction of μ, there exist $\gamma_{q^*} < \gamma < \mu$, a condition $r^* \leq q^*$ in $\mathbb{P}_{\gamma_{q^*}}$, and $\mathbf{S} \in C^{\gamma_{q^*}}$ such that $\omega \cdot \mathrm{rk}(\varphi^{\gamma_{q^*}}(\mathbf{S})) + \omega^2 + \omega \leq \gamma$ and $r^* \Vdash \varphi^{\gamma_{q^*}}(\mathbf{S})$.

Define F' to be the union of F, $\mathrm{dom}(f^{r^*})$, and $\mathrm{dom}(\mathbf{S})$. Since $\omega \cdot \mathrm{rk}(\varphi^{\gamma_{q^*}}(\mathbf{S})) + \omega^2 + \omega \leq \gamma < \mu$ and q and q^* are μ-F'-retaggings, by Lemma 3.5, there is some $r \leq q$ such that r and r^* are $(\omega \cdot \mathrm{rk}(\varphi^{q^*}(\mathbf{S})) + \omega^2)$-$F'$-retaggings. By Lemma 3.11, we have $r \Vdash \varphi^{\gamma_{q^*}}(\mathbf{S})$ and so $r \Vdash \varphi^\mu(\mathbf{S})$ as desired. $\dashv$

Following [10, Lemma 2.9(2)], we conclude the following corollary.

COROLLARY 3.14. *For each $F \subset_f \mathbb{N}$, $M_F \models \Sigma_1^1$-AC_0. It follows that $M_F = \mathrm{HYP}(T^G \oplus \langle f_i^G \rangle_{i \in F})$.*

PROOF. Suppose that $M_F \models \forall n \exists X \varphi(n, X)$, where $\varphi(n, X)$ is arithmetic with parameters from M_F. Fix some $p \in G$ such that $p \Vdash \forall n \exists X \varphi(n, X)$. By Lemma 3.13, fix some $\mu < \omega_1^{CK}$ (greater than the rank of every constant symbol in φ) such that $p \Vdash \forall n \exists X^\mu \varphi(n, X^\mu)$. It follows that $M_F \models \forall n \exists X^\mu \varphi(n, X^\mu)$.

For each n, let e_n be least such that $M_F \models \varphi(n, S_{\mu,F,e_n})$. For each e, $H_{\mu+\omega,F}$ computes whether $M_F \models \varphi(n, S_{\mu,F,e})$, so $\langle S_{\mu,F,e_n} \rangle_n$ is a Σ_1^1-AC_0 solution which lies in M_F. $\dashv$

We are ready to prove that M_∞ satisfies Δ_1^1-CA_0. The overall strategy follows [16, Lemma 12], with modifications in order to account for (C2).

3.7. Proof that M_∞ satisfies Δ_1^1-comprehension. Suppose $\varphi(n)$ and $\psi(n)$ are Σ-over-$\mathcal{L}_F$ with only n free and $M_\infty \models \forall n(\psi(n) \leftrightarrow \neg\varphi(n))$. We want to define $D \in M_\infty$ such that

$$M_\infty \models \forall n(\psi(n) \leftrightarrow n \in D).$$

A naive attempt is to consider

$$\{n \in \mathbb{N} : \exists q \in G(q \Vdash \psi(\mathbf{n}))\},$$

but there are two obstacles preventing us from showing that the above set lies in M_∞:

- M_∞ does not contain G so we cannot search over $q \in G$;
- deciding whether $q \Vdash \psi(\mathbf{n})$ is not hyperarithmetic in general.

The second obstacle is overcome using Lemma 3.13. To overcome the first obstacle, we shall use retagging to change the scope of our search to a class of conditions which look like they might lie in G, based on information from T^G and finitely many branches f_i^G (all of which lie in M_∞, unlike G).

Recall that (by genericity) h^G is the well-founded rank function of T^G. We say that a rank function $h : T \to \omega_1^{CK} \cup \{\infty\}$ is v-*good* if $T \subset T^G$ and for all $\sigma \in T$:

$$h^G(\sigma) < v \quad \Rightarrow \quad h(\sigma) = h^G(\sigma),$$
$$h^G(\sigma) \geq v \quad \Leftrightarrow \quad h(\sigma) \geq v.$$

We will use the notion of v-goodness in our definition of D.

We begin the proof by fixing $p \in G$ such that $p \Vdash \forall n(\psi(n) \leftrightarrow \neg\varphi(n))$. By genericity and by expanding F if necessary, we may assume that $F = \mathrm{dom}(f^p)$. Since $p \Vdash \forall n(\psi(n) \vee \varphi(n))$ (which is Σ-over-$\mathcal{L}_F$), by Lemma 3.13, we may fix $\mu < \omega_1^{CK}$ large enough such that $p \Vdash \forall n(\psi^\mu(n) \vee \varphi^\mu(n))$ and μ is greater than the rank of any constant in φ and ψ. Since $p \Vdash \forall n(\neg\psi(n) \vee \neg\varphi(n))$ and μ is large enough, we also have $p \Vdash \forall n(\neg\psi^\mu(n) \vee \neg\varphi^\mu(n))$ and so $p \Vdash \forall n(\psi^\mu(n) \leftrightarrow \varphi^\mu(n))$.

Next, fix $v < \omega_1^{CK}$ large enough such that $p \in \mathbb{P}_v$ and $\mathrm{rk}(\varphi^\mu(\mathbf{d}) \vee \psi^\mu(\mathbf{d})) < v$ for all $d \in \mathbb{N}$. Now we define $D \subseteq \mathbb{N}$ as follows: $d \in D$ if and only if there is some $q \in \mathbb{P}_{\omega v + \omega^2 + \omega}$ extending p such that:

(1) $q \Vdash \psi^\mu(\mathbf{d})$;
(2) $T^q \subset T^G$ (this is implied by (3) but we include it for emphasis);
(3) h^q is $(\omega v + \omega^2 + \omega)$-good;
(4) $\forall i \in F(f^q(i) = f_i^G \cap T^q)$.

Note that if $q \in G$, then q satisfies (2) (by (E1)) and (4) (by (E2)). (3) must hold as well, but even more is true: we must have $h^q \subseteq h^G$. Since M_∞ does not have access to h^G, we make do with sufficiently good approximations. Observe that D is hyperarithmetic in $T^G \oplus \langle f_i^G \rangle_{i \in F}$, so $D \in M_F \subseteq M_\infty$. It remains to show that $M_\infty \models \psi(\mathbf{d})$ if and only if $d \in D$.

The forward direction is straightforward: If $M_\infty \models \psi(\mathbf{d})$, then $M_\infty \models \psi^\mu(\mathbf{d})$, so we may fix $q' \in G$ extending p which forces $\psi^\mu(\mathbf{d})$. Using q', we define q as we did in the proof of Lemma 3.4: $T^q = T^{q'}$, $f^q = f^{q'}$, and

$$h^q(\sigma) = \begin{cases} \infty, & \text{if } h^{q'}(\sigma) \geq \omega v + \omega^2 + \omega, \\ h^{q'}(\sigma), & \text{otherwise.} \end{cases}$$

It is straightforward to check that q witnesses $d \in D$ (using Lemma 3.11 to show that $q \Vdash \psi^\mu(\mathbf{d})$).

To prove the backward direction, suppose $M_\infty \models \varphi(\mathbf{d})$. Then $M_\infty \models \varphi^\mu(\mathbf{d})$, so we may fix $r \in G$ extending p which forces $\varphi^\mu(\mathbf{d})$. Suppose towards a contradiction that $d \in D$, as witnessed by some q. Our plan is to construct conditions q^* extending q, r^* extending r, and s^* extending p, as illustrated in Figure 3.

We begin by using an automorphism of $\mathbb{P}$ to avoid conflict between f^q and f^r. Consider a permutation π which fixes F and moves $\mathrm{dom}(f^q) \backslash F$ to some set which is disjoint with $\mathrm{dom}(f^r)$. Since $F = \mathrm{dom}(f^p)$, we have $\hat{\pi} q \leq p$. It is straightforward to check that $\hat{\pi} q$ witnesses $d \in D$. Therefore, by replacing q with $\hat{\pi} q$, we may assume that $\mathrm{dom}(f^q) \backslash F$ and $\mathrm{dom}(f^r)$ are disjoint. It follows that $\mathrm{dom}(f^q) \cap \mathrm{dom}(f^r) = F$.

Next, fix finite partial functions g_{q^*} and g_{r^*} on $\mathbb{N}$ such that:

– $\mathrm{dom}(g_{q^*})$, $\mathrm{dom}(g_{r^*})$, and $\mathrm{dom}(f^q) \cup \mathrm{dom}(f^r)$ are pairwise disjoint;
– $\mathrm{range}(g_{q^*}) = \{\sigma \in T^r \backslash T^q : |\sigma| = 1\}$;
– $\mathrm{range}(g_{r^*}) = \{\sigma \in T^q \backslash T^r : |\sigma| = 1\}$.

We may now define q^* as follows:

– $T^{q^*} = T^q \cup T^r$;
– $f^{q^*}(i) = \begin{cases} f^q(i), & \text{if } i \in \mathrm{dom}(f^q) \backslash F; \\ f_i^G \cap T^{q^*}, & \text{if } i \in F; \\ g_{q^*}(i), & \text{if } i \in \mathrm{dom}(g_{q^*}); \end{cases}$

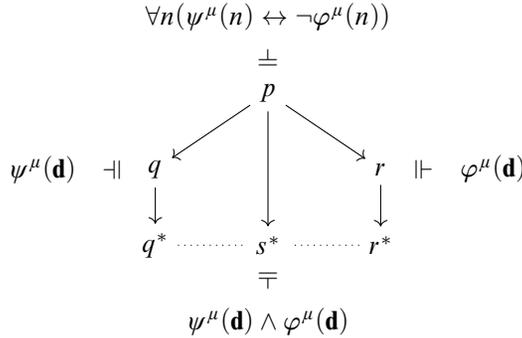$$\forall n(\psi^\mu(n) \leftrightarrow \neg\varphi^\mu(n))$$

FIGURE 3. p and r lie in G, while q "looks like" it lies in G.

– h^{q^*} is defined by cases:

$$h^{q^*}(\tau) = \begin{cases} h^q(\tau), & \text{if } \tau \in T^q, \\ \infty, & \text{if } \exists i(\tau \subseteq f^{q^*}(i)), \\ h^r(\tau), & \text{if } h^r(\tau) < \omega v + \omega^2, \\ \omega v + \omega^2 + |\tau|_Q, & \text{otherwise,} \end{cases}$$

where $Q = \{\sigma \in T^r \backslash T^q : h^r(\sigma) \geq \omega v + \omega^2\}$.

Next, define r^* as follows:

– $T^{r^*} = T^q \cup T^r$;
– $f^{r^*}(i) = \begin{cases} f^r(i), & \text{if } i \in \text{dom}(f^r)\backslash F; \\ f_i^G \cap T^{r^*}, & \text{if } i \in F; \\ g_{r*}(i), & \text{if } i \in \text{dom}(g_{r*}); \end{cases}$
– $h^{r^*} = h^G \upharpoonright T^{r^*}$.

Finally, define s^* as follows:

– $T^{s^*} = T^q \cup T^r$;
– $f^{s^*} = f^{q^*} \cup f^{r^*}$;
– $h^{s^*}(\sigma) = \begin{cases} h^G(\sigma), & \text{if } h^G(\sigma) < \omega v + \omega^2; \\ \infty, & \text{otherwise.} \end{cases}$

One may verify that:

– q^* is a condition which extends q;
– r^* is a condition which extends r;
– s^* is a condition which extends p;
– q^* and s^* are $(\omega v + \omega^2)$-$\text{dom}(f^{q^*})$-retaggings;
– r^* and s^* are $(\omega v + \omega^2)$-$\text{dom}(f^{r^*})$-retaggings.

The verification is the same as that for ordinary Steel forcing, with minor additions in order to handle the new paths contributed by g_{q*} and g_{r*}. We only discuss g_{q*} here; the argument for g_{r*} is similar. First, note that g_{q*} was defined to ensure that q^* satisfies (C2). Second, (E3) is satisfied for $q^* \leq q$ because for each $i \in \text{dom}(f^{q^*})\backslash\text{dom}(f^q)$ $(= \text{dom}(g_{q*}))$, we have $g_{q*} \notin T^q$ and $|g_{q*}(i)| = 1$.

With the above facts, we may complete the proof that if $M_\infty \models \varphi(\mathbf{d})$, then $d \notin D$. Since $q^* \leq q$ and $q \Vdash \psi^\mu(\mathbf{d})$, we have $q^* \Vdash \psi^\mu(\mathbf{d})$. It follows from Lemma 3.11 that $s^* \Vdash \psi^\mu(\mathbf{d})$ (note $F \subseteq \operatorname{dom}(f^{q^*})$ so $\psi^\mu(\mathbf{d})$ is Σ-over-$\mathcal{L}_{\operatorname{dom}(f^{q^*})}$). Similarly, we have $s^* \Vdash \varphi^\mu(\mathbf{d})$. But since $M_\infty \models \forall n(\psi^\mu(n) \to \psi(n))$ and $M_\infty \models \forall n(\varphi^\mu(n) \to \varphi(n))$, it follows that $s^* \Vdash \psi(\mathbf{d}) \wedge \varphi(\mathbf{d})$. This contradicts the fact that $p \Vdash \forall n(\psi(n) \leftrightarrow \neg\varphi(n))$ and $s^* \leq p$.

This completes the proof that M_∞ satisfies $\Delta^1_1\text{-CA}_0$.

REMARK 3.15. We are unable to extend our proof to show that M_∞ satisfies the Π^1_1-separation principle (studied by Montalbán [10]). We do not know if Π^1_1-separation implies finite choice. If there is a model of Π^1_1-separation and $I\Sigma^1_1$ (such as an ω-model of Π^1_1-separation) which does not satisfy finite choice, then we would obtain a negative answer to [2, Question 1.11] (using Theorem 1.5).

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