Notes

Chapter 1

1 To avoid cluttering of brackets, we use the notation $e^2/4\pi \equiv e^2/(4\pi)$, etc. Furthermore, units $\hbar = 1$, $c = 1$ will be used. Then dimensions are like [mass] = [energy] = [momentum] = $[(\text{length})^{-1}] = [(\text{time})^{-1}]$, etc.

2 As a model for mesons we have to take the spins of the quarks into account. In a first approximation we can imagine neglecting spin-dependent forces. Then the maximum spin is $J = L + S$, with L the orbital angular momentum and $S = 0, 1$ the total spin of the quark–antiquark system. The π has the $q\bar{q}$ spins antiparallel, $S = 0$, the ρ has parallel $q\bar{q}$ spins, $S = 1$. In a second approximation spin-dependent forces have to be added, which split the π and ρ masses. In picking the right particles out of the tables of the Particle Data Group [2], we have to choose quantum numbers corresponding to the same S but changing L. This means that the parity and charge-conjugation parity flip signs along a Regge trajectory. The particles on the ρ trajectory in figure 1.3 are $\rho(769)$, $a_2(1320)$, $\rho_3(1690)$, and $a_4(2040)$, those on the π trajectory are $\pi(135)$, $\pi(135)$, $b_1(1235)$, and $\pi_2(1670)$. The mass m_q used in this model is an effective ('constituent') quark mass, $m_u \approx m_d \approx m_\rho/2 = 385$ MeV, which is much larger than the mass parameters appearing in the Lagrangian (the so-called 'current masses'), which are only a few MeV. In the last chapter we shall arrive at an understanding of this in terms of chiral-symmetry breaking.

Chapter 2

1 The formal canonical quantization of the scalar field in the continuum is done as follows. Given the Lagrangian of the system

$$L(\varphi, \dot{\varphi}) = \int d^3x \, \tfrac{1}{2}(\dot{\varphi})^2 - V(\varphi), \tag{N.1}$$

the canonical momentum follows from varying with respect to $\dot{\varphi}$,

$$\delta_{\dot{\varphi}} L = \int d^3x \, \dot{\varphi} \, \delta\dot{\varphi} \Rightarrow \pi \equiv \frac{\delta L}{\delta \dot{\varphi}} = \dot{\varphi}. \tag{N.2}$$

https://doi.org/10.1017/9781009402705.014 Published online by Cambridge University Press

Solving for $\dot{\varphi}$ in terms of π, the Hamiltonian is given by the Legendre transformation

$$H(\varphi, \pi) = \int d^3x\, \pi\dot{\varphi} - L(\varphi, \dot{\varphi}) = \int d^3x\, \tfrac{1}{2}\pi^2 + V(\varphi). \qquad \text{(N.3)}$$

Defining the Poisson brackets as

$$(A, B) = \int d^3x\, \frac{\delta A}{\delta\varphi(x)}\, \frac{\delta B}{\delta\pi(x)} - A \leftrightarrow B, \qquad \text{(N.4)}$$

the canonical (equal time) Poisson brackets are given by

$$(\varphi(\mathbf{x}), \pi(\mathbf{y})) = \delta(\mathbf{x} - \mathbf{y}), \quad (\varphi(\mathbf{x}), \varphi(\mathbf{y})) = 0 = (\pi(\mathbf{x}), \pi(\mathbf{y})). \qquad \text{(N.5)}$$

The Lagrange (stationary-action) equations of motion are then identical to Hamilton's equations

$$\dot{\varphi} = (\varphi, H), \quad \dot{\pi} = (\pi, H). \qquad \text{(N.6)}$$

The canonically quantized theory is obtained by considering the canonical variables as operators $\hat{\varphi}$ and $\hat{\pi}$ in Hilbert space satisfying the canonical commutation relations obtained from the correspondence principle Poisson bracket → commutator:

$$[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \quad [\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{y})] = 0 = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})]. \qquad \text{(N.7)}$$

Observables such as the Hamiltonian become operators (after symmetrizing products of $\hat{\varphi}$ and $\hat{\pi}$, if necessary). The quantum equations of motion then follow from Heisenberg's equations

$$\partial_0\hat{\varphi} = i[\hat{H}, \varphi], \quad \partial_0\hat{\pi} = i[\hat{H}, \hat{\pi}]. \qquad \text{(N.8)}$$

These need not, but often do, coincide with the classical equations of motion transcribed to $\hat{\varphi}$ and $\hat{\pi}$. From (N.7) one observes that the quantum fields are 'operator-valued distributions', hence products like $\hat{\pi}^2$ occuring in the formal Hamiltonian are mathematically ill-defined.

Chapter 4

1 The derivation leading to (4.72) is how I found the lattice gauge-theory formulation in 1972 (cf. [42]). I still find it instructive how a pedestrian approach can be brought to a good ending.

Chapter 8

1 Only Abelian chiral transformations form a group: if V and W are two chiral transformations, then $U = VW = V_{\rm L}W_{\rm L}P_{\rm L} + V_{\rm L}^\dagger W_{\rm L}^\dagger P_{\rm R}$ has $U_{\rm L} = V_{\rm L}W_{\rm L} \neq U_{\rm R}^\dagger = W_{\rm L}V_{\rm L}$, unless $V_{\rm L}$ and $W_{\rm L}$ commute.

2 This can be checked here by re-installing the lattice spacing, writing $M_f = m_f + 4r/a$, and $\psi_{fx} = a^{3/2}\psi_f(x)$, etc. with continuum fields $\psi(x)$, $\bar{\psi}(x)$ that are smooth on the lattice scale (the emerging overall factor a^3 must be dropped to get the continuum currents and divergences). Using for convenience the two-index notation for the lattice gauge field

($U_{\mu x} = U_{x,x+\mu}$, $U^{\dagger}_{\mu x-\hat{\mu}} = U_{x,x-\mu}$), we may write $U_{x,x\pm\hat{\mu}a}\psi_g(x \pm \hat{\mu}a) = \psi_g(x) \pm aD_\mu\psi_g(x) + \frac{1}{2}a^2D_\mu^2\psi_g(x) + \cdots$, with $D_\mu\psi_g(x) = [\partial_\mu - igG_\mu(x)]\psi_g(x)$ the continuum covariant derivative, this gives the expected result.

3 The way Σ is introduced here corresponds to four staggered flavors, $\Sigma = \sum_{f=1}^{4}\langle\bar{\psi}_f\psi_f\rangle$. Using the $SU(2)$ value $a\sqrt{\sigma} = 0.2634(14)$ [69] and $\sqrt{\sigma}$ = 420 MeV, the ratio $(0.00863/4)^{1/3}/0.263 = 0.491$ corresponds to 206 MeV or $\Sigma = 4(206\,\text{MeV})^3$. This number appears somewhat small, but we have to keep in mind that this is for $SU(2)$, not $SU(3)$, and it also has to be multiplied by the appropriate renormalization factor.

4 For staggered fermions to be sensitive to topology, quenched $SU(3)$ gauge couplings need to be substantially smaller than the value $\beta = 6/g^2 = 5.1$ used in [143, 144]. Vink [116, 117] found that values $\beta \gtrsim 6$ were needed in order to obtain reasonable correlations between the 'fermionic topological charge' and the 'cooling charge' (cf. figure 8.2). Note that the change $\beta = 5.1 \to 6$ corresponds to a decrease in lattice spacing by a factor of about four.

5 Ironically, when the mechanism of canceling the anomalies out between different fermion species was proposed [148], I doubted that it was necessary, and this was one of the reasons (apart from a non-perturbative formulation of non-Abelian gauge theory) why I attempted to put the electroweak model on the lattice. On calculating the one-loop gauge-field self-energy and the triangle diagram, I ran into the species-doubling phenomenon, without realizing that the lattice produced the very cancellation mechanism I had wanted to avoid.

6 At the time of writing the direct Euclidean approach is considered suspect and a Lorentzian formulation is being pursued [176]. For an impression of what is involved in a non-perturbative computation of gravitational binding energy, see [177].

7 The problem here is that, in order to deal with the oscillating phase $\exp(iS)$ in the path integral, one has to make approximations right from the beginning. To incorporate sphalerons, kinks, etc. one needs a lattice formulation that allows arbitrarily *in*homogeneous field configurations [178, 179].

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