Corrigenda to "Drift"

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In the author's paper "Drift" (Lighthill 1956) there are mistakes which need correction. Equation (28) should read

$$\phi \sim -\frac{m}{4\pi U r} \quad \text{as } r \to \infty, \tag{28}$$

since the definition of the disturbance potential ϕ (p. 35) states that the full velocity potential is $U(x+\phi)$. Hence by (18) equation (29) for the asymptotic form of the secondary flow due to a source should read

$$v_x = -\frac{Amy}{4\pi U(y^2+z^2)}\left(1+\frac{x}{r}\right), \qquad v_y = -\frac{Am}{4\pi U r}, \qquad v_z = 0, \tag{29}$$

where as well as the U's a 4π was accidentally missing from the version printed and a y misprinted as x.

A still graver error is made on p. 37 in the calculation of $\mathbf{v}_2$, the part of the asymptotic form of the secondary velocity field due to the trailing vorticity. The result given by equation (20) is doubly wrong; there should be a minus sign in front of this Biot–Savart integral, but more seriously the limiting process has been carried out incorrectly. This can be seen at once from the fact that the derived result (22) is a velocity field which is not irrotational in the region upstream of the vorticity which generates it.

The full Biot–Savart integral is

$$\mathbf{v}_2 = -\frac{1}{4\pi}\int \boldsymbol{\omega}_2(\mathbf{r}_1) \wedge \nabla \frac{1}{|\mathbf{r}-\mathbf{r}_1|}\, d\mathbf{r}_1, \tag{82}$$

and for $\nabla(1/|\mathbf{r}-\mathbf{r}_1|)$ in this to be replaced by $\nabla(1/r)$ for large r, as in (20), $\boldsymbol{\omega}_2(\mathbf{r}_1)$ must tend to 0 sufficiently rapidly as $r_1 \to \infty$. It is not sufficient that the part which is $O(r_1^{-3})$ be removed, as was done in the paper. The trailing vorticity itself must be removed, since it gives a contribution which cannot be estimated correctly by replacing $\nabla(1/|\mathbf{r}-\mathbf{r}_1|)$ by $\nabla(1/r)$.

The correct answer may be obtained by subtracting a distribution of 'horseshoe vortices', to remove the trailing vorticity. If $\mathbf{v}(\mathbf{r};\, y_0, z_0)$ is the velocity field of a horseshoe vortex with straight arms stretching from $(\infty, 0, 0)$ to $(0,0,0)$, thence to $(0, y_0, z_0)$, and thence to (∞, y_0, z_0), and unit circulation in the positive sense about each arm, and we put

$$\mathbf{v}_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\,\frac{\partial X(y_0, z_0)}{\partial z_0}\,\mathbf{v}(\mathbf{r};\, y_0, z_0)\, dy_0\, dz_0 \tag{83}$$

and $\boldsymbol{\omega}_3 = \operatorname{curl} \mathbf{v}_3$, then $\boldsymbol{\omega}_3$ has the same distribution of trailing vorticity as $\boldsymbol{\omega}_2$, and $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_3$ tends to zero as $r \to \infty$ in *all* directions. It is also a closed system of vortex lines, for the argument of equation (21) applied to $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_3$

shows that $\int(\boldsymbol{\omega}_2-\boldsymbol{\omega}_3)\,d\tau = 0$. Hence its velocity field falls off like r^{-3} and may be neglected in the present context, so that $\mathbf{v}_2$ and $\mathbf{v}_3$ are asymptotically equal.

Now, the theory of the horseshoe vortex tells us that far from the vortex

$$\mathbf{v}(\mathbf{r}\,;\,y_0, z_0) \sim \nabla\left\{\frac{y_0 z - z_0 y}{4\pi(y^2+z^2)}\left(1+\frac{x}{r}\right)\right\}, \tag{84}$$

the curly bracket being the potential of a line of doublets of uniform stength $(0, -z_0, y_0)$ per unit length stretching along the positive x-axis. Substitution of (84) in (83), with use of (14), gives

$$\mathbf{v}_2 \sim \mathbf{v}_3 \sim \frac{AV_h}{4\pi}\nabla\left\{\frac{y}{y^2+z^2}\left(1+\frac{x}{r}\right)\right\}. \tag{85}$$

Hence the corrected form of (24) is

$$\mathbf{v}_1 \sim \frac{A(V_b+V_h)}{4\pi}\left(-\frac{y}{r^3}, \frac{x}{r^3}, 0\right) + \frac{AV_h}{4\pi}\nabla\left\{\frac{y}{y^2+z^2}\left(1+\frac{x}{r}\right)\right\}. \tag{24}$$

Of particular interest is the value of (24) on $y = z = 0$ for negative x. Since the curly bracket becomes $y/2x^2$ under these circumstances, we see that

$$\mathbf{v}_1(x, 0, 0) \sim \left\{0, -\frac{A(V_b+\frac{1}{2}V_h)}{4\pi x^2}, 0\right\} \tag{86}$$

as $x \to -\infty$. The uncorrected (24) gave V_b+2V_h instead of $V_b+\frac{1}{2}V_h$ in (86).

The above correction affects the calculation of the asymptotic form of the so-called 'tertiary flow' (equation (25)). But this can in any case be criticized, as being based on a 'tertiary vorticity' field derived by considering only the stretching and rotation of the undisturbed vorticity by the sum of the primary and secondary flows, and not the shearing of the secondary vorticity pattern by the undisturbed shear flow, which gives tertiary-vorticity terms of the same order at large distances. Equation (26) for the type of equation which needs to be solved to get a better picture of the flow at large distances may be criticized on the same ground; the full linearized form of Helmholtz's equation for the vorticity is

$$(U+Ay)\frac{\partial\boldsymbol{\omega}}{\partial x} = -A\frac{\partial\mathbf{v}}{\partial z} + (A\omega_y, 0, 0),$$

and the last term (representing the effect just mentioned) should not be left out. However, the general conclusion that there are difficulties here, which need to be put right by some method of the general character described, is right; the correct answers about the form of the disturbances at large distances will be presented in a forthcoming paper in this journal.

Finally, the last formula of § 3 must be recalculated in the same way as (85) above; it becomes

$$\mathbf{v}_2 \sim \nabla\left[\frac{1+x/r}{4\pi(y^2+z^2)}\left\{x\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial V}{\partial z}X\,dydz + y\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial V}{\partial y}X\,dydz\right\}\right]. \tag{34}$$

Reference

Lighthill, M. J. 1956 *J. Fluid Mech.* **1**, 31.