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RADICAL CLASSES NEED NOT HAVE A UNIQUE MAXIMAL R_0-CLOSED SUBCLASS

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1. **Introduction.** It was shown in [1] that certain classes of groups which are closed under quotients, and extensions contain a unique maximal R_0-closed subclass. These results prompted the question whether there exists a class of groups which is closed under quotients and extensions and yet does not have a unique maximal R_0-closed subclass. This note provides an example of such a class.

2. **Notations and Definitions.** We will use the closure operations notation for classes of groups as described for instance in [2] section 1.1. In particular if $\mathscr{X}$ is any given class of groups, $P\mathscr{X}$ will be the class of all groups which are extensions of $\mathscr{X}$-groups, $Q\mathscr{X}$ the class of all groups occurring as quotients of $\mathscr{X}$-groups, $R_0\mathscr{X}$ the class of all groups G containing a finite collection of normal subgroups $H_1, \ldots, H_n$ such that $\bigcap_{i=1}^{n} H_i = e$ and the factor groups G/H_i are $\mathscr{X}$-groups.

By a radical class we mean a class closed under quotients, extensions and normal joins, and by the radical class generated by a class $\mathscr{X}$ we mean the {Q,P,N}-closure of $\mathscr{X}$.

3. **The Example.** Let $\mathscr{A}$ be the class of groups which are extensions of elementary abelian 3-groups by elementary abelian 2-groups with the property that all their central factors are 2-groups. Similarly define $\mathscr{B}$ to be the class of extensions of elementary abelian 5-groups by elementary abelian 2-groups such that all the central factors are 2-groups. It is easy to see that the classes $\mathscr{A}$ and $\mathscr{B}$ are Q and R_0-closed. Therefore the class $\mathscr{A} \cup \mathscr{B}$ is Q-closed and, since $QP \leq PQ$, the class $\mathscr{X} = P(\mathscr{A} \cup \mathscr{B})$ is both P and Q-closed.

If there existed a unique maximal R_0-closed subclass of $\mathscr{X}$, it would contain both $\mathscr{A}$ and $\mathscr{B}$, hence it would contain also $\mathscr{A} \cup \mathscr{B}$ and $R_0(\mathscr{A} \cup \mathscr{B})$, and this would lead to a contradiction because $R_0(\mathscr{A} \cup \mathscr{B})$ is not contained in $\mathscr{X}$. In fact consider the group

$$G = \langle a, b, x; a^3 = b^5 = x^2 = e, [a, b] = e, a^x = a^{-1}, b^x = b^{-1}\rangle.$$

G contains the two disjoint normal subgroups $\langle a\rangle$ and $\langle b\rangle$; $G/\langle a\rangle$ belongs to $\mathscr{B}$ and $G/\langle b\rangle$ belongs to $\mathscr{A}$, thus $G \in R_0(\mathscr{A} \cup \mathscr{B})$. Clearly G does not belong to $\mathscr{A} \cup \mathscr{B}$, moreover it does not contain any proper subnormal $(\mathscr{A} \cup \mathscr{B})$-subgroup, therefore it cannot belong to $\mathscr{X}$ either.

It is worthwhile noticing that G does not belong even to the radical class generated by $\mathscr{X}$, for this is equal to $\{Q, P, N\}\mathscr{X}$, and, since G is finite, $G \in \{Q, P, N\}\mathscr{X}$

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implies $G \in \{Q, P, N_0\}\mathscr{X} = \mathscr{X}$ because $N_0 \leq PQ$. Thus we may conclude that not even radical classes need have a unique maximal R_0-closed subclass.

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References

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