A SIMPLE RING SEPARATING CERTAIN RADICALS

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All rings considered will be associative. For a class M of rings let UM be the class of all rings having no non-zero homomorphic image in M. A hereditary class M of prime rings is called a "special class" [see **1**, p. 191] if it has the property that when $I \in M$ with I an ideal of a ring R, then $R/I^* \in M$ where I^* is the annihilator of I in R, and the corresponding radical class UM is then a "special radical". Let S be the class of all subdirectly irreducible rings with simple heart.

PROPOSITION 1 [**1**; Theorem 7, p. 202]. *For any class W of simple rings the class M of all $R \in S$ with heart in W is a special class so UM is a special radical.*

For the class

$$T = \{R \in S \text{ whose heart contains an idempotent}\},$$

the special radical UT has been called the "Behrens radical" [see **3**]. Notice that if D is the class of all rings with unit, then the Brown–McCoy radical $UD \supseteq UT$. In attempting to characterize the Behrens radical one might consider the classes:

$$V = \{R \in S \text{ with von Neumann regular heart}\},$$

$$N = \{R \in S \text{ whose heart contains a minimal left ideal}\}.$$

Since both $V \subseteq T$ and $N \subseteq T$, we have

$$UT \subseteq UD \cap UV \cap UN. \tag{1}$$

Note that since a semiprime ring contains a minimal left ideal if and only if it contains a minimal right ideal [**3**, p. 65], we actually have

$$N = \{R \in S \text{ whose heart contains a minimal left and a minimal right ideal}\}.$$

Using the following proposition, we will show that the inequality in (1) is proper.

PROPOSITION 2. *The inclusion* (1) *is proper if and only if there exists a simple ring without unit, not von Neumann regular, and not containing a minimal one-sided ideal, but which does contain an idempotent.*

Proof. The sufficiency is clear, so suppose that there exists some $R \in UD \cap UV \cap UN$ such that $R \notin UT$. Since radical classes are homomorphically closed, we may assume that $R \in T$. But T is hereditary and since the radicals in (1) are special, they are also hereditary [**1**, Corollary 5, p. 195]. Thus the heart of R is a simple ring in $T \cap UD \cap UV \cap UN$.

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The following two propositions are well-known, but we give the proofs for completeness:

PROPOSITION 3. *If I is a proper right ideal of a simple ring R with unit, then $I/I \cap l(I)$ is a simple ring, where $l(I)$ is the left annihilator of I in R.*

Proof. If $x \notin l(I)$ then $xI \neq 0$ so by simplicity $0 \neq RxI = R$. Also since R has a unit, $IR = I$. Thus if $x \in I$ and (x) is the ideal of I generated by x, we have $IRxI = IR = I \subseteq (x)$. Thus $I = (x)$ and so $I/I \cap l(I)$ is simple.

PROPOSITION 4. *No proper right ideal of a prime ring has a unit.*

Proof. Let $0 \neq I$ be a right ideal of a prime ring R. If e is the unit of I then $R = I \oplus V$, where $V = \{x - ex\}$ for all $x \in R$. Since $ex = exe$, we have $ex(y - ey) = 0$ and so $IV = 0$. But in a prime ring this implies that $V = 0$, so that $I = R$ is not proper.

COROLLARY. *If the left annihilator $l(I) = 0$ for a proper right ideal I of a simple ring R with unit, then I is a simple ring without unit.*

Now consider the ring $R = Z_2[x, y, u, v]$ in non-commuting variables with relations $xu = yv = 1$, $xv = yu = 0$, and $ux = 1 + vy$. Note: R can be regarded as a polynomial ring in which all elements are "reduced"; that is, no term contains xu, yv, xv, yu, or ux and if such a term occurs in a product it is immediately reduced (using the above relations) [see **4** for details]. R is a ring with unit which has been shown [**5**, Theorem 2, p. 307] to be simple.

We consider the right ideal $I = (u+1)R$.

LEMMA 1. *I is a proper right ideal of R.*

Proof. Certainly $I \neq 0$. Suppose that $I = R$. It will follow that $(u+1)a = 1$ for some $a \in R$. But any longest term w of a will produce a longest term uw (or vyu' if $w = xu'$). In either case it will not be equal to any other term and so will not be cancelled in the product. Thus the product cannot equal 1.

LEMMA 2. $l(I) = 0$.

Proof. Suppose that $(h_0 + hx + gy)(u+1) = 0$, where h_0 is a function of u, v alone. We obtain $h_0(u+1) + h + hx + gy = 0$. A longest term w of h would produce an uncancelled longest term wx and thus $h = 0$. But then $gy = 0$, so that $g = 0$, and clearly $h_0(u+1) = 0$ implies that $h_0 = 0$.

LEMMA 3. *I is not von Neumann regular.*

Proof. If $(u+1)(u+1)a(u+1) = u+1$ for some $a \in R$, then, since $l(I) = 0$, we have $(u+1)^2 a = 1$, contradicting the fact that I is proper.

LEMMA 4. *I does not contain a minimal right ideal.*

Proof. We show first that any non-zero right ideal J of I contains a non-zero element $(u+1)b$, where b is a function of u and v alone. Suppose $0 \neq (u+1)a \in J$, where $a = h_0 + h_1x + h_2y$ with h_0 a function of u and v alone. Suppose first that $h_1 = 0$. Then if $h_0 \neq 0$ we can multiply $(u+1)a$ on the right by $(u+1)u$, or if $h_0 = 0$ multiply by $(u+1)v$ and use induction on the length of a.

Thus suppose that $h_1 \neq 0$. We use induction on the longest term of a ending in x. We have $(u+1)a(u+1)u = (u+1)(h_0u^2 + h_0u + h_1u + h_1)$. If any term ending in x were to remain, it would be in $h_1u + h_1$ shorter than the longest such term in h_1x, and the result would follow by induction. If no term ending in x remains but there is one ending in y, then again we can complete the proof by the argument of the first paragraph. We will thus have the desired result unless the product reduces to zero, that is unless $h_0u^2 + h_0u + h_1u + h_1 = 0$. But then the terms of h_1 would end in u (which is not permitted, since h_1x is reduced).

It is now clear that J cannot be minimal; for if $(u+1)b \in J$ with b a function of u and v alone, then this would mean $(u+1)b\,I = J$. Thus $(u+1)b(u+1)a = (u+1)b$, for some $a \in R$, which would then also have to be a function of u and v alone, giving terms on the left that are too long.

COROLLARY. *I contains no minimal left ideal.*

THEOREM. *The inclusion* (1) *is proper.*

Proof. From Lemmas 1 and 2, the ring I is simple without unit, and Lemmas 3 and 4 say that it is not von Neumann regular and does not contain a minimal one-sided ideal. However, I does contain the idempotent $(u+1)vy$.

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