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On vector spaces of certain modular forms of given weights

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Let p be a rational prime and Q_p be the field of p-adic numbers. Jean-Pierre Serre [Lecture Notes in Mathematics, **350**, 191-268 (1973)] had defined p-adic modular forms as the limits of sequences of modular forms over the modular group $SL_2(Z)$. He proved that with each non-zero p-adic modular form there is associated a unique element called its weight k. The p-adic modular forms having the same weight form a Q_p-vector space. The object of this paper is to obtain a basis of p-adic modular forms and thus to know precisely all p-adic modular forms of a given weight k. The dimension of such modular forms as a Q_p-vector space is countably infinite.

1. Notations and definitions

Let Z be the ring of rational integers and Q the field of rational numbers. From now on we will write for the modular forms over $SL_2(Z)$ simply "modular forms".

Let ν_p denote the valuation of the field of p-adic numbers Q_p which is normalised so that $\nu_p(p) = 1$. Let Z_p be the ring of p-adic integers; that is, $Z_p = \{x \mid x \in Q_p, \nu_p(x) \geq 0\}$.

For an even integer $k \geq 2$, take

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$$E_k = 1 - \frac{2k}{b_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n ,$$

where b_k is the kth Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. For $k \geq 4$, E_k is a modular form of weight k . We will denote E_2, E_4 , and E_6 by P, Q, R respectively.

As usual, we take,

$$\Delta = 2^{-6}3^{-3}\left(Q^3-R^2\right) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{1}^{\infty} \left(1-q^n\right)^{24} .$$

Then Δ is a cusp form of weight 12 .

Let $\mathbb{Q}_p[[q]]$ be the ring of formal power series in q with coefficients in $\mathbb{Q}_p$.

DEFINITION 1.1. Let $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{Q}_p[[q]]$. Define

$$\nu_p(f) = \inf_n \nu_p\left(a_n\right) .$$

DEFINITION 1.2. Let $\{f_i\}$ be a sequence of elements of $\mathbb{Q}_p[[q]]$. We say that $f_i \to f$, if the coefficients of f_i tend uniformly to those of f ; that is, $\nu_p\left(f-f_i\right) \to \infty$ as $i \to \infty$.

DEFINITION 1.3. Let $\{f_i\}$, $f_i = \sum_{n=0}^{\infty} a_n^{(i)} q^n$, be a sequence of modular form with coefficients $a_n^{(i)}$ rational. Let $f_i \to f = \sum_{n=0}^{\infty} a_n q^n$, with a_n in $\mathbb{Q}_p$, in the sense of Definition 1.2. Then f is called a *p-adic modular form*.

DEFINITION 1.4. Let m be an integer greater than or equal to 1 if $p \neq 2$ and $m \geq 2$ if $p = 2$. Define

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$$X_m = \begin{cases} (Z/p^{m-1}Z) \times Z/(p-1)Z\ , & \text{if } p \neq 2\ , \\ Z/2^{m-2}Z\ , & \text{if } p = 2\ . \end{cases}$$

Let X be the limit of the projective system $\{X_m\}$. Then

$$X = \varprojlim X_m = \begin{cases} Z_p \times Z/(p-1)Z\ , & \text{if } p \neq 2\ , \\ Z_2\ , & \text{if } p = 2\ . \end{cases}$$

Now we have the following theorem proved in [1].

THEOREM (Serre). *Let f be a p-adic modular form and let $\{f_i\}$ be a sequence of modular forms with rational coefficients. Let the weight of f_i be k_i and let $f_i \to f$. Then $\{k_i\}$ has a limit k in the group X . This limit depends only on f and not on the choice of the sequence $\{f_i\}$.*

DEFINITION 1.5. The limit k in X of $\{k_i\}$ as stated in the above theorem of Serre is called the *weight* of the p-adic modular form f .

DEFINITION 1.6. A series $\sum_{n=0}^{\infty} g_n$, where the g_n's are p-adic modular forms is called an *isobaric* series if each g_n has the same weight.

REMARK 1. In the theorem of Serre, since each k_i is an even integer, the limit k of $\{k_i\}$ is an element of $2X$. This implies that

(i) if p is an odd prime and $k = (s, u)$, then $s \in Z_p$ and u is an even integer $\bmod (p-1)$, and

(ii) if $p = 2$, then $k = s$, with $s \in Z_2$.

REMARK 2. If $\{f_i\}$ is a sequence of p-adic modular forms with

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weights k_i, and $f_i \to f$ then f is a p-adic modular form. We prove the following elementary lemma.

LEMMA. *Let f and g be any two formal series in q with coefficients from $\mathbb{Q}_p$. Then,*

$$\nu_p(fg) = \nu_p(f) + \nu_p(g) .$$

Proof. Let

$$f = \sum_{n=0}^{\infty} a_n q^n$$

and

$$g = \sum_{n=0}^{\infty} b_n q^n ,$$

where $a_n, b_n \in \mathbb{Q}_p$. Also let $\nu_p(f) = A$ and $\nu_p(g) = B$. Let

(i) $\nu_p(a_k) = A$, $\nu_p(a_n) > A$ for $n = 0, 1, \ldots, k-1$, and

(ii) $\nu_p(b_l) = B$, $\nu_p(b_n) > B$ for $n = 0, 1, \ldots, l-1$.

Now $fg = \sum_{n=0}^{\infty} c^n q^n$, where $c_n = \sum_{i+j=n} a_i b_j$. So $\nu_p(fg) = \inf_n \nu_p(c_n) \geq A + B$. Also $\nu_p(c_{k+l}) = \nu_p\left(\sum_{i+j=k+l} a_i b_j\right) = A + B$, whence $\nu_p(fg) \leq A + B$. Combining the two inequalities, we obtain the desired result. //

We can imbed $\mathbb{Z}_p$ in X if $p \neq 2$ by mapping $s \in \mathbb{Z}_p$ to $(s, 0)$ in X. Now we prove the following theorem.

THEOREM 1. *Let p be a prime number greater than or equal to 5 and let $s \in \mathbb{Z}_p$. Let $\{s_n\}$ be a sequence of non-negative rational integers such that $s_n \to s$ in $\mathbb{Z}_p$. Then the sequence $\left\{E_{p-1}^{s_n}\right\}$ of modular forms (the weight of $E_{p-1}^{s_n}$ is $(p-1)s_n$), is convergent in the sense of Definition 1.2 and its limit is a p-adic modular form of weight $(p-1)s$.*

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Proof. Since $s_n \to s$ in Z_p, so $s_{n+1} \equiv s_n \bmod p^{n+1}$ for $n = 0, 1, 2, \ldots$. Let $|s_{n+1}-s_n| = \lambda_{n+1}p^{n+1}$, where λ_{n+1} is an integer greater than or equal to 0. Hence

$$E_{p-1}^{s_{n+1}} - E_{p-1}^{s_n} = \varepsilon E_{p-1}^{t_n}\left(E_{p-1}^{\lambda_{n+1}p^{n+1}} - 1\right),$$

where $t_n = \min(s_{n+1}, s_n)$, and $\varepsilon = 1$ or -1.

Therefore

$$\nu_p\left(E_{p-1}^{s_{n+1}} - E_{p-1}^{s_n}\right) = \nu_p\left(E_{p-1}^{\lambda_{n+1}p^{n+1}} - 1\right) \text{ by Lemma 1 and the fact that } \nu_p(E_{p-1}) = 0$$
$$\geq (n+1) .$$

Thus $\left\{E_{p-1}^{s_n}\right\}$ is a convergent sequence of modular forms. Let its limit be denoted by E_{p-1}^{s} in the sense of Definition 1.2. Hence E_{p-1}^{s} is a p-adic modular form and its weight is $\lim_{n\to\infty} (p-1)s_n = (p-1)s$.

REMARK. The case $p = 2, 3$. If we take E_2 in place of E_{p-1} and replace the word "modular forms" by p-adic modular forms in the above theorem then by using Corollaire 2 of Théorème 21' of Serre [*1*] we find the theorem holds in these cases too.

Let p be an odd prime and let f be any p-adic modular form of weight $k = (s, u)$, $s \in Z_p$, $0 \leq u < p-1$, and u is even for $n = 0, 1, 2, \ldots$. Choose any non-negative integers $a(n)$ and $b(n)$ satisfying

$$4a(n) + 6b(n) + 12n \equiv u \bmod (p-1) .$$

Consider

$$(1) \qquad f_n = Q^{a(n)}R^{b(n)}\Delta^n E_{p-1}^{s_n}$$

for $n = 0, 1, 2, \ldots$, where

$$s_n = \frac{s-[12n+4a(n)+6b(n)]}{p-1} \in Z_p .$$

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From Theorem 1, it follows that $E_{p-1}^{s_n}$ is a p-adic modular form of weight $(p-1)s_n$ and hence f_n is a p-adic modular form of weight $k = (s, u)$. Let

$$(2) \qquad f_n = \sum_{m=0}^{\infty} a_m^{(n)} q^m .$$

Since $\Delta^n = q^n + \ldots$, and the constant term in each of Q, R, and E_{p-1} is 1, so

$$(3) \qquad a_m^{(n)} = 0 \text{ for } 0 \le m < n \text{ and } a_n^{(n)} = 1 .$$

With the notations as above for f_n we have the following theorem.

THEOREM 2. *f is a p-adic modular form of weight k iff $f = \sum a_n f_n$ with $\nu_p(a_n) \to \infty$ as $n \to \infty$.*

Proof. Let f be a p-adic modular form and let

$$f = \sum b_m^{(0)} q^m , \quad b_m^{(0)} \in \mathbb{Q}_p .$$

The series $f - b_0^{(0)} f_0$ has no constant term and is a p-adic modular form of weight k. Let

$$f - b_0^{(0)} f_0 = \sum_{m=1}^{\infty} b_m^{(1)} q^m .$$

Now consider

$$f - b_0^{(0)} f_0 - b_1^{(1)} f_1 \left(= \sum_{m=2} b_m^{(2)} q^m \text{ (say)}\right) .$$

It is a p-adic modular form of weight k. Continuing this process, we see that $f - \sum_{\gamma=0}^{t} b_\gamma^{(\gamma)} f_\gamma$ is a p-adic modular form of weight k for every non-negative integer t, and has no terms containing q^m for $m = 0, 1, \ldots, t$. Hence we can find $b_\gamma^{(\gamma)} \in \mathbb{Q}_p$ $(\gamma = 0, 1, 2, \ldots)$ such that as formal series in q, $f = \sum_{n=0}^{\infty} b_n^{(n)} f_n$; that is, $f = \varinjlim g_t$ where $g_t = \sum_{\gamma=0}^{t} b_\gamma^{(\gamma)} f_\gamma$. Now each g_t is a p-adic modular form of weight k and $\{g_t\}$ is a convergent sequence, so $\nu_p(g_{t+1}-g_t) \to \infty$ as $t \to \infty$. Hence $\nu_p\left(b_{t+1}^{(t+1)} f_{t+1}\right) \to \infty$ with t. Now $\nu_p(f_{t+1}) = 0$ for each t, so $\nu_p\left(b_{t+1}^{(t+1)}\right) \to \infty$. Hence taking $b_n^{(n)} = a_n$, we get $f = \sum a_n f_n$ with $\nu_p(a_n) \to \infty$ as $n \to \infty$.

Conversely let $f = \sum_{n=0}^{\infty} a_n f_n$, with $(a_n) \to \infty$ as $n \to \infty$. As above, taking $g_t = \sum_{\gamma=0}^{t} a_\gamma f_\gamma$, which is a p-adic modular form of weight k. Since $\nu_p(a_n) \to \infty$ with n and $\nu_p(f_n) = 0$, so $\{g_t\}$ is a convergent sequence with its limit equal to f. Hence f is a p-adic modular form.

COROLLARY 1. *Any p-adic modular form can be written as an isobaric series in Q, R, and E_{p-1}.*

Proof. Obvious.

COROLLARY 2. *The dimension of the Q_p-vector space of p-adic modular forms of weight k is countably infinite.*

Proof. In view of property (3) above the f_n's are linearly independent over $\mathbb{Q}_p$. Also from Theorem 2 any p-adic modular form can be written as a linear combination over $\mathbb{Q}_p$ of f_n's. So $\{f_n \mid n = 0, 1, 2, \ldots\}$ is a basis of p-adic modular forms of the given weight k.

REMARK 1. The case $p = 2$. Here we take

$$f_n = \Delta^n E_2^{s_n}, \quad s_n = \frac{s-12n}{2}. \tag{4}$$

Since $s \in 2\mathbb{Z}_2$, so $s_n \in \mathbb{Z}_2$. These f_n's have the property (3) and

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Theorem 2 and its corollaries are true if we replace E_{p-1} by E_2 $(= P)$.

REMARK 2. With the notations of [1], $\left\{\Delta^n E^*_{k-12n} \mid n = 0, 1, 2, 3, \ldots\right\}$ also forms a basis of p-adic modular forms of weight k .

Reference

[1] Jean-Pierre Serre, "Formes modulaires et fonctions zêta p-adiques", *Modular functions of one variable III*, 191-268 (Proc. Internat. Summer School, University of Antwerp, RUCA, 1972. Lecture Notes in Mathematics, **350**. Springer-Verlag, Berlin, Heidelberg, New York, 1973).

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