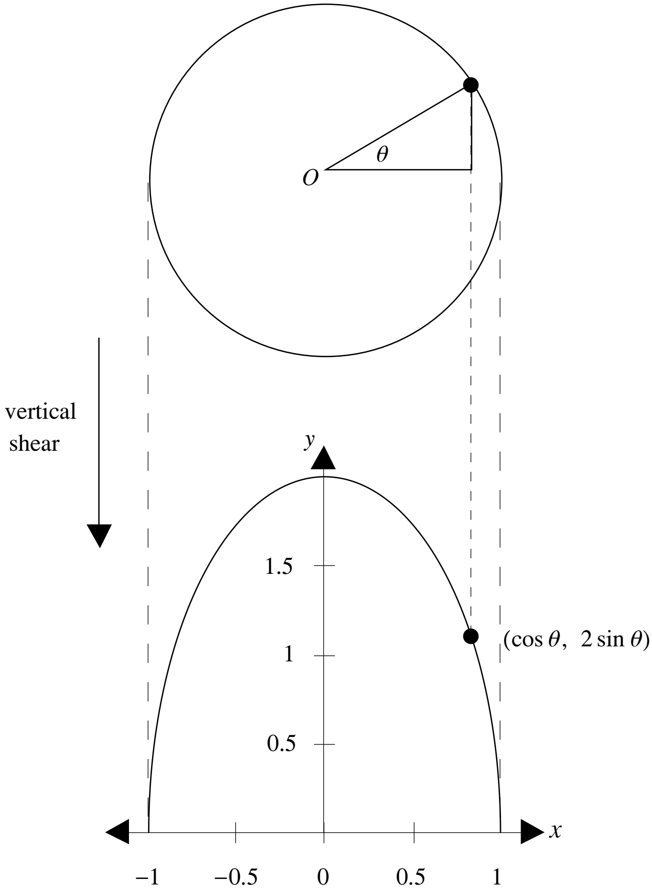
Teaching Note

$\int_0^\pi \sin^2\theta \, d\theta$ from a shear-like transformation of a circle

Shearing is an area-preserving transformation sometimes used in school mathematics to demonstrate that all parallelograms with the same base and height have the same area. Using the fact that the area of any shape is preserved under a shear, I consider here applying a shearing-like transformation to a unit circle. By symmetry, the direction of the shear is arbitrary, and we obtain the curve $(\cos\theta, \ 2\sin\theta)$, where θ is the angle at the centre of the circle shown below.



We know that the area of the unit circle is π, so this must also be the area under the curve. So,

$$\pi = \int_{-1}^{1} 2\sin\theta \, dx.$$

Since $x = \cos\theta$, $dx = -\sin\theta \, d\theta$, so

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$$\pi = -\int_{x=-1}^{x=1} 2\sin^2\theta\, d\theta = 2\int_0^{\pi} \sin^2\theta\, d\theta.$$

This means that

$$\int_0^{\pi} \sin^2\theta\, d\theta = \frac{\pi}{2}.$$

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Feedback

On Note 108.28: Erik Vigren writes: At the time of submitting and proof reading my Note [1], I was unaware of [2] and references within.

In [1], we have

$$\gamma(x, y) = \frac{y}{\sqrt{\frac{y}{x} - 1}} \tan^{-1}\sqrt{\frac{y}{x} - 1}, \tag{1}$$

as the limit of a recursively defined sequence in which the initial entries are x and y.

In [2] two sequences are considered generated by $a_{n+1} = M(a_n, b_n)$ and $b_{n+1} = M'(a_{n+1}, b_n)$ where M and M' are means. My construction is in essence the same, which can be realised by viewing the sequence entries with even and odd indices as parts of two separate, yet connected, sequences. Of particular relevance is (1) in [1], equivalent to (but not identical to) (12) in [2]. A proof of (12) in [2] appears in [3] and relations reminiscent of the Corollary (4) in my Note are also to be found in both [2] and [3].

References

1. Erik Vigren, π is a mean of 2 and 4, *Math. Gaz.* **108** (July 2024) pp. 331-334.
2. D. M. E. Foster and G. M. Phillips, The Arithmetic-Harmonic Mean, *Mathematics of Computation* **42** (165) (1984) pp. 183-191.
3. G. M. Phillips, Archimedes the numerical analyst, *Amer. Math. Monthly* **88** (1981) pp. 165-169.

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