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**PAPERS**

# Propagation dynamics of time-dependent reaction-diffusion equations under climate change in an infinite cylinder

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**Abstract**

In this paper, we consider a reaction-diffusion equation that models the time-almost periodic response to climate change within a straight, infinite cylindrical domain. The shifting edge of the habitat is characterised by a time-almost periodic function, reflecting the varying pace of environmental changes. Note that the principal spectral theory is an important role to study the dynamics of reaction-diffusion equations in time heterogeneous environment. Initially, for time-almost periodic parabolic equations in finite cylindrical domains, we develop the principal spectral theory of such equations with mixed Dirichlet–Neumann boundary conditions. Subsequently, we demonstrate that the approximate principal Lyapunov exponent serves as a definitive threshold for species persistence versus extinction. Then, the existence, exponential decay and stability of the forced wave solutions $U(t,x_1,y)=V\left(t,x_1-\int_0^t c(s)ds,y\right)$ are established. Additionally, we analyse how fluctuations in the shifting speed affect the approximate top Lyapunov exponent.

## 1. Introduction

This paper considers the following reaction-diffusion equation with Neumann boundary condition under climate change in an infinitely long straight cylinder:

$$\begin{cases} \partial_t u = \Delta u + f\left(t, x_1 - \int_0^t c(s)ds, y, u\right), \ t>0, x_1\in\mathbb{R}, y\in\omega, \\ \partial_\nu u(t,x_1,y)=0, \quad t>0, x_1\in\mathbb{R}, y\in\partial\omega. \end{cases} \tag{1.1}$$

We point that in (1.1) $\omega\subset\mathbb{R}^{N-1}$ is a bounded smooth domain. Here, $c:\mathbb{R}\to\mathbb{R}$ is Hölder continuous and almost periodic in the sense that for any $\epsilon>0$, the set

$$T(c,\epsilon)=\{\tau\in\mathbb{R}: |c(t+\tau)-c(t)|<\epsilon,\ t\in\mathbb{R}\}$$

is relatively dense in $\mathbb{R}$. Set

$$\Omega=\{(x_1,y)\in\mathbb{R}\times\mathbb{R}^{N-1}: x_1\in\mathbb{R}, y\in\omega\}.$$

Then, $\nu$ in (1.1) denotes the exterior unit normal vector field to $\Omega$ and $\partial_\nu := \nu\cdot\nabla$. In (1.1), $c(t)$ is the shifting speed of the climate envelope and $\int_0^t c(s)ds$ is the location of the climate envelope. Let $f(t,x,u)=ug(t,x,u)$, where the function $g:\mathbb{R}\times\Omega\times[0,+\infty)\to\mathbb{R}$ is continuously differentiable, uniformly continuous, and bounded on $\mathbb{R} \times \Omega \times [0, \delta]$ for any $\delta > 0$. Then, we have the following standard assumptions for $g(t, x, u)$:

(H1) $g(t, x, u)$ is almost periodic in $t$ uniformly with respect to $x \in \Omega$ and $u$ in bounded sets of $[0, +\infty)$;

(H2) $g_u(t, x, u) \le 0$ for all $(t, x, u) \in \mathbb{R} \times \Omega \times [0, +\infty)$, and there is a domain $D \subset \Omega$ with $|D| > 0$ such that $u \mapsto g(t, x, u)$ is decreasing on $\mathbb{R}^+$ for each $x \in D$ and $t \in \mathbb{R}$;

(H3) there is $M_0 > 0$ such that $\sup_{t\in\mathbb{R}, x\in\Omega} g(t, x, u) < 0$ for all $u \ge M_0$;

(H4) $\lim_{r\to\infty} \sup_{|x_1|>r, t\in\mathbb{R}, y\in\omega} g(t, x_1, y, 0) < 0$.

Here, (H1)–(H2) say that $f(t, x, u)$ is of logistic type and $M_0$ in (H3) is an upper bound for the carrying capacity. Assumption (H4) means that the favourable zone where species can grow is bounded. A typical example of $g(t, x, u)$ satisfying (H1)–(H4) is $g(t, x, u) = a(t, x) - b(t, x)u$ with $a(t, x), b(t, x) \in L^\infty(\mathbb{R} \times \Omega)$ such that $b(t, x) \ge 0$ a.e. in $\mathbb{R} \times \Omega$, $b(t, x) > 0$ in $\mathbb{R} \times D$ with $|D| > 0$ and $D \subset \Omega$,

$$\inf_{(t,x)\in\mathbb{R}\times\Omega\,:\,a(t,x)>0} b(t, x) > 0, \quad \lim_{r\to\infty} \sup_{|x_1|>r, t\in\mathbb{R}, y\in\omega} a(t, x_1, y) < 0,$$

where $a(t, x)$ and $b(t, x)$ are almost periodic in $t$ uniformly with respect to $x \in \Omega$. Recall that a function $f(t, x) \in C(\mathbb{R} \times E, \mathbb{R})$ is said to be almost periodic in $t$ uniformly with respect to $x \in E \subset \mathbb{R}^N$, if it is uniformly continuous on $\mathbb{R} \times E$ and for any fixed $x \in E$, $f(t, x)$ is an almost periodic function of $t$.

In the past years, to describe the impacts of global warming on biological species, a vast amount of research has been carried out regarding the population dynamics of reaction-diffusion equation system under climate change. See, for example, [1, 4, 5, 7, 8, 11, 13, 15, 20–22, 28] and references therein. We also refer to see [28] for more results and understanding on the diffusion and propagation in shifting environments. It should be pointed out that in Berestycki and Rossi [2], the authors proved the existence and uniqueness of travelling wave solutions of the type $u(t, x) = U(x - cte)$ for the reaction-diffusion equation

$$\partial_t u = \Delta u + f(x - cte, u), \qquad t > 0,\ x \in \mathbb{R}^N$$

and the large time behaviour of solutions with arbitrary nonnegative bounded initial datum depend on the sign of the generalised principle eigenvalue in $\mathbb{R}^N$ of an associated linear operator. These problems with $N = 1$ have been studied in [4] to investigate the impact of climate shift on the dynamics of a biological species. Berestycki and Rossi [3] further established analogous results for the Neumann problem in domains which are asymptotically cylindrical, as well as for the problem in the whole space with $f$ periodic in some space variables, orthogonal to the direction of the life. When the environment is assumed to be only globally unfavourable, with favourable pockets extending to infinity, Vo [27] studied the persistence versus extinction of species in reaction-diffusion equation with an infinite cylindrical domain. Boubours and Giletti [6] also studied the spreading and vanishing situations for a monostable reaction-diffusion equation when the initial datum is compactly supported. Recently, in a time-periodic shifting environment, Fang et al. [9] studied a nonautonomous reaction-diffusion equation $u_t = u_{xx} + ug(t, x - ct, u)$ and showed that there exists $c^* > 0$ such that a unique forced time-periodic wave exists if and only if $|c| < c^*$ and it attracts other solutions in a certain sense according to the tail behaviour of initial values. Later, Zhang and Zhao [30] extended the above results to the nonlocal dispersal equation in time-periodic shifting environment.

From above works, we find that the shift environment is considered by $x - ct$ with a constant speed. In real environments, climate change is influenced by various seasonal variations and is rather fluctuating and unpredictable, leading to fluctuations in the size, shifting speed, and location of the climate envelope. Very recently, Shen et al. [26] investigated the population dynamics of a reaction-diffusion equation under climate change in a spatiotemporally heterogeneous environment, which is described by a climate envelope shifting with a time-dependent speed function. They established the persistence criterion in term of the sign of the approximate top Lyapunov exponent and the existence of a unique forced wave solution that dominates the population profile of species in the long run.

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Motivated by [3] and [26], in the current paper, we try to study the persistence criterion and the existence of forced waves solution of equation (1.1) with Neumann boundary conditions in time-dependent environment. Set $\partial_1 := \frac{\partial}{\partial x_1}$. To consider (1.1) in the moving frame, we introduce the change of variable $v(t, x_1, y) = u(t, x_1 + \int_0^t c(s)ds, y)$, and $v(t, x_1, y)$ satisfies the following reaction-diffusion equation with Neumann boundary condition

$$\begin{cases} \partial_t v = \Delta v + c(t)\partial_1 v + f(t, x, v), \; t > 0, x \in \Omega, \\ \partial_\nu v(t, x) = 0, \quad t > 0, x \in \partial\Omega. \end{cases} \tag{1.2}$$

Note that $v = 0$ is a solution of (1.2). We first consider the stability of $v = 0$. Then the linearisation of (1.2) at $v = 0$ is

$$\begin{cases} \partial_t w = \Delta w + c(t)\partial_1 w + g(t, x, 0)w, \quad t > 0, \; x \in \Omega, \\ \partial_\nu w = 0, \quad t > 0, \; x \in \partial\Omega. \end{cases} \tag{1.3}$$

Our main results depend on the sign of Neumann Lyapunov exponent of Neumann problem (1.3). The results on the Neumann principal Lyapunov exponent of (1.3) are given in Section 2. Set $L > 0$. Denote by $\lambda_L$ the principal Lyapunov exponent of (1.3) restricted on $(-L, L) \times \omega$ and equiped with Neumann boundary condition $\partial_\nu w = 0$ on $(-L, L) \times \partial\omega$ and Dirichlet boundary condition on $\pm L$,

$$\begin{cases} \partial_t w = \Delta w + c(t)\partial_1 w + g(t, x, 0)w, \quad t > 0, \; x \in (-L, L) \times \omega, \\ \partial_\nu \omega = 0, \quad t > 0, \; x \in (-L, L) \times \partial\omega, \\ w(t, -L, y) = w(t, L, y) = 0, \quad t > 0, \; y \in \omega. \end{cases} \tag{1.4}$$

We refer the reader to Section 2 for more details on the principal Lyapunov exponent $\lambda_L$. It is shown in Lemma 2.2 that $\lambda_L$ is non-decreasing in $L$ and bounded above by $\sup_{t\in\mathbb{R}, x\in[-L,L]\times\omega} g(t, x, 0)$, then

$$\lambda_\infty := \lim_{L\to\infty} \lambda_L$$

is well defined and called the approximate principal Lyapunov exponent of (1.3). In fact, $\lambda_\infty$ always has the same sign as that of the principal Lyapunov exponent $\lambda$ of (1.3), see Lemma 2.3 later.

Let

$$X = \left\{ u \in C_b^1(\mathbb{R} \times \omega) : \nabla u \text{ is uniformly continuous}, \, \partial_\nu u(x_1, y) = 0, \; y \in \partial\omega \right\}$$

with the $C^1$ norm $\|u\| = \sup_{x\in\mathbb{R}\times\omega} (|u(x)| + |\nabla u(x)|)$, where $|\nabla u(x)| = \Sigma_{i=1}^N |u_{x_i}(x)|$. Let

$$X^+ = \{u \in X : u(x) \geq 0, x \in \mathbb{R} \times \omega\}.$$

For any $u_0(\cdot) \in X^+$, let $u(t, x; u_0)$ be the unique solution of (1.1) with $u(0, x; u_0) = u_0(x)$ for all $t \geq 0$.

**Theorem 1.1.** *Assume that (H1)-(H4) hold true. Then the following properties hold:*

(1) *If $\lambda_\infty < 0$, for any $u_0 \in X^+$, there is*

$$\lim_{t\to\infty} u(t, \cdot; u_0) = 0$$

*uniformly with respect to $x \in \Omega$.*

(2) *If $\lambda_\infty > 0$, then for any $u_0 \in X^+ \setminus \{0\}$ and any $L > 0$, there holds*

$$\liminf_{t\to\infty} \inf_{x_1\in[-L,L], y\in\omega} u\left(t, x_1 + \int_0^t c(s)ds, y; u_0\right) > 0.$$

A forced wave solution $U(t, x_1, y)$ of (1.1) is a positive, bounded and almost-periodic entire solution $V(t, x_1, y)$ of (1.2) with the form

$$U(t, x_1, y) = V\left(t, x_1 - \int_0^t c(s)ds, y\right), \quad t \in \mathbb{R}, (x_1, y) \in \mathbb{R} \times \omega. \tag{1.5}$$

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**Theorem 1.2.** *Assume that (H1)-(H4) hold true. If $\lambda_\infty > 0$, then (1.1) admits a unique forced wave solution $U(t, x_1, y)$ with*

$$\lim_{|x_1|\to\infty} U(t, x_1, y) = 0$$

*uniformly with respect to $t \in \mathbb{R}$ and $y \in \omega$. Moreover, for any $u_0 \in X^+ \setminus \{0\}$, there holds*

$$\lim_{t\to\infty} (u(t, x_1, y; u_0) - U(t, x_1, y)) = 0$$

*uniformly with respect to $x_1 \in \mathbb{R}$ and $y \in \omega$.*

We point out that Theorem 1.1 follows from Theorems 3.1 and 3.2. Theorem 1.2 follows directly from Theorem 4.1. From Theorem 1.1, we know that the sign of approximate principal Lyapunov exponent $\lambda_\infty$ is the sharp threshold for the persistence criterion. When $\lambda_\infty > 0$, the species will persist and (1.1) exists a unique, bounded and positive forced wave solution $U(t, x_1, y)$ in cylindrical type domain.

In order to analyse the effect of the shift speed on $\lambda_\infty$, we set $c(t) = c + A\sigma(t)$, where $A$ is positive constant and $A\sigma(t)$ is the fluctuation on the shifting speed $c$. Let $\lambda^A_\infty(c)$ be the approximate principal Lyapunov exponent depend on $c$ and $A$. By using the Liouville transform, Theorem 4.2 in Section 4 shows that there is a critical speed $c^* > 0$ such that $\lambda^0_\infty(c) > 0$ for $c \in [0, c^*)$ and $\lambda^0_\infty(c) < 0$ for $c > c^*$ when $A = 0$. This means that the species will persist when the climate envelope shifts with a slower shift speed. In this case, we also show that the force wave solution $U(t, x_1, y)$ is exponentially decay, see Theorem 4.3. When $A > 0$, we only show that $\lambda^A_\infty(c)$ has a lower bounded depend on $c$, see Theorem 4.2 (2). Due to the presence of cylinder domain and time almost periodic case, the characterisation of influence of the fluctuation on the shifting speed $A\sigma(t)$ on $\lambda^A_\infty(c)$ tends to be much more difficult. We leave it as an open problem.

We also remark here that we only consider the propagation dynamics of time-dependent reaction-diffusion equations (1.1) under (H4). Specifically, (H4) implies that the favourable habitat for species growth is contained in a bounded region, i.e., the favourable environment is surrounded by an unfavourable one. When

$$\lim_{r\to\infty} \sup_{x_1 > r, t\in\mathbb{R}, y\in\omega} g(t, x_1, y, 0) < 0, \quad \lim_{r\to\infty} \sup_{x_1 < -r, t\in\mathbb{R}, y\in\omega} g(t, x_1, y, 0) > 0,$$

the environment is favourable at $+\infty$ and unfavourable at $-\infty$. Moreover, the case

$$\lim_{r\to\infty} \sup_{|x_1| > r, t\in\mathbb{R}, y\in\omega} g(t, x_1, y, 0) > 0$$

implies the scenario that unfavourable environment is surrounded by favourable one. There is still few works on these two cases for time-dependent reaction-diffusion equations in an infinite cylinder and we will continue to study these two cases in the future. Very recently, what is worth mentioning is that Zhang and Zhao [31] studied spreading properties and forced travelling waves of reaction-diffusion equation in a time-heterogenous shifting environment in the case that favourable environment is surrounded by unfavourable oneenvironment is favourable at $+\infty$ and unfavourable at $-\infty$.

The rest of the paper is organised as follows. In Section 2, we will study the Lyapunov exponent theory of Neumann problem (1.3) and mixed boundary conditions problem (1.4). In Section 3, we consider the long-time dynamical behaviours of (1.1) with the initial value. and prove Theorem 1.1. We will prove the existence, uniqueness, stability and exponential decay of forced wave solutions of (1.1) in Section 4.

## 2. Lyapunov exponent theory with mixed boundary conditions

In this section, we study the principal spectral theory for time almost periodic parabolic equation with mixed Dirichlet–Neumann boundary condition in the finite cylinders $(-L, L) \times \omega$ for any $L > 0$. Recall that $\Omega = \mathbb{R} \times \omega$. Denote

$$\Omega_L = (-L, L) \times \omega \quad \text{for any } L > 0.$$

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Let $b_i(t,x)\,(i=1,\ldots,N), b_0(t,x)$ be bounded and continuous in $\mathbb{R}\times\Omega$. Assume that $b_i(t,x)\,(i=1,\ldots,N)$ and $b_0(t,x)$ are time almost-periodic in $t$ uniformly for $x\in\Omega$. For each $L>0$, we consider the following linear problem

$$\begin{cases}\partial_t u=\Delta u+\sum_{i=1}^N b_i(t,x)\frac{\partial u}{\partial x_i}+b_0(t,x)u, & t>0,\ x\in(-L,L)\times\omega,\\ \partial_\nu u=0, \quad t>0,\ x\in(-L,L)\times\partial\omega, & \\ u(t,-L,y)=u(t,L,y)=0, \quad t>0,\ y\in\omega. & \end{cases} \tag{2.1}$$

The principal spectral theory of linear parabolic problem with Dirichlet, Neumann boundary conditions or periodic boundary condition has been widely studied, for example, see [12, 16–18, 23–25, 29] and so on. Note that the linear problem (2.1) is considered in the finite cylinders $\Omega_L$ with Neumann boundary conditions on the sides $(-L,L)\times\partial w$ and Dirichlet boundary conditions on the bases $\{\pm L\}\times\omega$, which can be called as mixed Dirichlet–Neumann boundary conditions as that in [3]. In this section, we will develop the principal spectral theory for time almost periodic parabolic equation in finite cylinder with mixed Dirichlet–Neumann boundary conditions.

Let $b:=(b_i,b_0):=(\{b_i\}_{i=1}^N,b_0)$. Define

$$\mathrm{Y}(b)=cl\{b\cdot s: s\in\mathbb{R}\},$$

where $b\cdot s(t,x)=\sigma_s b(t,x)=b(t+s,x)$ and the closure is taken under the topology of local uniform convergence. Then, $(\mathrm{Y}(b),(\sigma_t)_{t\in\mathbb{R}})$ is a compact flow and unique ergodic, which implies that $\mathrm{Y}(b)$ is unique ergodic and minimal. For convenience, we write Y for $\mathrm{Y}(b)$.

Let

$$X_L=\{u\in C^1([-L,L]\times\omega,\mathbb{R}):\partial_\nu u=0 \text{ on } (-L,L)\times\partial\omega,\ u(-L,y)=u(L,y)=0, y\in\omega\}$$

be equipped with the $C^1$ norm $\|u\|_{X_L}=\max_{[-L,L]\times\omega}(|u(x)|+|\nabla u(x)|)$, where $|\nabla u(x)|=\Sigma_{i=1}^N|u_{x_i}(x)|$. Let

$$X_L^+=\{u\in X_L: u\geq 0, x\in[-L,L]\times\omega\}.$$

Assume $\tilde{X}_L\subset L^p(\Omega_L)(p>n)$ be a fractional power space of $-\Delta:\mathcal{D}\to L^p(\Omega_L)$ satisfying $\tilde{X}_L\hookrightarrow C^\alpha(\Omega_L)$ for some $1<\alpha<2$, where $\mathcal{D}=\{v\in H^{2,p}(\Omega_L): u(-L,y)=u(L,y)=0, \partial_\nu u(x_1,y)=0, y\in\partial\omega\}$. Then, for any $\widetilde{b}\in\mathrm{Y}$ and $u_0\in\tilde{X}_L$, there is a unique classical solution $u(t,\cdot;u_0,\widetilde{b})$ of (2.1) with the initial condition $u(0,\cdot;u_0,\widetilde{b})=u_0\in\tilde{X}_L$. Put

$$U_L(t,\widetilde{b})u_0 := u(t,\cdot;u_0,\widetilde{b}) \quad \text{for } u_0\in\tilde{X}_L.$$

For convenience of expression, in the following, we still write $b$ for $\widetilde{b}$. Then, (2.1) generates a skew-product semiflow on $\tilde{X}_L\times\mathrm{Y}(b)$:

$$\Pi_t:\tilde{X}_L\times\mathrm{Y}\to\tilde{X}_L\times\mathrm{Y},\ t\geq 0,$$
$$\Pi_t(u_0,b)=(U_L(t,b)u_0, b\cdot t).$$

Let

$$\tilde{X}_L^+=\{u\in\tilde{X}_L: u\geq 0, x\in[-L,L]\times\omega\}.$$

Then, we have the following exponential separation theorem from [17].

**Theorem 2.1** (Exponential Separation). *There are subspace $\tilde{X}_L^1(b),\tilde{X}_L^2(b)$ of $\tilde{X}_L$ such that $\tilde{X}_L^1(b),\tilde{X}_L^2(b)$ are continuous in $b\in\mathrm{Y}$, and satisfy the following properties:*

(1) $\tilde{X}_L=\tilde{X}_L^1(b)\oplus\tilde{X}_L^2(b)$ *for any* $b\in\mathrm{Y}$.

(2) $\tilde{X}_L^1(b)=span\{\widetilde{\phi}_L(b)\}$, $\widetilde{\phi}_L(b)\in Int(\tilde{X}_L^+)$ *and is continuous in* $b$ *and* $\|\widetilde{\phi}_L(b)\|_{\tilde{X}_L}=1$ *for any* $b\in\mathrm{Y}$.

(3) $\tilde{X}_L^2(b)\cap Int(\tilde{X}_L^+)=\emptyset$ *for any* $b\in\mathrm{Y}$.

(4) $U_L(t,b)\tilde{X}_L^1(b)=\tilde{X}_L^1(\sigma_t b)$ *and* $U_L(t,b)\tilde{X}_L^2(b)\subset\tilde{X}_L^2(\sigma_t b)$ *for any* $b\in\mathrm{Y}$ *and* $t>0$.

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(5) *There are* $M, \gamma > 0$ *such that*

$$\frac{\|U_L(t,b)\widetilde{\phi}_L(b)\|_{\tilde{X}_L}}{\|U_L(t,b)w\|_{\tilde{X}_L}} \le Me^{-\gamma t}$$

*for any* $t > 0$, $b \in \mathrm{Y}$ *and* $w \in \tilde{X}_L^2(b)$ *with* $\|w\|_{\tilde{X}_L} = 1$.

Let

$$\phi_L(b) = \frac{\widetilde{\phi}_L(b)}{\|\widetilde{\phi}_L(b)\|_{L^2(\Omega_L)}}$$

and

$$\kappa_L(b) = \langle \Delta\phi_L(b) + \sum_{i=1}^N b_i(t,x)\frac{\partial \phi_L(b)}{\partial x_i} + b_0(t,x)\phi_L(b), \phi_L(b)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\Omega_L)$. Then, $\kappa_L(b)$ is continuous with respect to $b \in \mathrm{Y}$. Set

$$\eta_L(t,b) = \|U_L(t,b)\phi_L(b)\|_{L^2(\Omega_L)}.$$

Then, we have

$$\frac{d\eta_L(t,b)}{dt} = \kappa_L(\sigma_t b)\eta_L(t,b)$$

and

$$\eta_L(t,b) = e^{\int_0^t \kappa_L(\sigma_\tau b)d\tau}.$$

Then,

$$\phi_L(\sigma_t b)(x) := \frac{U_L(t,b)\phi_L(b)}{\|U_L(t,b)\phi_L(b)\|_{L^2}}$$

satisfies

$$\begin{cases} \dfrac{\partial \phi_L(\sigma_t b)}{\partial t} = \Delta\phi_L(\sigma_t b) + \sum_{i=1}^N b_i(t,x)\dfrac{\partial \phi_L(\sigma_t b)}{\partial x_i} + b_0(t,x)\phi_L(\sigma_t b) \\ \qquad\qquad -\kappa_L(\sigma_t b)\phi_L(\sigma_t b), \quad t > 0,\ x \in (-L,L)\times\omega, \\ \partial_\nu \phi_L(\sigma_t b) = 0, \quad t > 0,\ x \in (-L,L)\times\partial\omega, \\ \phi_L(\sigma_t b)(-L,y) = \phi_L(\sigma_t b)(L,y) = 0, \quad t > 0,\ y \in \omega. \end{cases} \tag{2.2}$$

Define $\|U_L(t,b)\| = \sup_{u\in\tilde{X}_L,\,\|u\|_{\tilde{X}_L}=1} \|U_L(t,b)u\|_{\tilde{X}_L}$. Note that Y is unique ergodic, then

$$\lim_{t-s\to\infty} \frac{1}{t-s}\ln\|U_L(t-s,\sigma_s b)\| = \lim_{t-s\to\infty}\frac{1}{t-s}\int_s^t \kappa_L(\sigma_\tau b)d\tau$$

for all $b \in Y$.

**Definition 2.1** (Principal Lyapunov exponent). *The number*

$$\lambda_L(\mathrm{Y}) := \lim_{t-s\to\infty}\frac{1}{t-s}\ln\|U_L(t-s,\sigma_s b)\|, \quad \forall b \in \mathrm{Y}$$

*is called the principal Lyapunov exponent of (2.1).*

Note that $U_L(t,b)$ can act on $X_L$, and the principal Lyapunov exponents of $U_L(t,b)$ on $\tilde{X}_L$ and $X_L$ coincide. Then, we have the following

**Lemma 2.1.** *There is a continuous function* $\phi_L : Y \to X_L^+ \setminus \{0\}$ *such that the following hold for any* $\tilde{b} \in Y$:

*(1)* $\|\phi_L(\tilde{b})\|_\infty = 1$,

*(2)* $U_L(t,\tilde{b})\phi_L(\tilde{b}) = \|U_L(t,\tilde{b})\phi_L(\tilde{b})\|_\infty \phi_L(\tilde{b}\cdot t)$,

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*(3)* $\lim_{t\to\infty} \dfrac{\ln \|U_L(t,\tilde{b})\phi_L(\tilde{b})\|_\infty}{t} = \lambda_L(Y)$,

*(4)* *For any* $\phi_0 \in X_L^+ \setminus \{0\}$, $\lim_{t\to\infty} \dfrac{\ln \|U_L(t,\tilde{b})\phi_0\|_\infty}{t} = \lambda_L(Y)$.

**Remark 2.1.**

(1) *Observing that* $\lambda_L(\mathrm{Y})$ *and* $\phi_L(\sigma_t b)(x)$ *are analogues of principal eigenvalue and principal eigenfunctions of elliptic problem, respectively. We call* $\mathrm{span}\{\phi_L(\sigma_t b)(x)\}_{t\in\mathbb{R}}$ *as the principal Floquet bundle of (2.1) associate to the principal Lyapunov exponent.*

(2) *The principal eigenvalue of the following elliptic problem with mixed Dirichlet–Neumann boundary conditions,*

$$\begin{cases} -\Delta u - \sum_{i=1}^N b_i(x)\frac{\partial u}{\partial x_i} - b_0(x)u = \lambda_L u, & x \in (-L,L)\times\omega, \\ \partial_\nu u = 0 \quad \text{on } (-L,L)\times\partial\omega, \\ u(-L,y) = u(L,y) = 0 \quad \text{on } y \in \overline{\omega} \end{cases} \tag{2.3}$$

*has been studied in [3, Theorem 3.1]. Their results show that for any* $L>0$, *there exists a unique principal eigenvalue* $\lambda_L$ *of (2.3) corresponding to a positive eigenfunction* $\psi_L \in W^{2,p}(\Omega_L)$, *for any* $p>1$. *Moreover,* $\lambda_L$ *converges to* $\lambda_1(\Omega)$ *as* $L\to\infty$, *where* $\lambda_1(\Omega)$ *is the generalised Neumann principal eigenvalue defined by*

$$\lambda_1(\Omega) := \sup\{\lambda\in\mathbb{R} : \exists\phi>0,\ \Delta\phi + \sum_{i=1}^N b_i(x)\frac{\partial\phi}{\partial x_i} + b_0(x)\phi + \lambda u \le 0,\ \text{a.e. in } \Omega,$$
$$\partial_\nu\phi \ge 0 \text{ on } \partial\Omega\}.$$

*We point out in* $\lambda_1(\Omega)$, *the function* $\phi$ *is understood to belong to* $W^{2,p}((-L,L)\times\omega)$ *for some* $p>N$ *and every* $L>0$.

**Lemma 2.2.** *Set* $\lambda(L) := \lambda_L(\mathrm{Y})$. *Then, the function* $\lambda(L) : \mathbb{R}^+ \to \mathbb{R}$ *is nondecreasing in* $L$ *and* $\lim_{L\to\infty}\lambda(L)$ *exists.*

**Proof.** For any $L>0$, let $\phi_L(\sigma_t b)(x)$ be as in Lemma 2.1. Let $0<L_1<L_2$ be fixed. Then, there exists $\epsilon_0\in(0,1)$ such that

$$\epsilon_0\phi_{L_1}(\sigma_t b)(x) < \phi_{L_2}(\sigma_t b)(x) \quad \text{on } [-L_1,L_1]\times\omega.$$

It then follows from the comparison principle that

$$\epsilon_0 U_{L_1}(t,b)\phi_{L_1}(\sigma_t b)(x) < U_{L_2}(t,b)\phi_{L_2}(\sigma_t b)(x) \quad \text{on } [-L_1,L_1]\times\omega.$$

Then

$$\lambda(L_2) = \lim_{t\to\infty}\frac{\ln\|U_{L_2}(t,b)\phi_{L_2}(\sigma_t b)\|_\infty}{t} \ge \lim_{t\to\infty}\frac{\ln\|\epsilon_0 U_{L_1}(t,b)\phi_{L_1}(\sigma_t b)\|_\infty}{t} = \lambda(L_1),$$

which implies that $\lambda(L)$ is non-decreasing in $L$.

Let $b_L := \sup_{t\in\mathbb{R},x\in\Omega_L} b_0(t,x)$. Then, $e^{b_L t}$ is a supersolution of (2.1), and thus,

$$U_L(t,b)\phi_L(\sigma_t b) \le e^{b_L t} \quad \text{in } \Omega_L$$

for all $t\ge 0$. It then follows that

$$\lambda(L) \le \lim_{t\to\infty}\frac{\ln e^{b_L t}}{t} = b_L.$$

Hence, $\lim_{L\to\infty}\lambda(L)$ exists. This completes the proof. □

From Lemma 2.2, we know that $\lim_{L\to\infty}\lambda_L(\mathrm{Y})$ exists. Set

$$\lambda_\infty(\mathrm{Y}) := \lim_{L\to\infty}\lambda_L(\mathrm{Y}).$$

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Then $\lambda_\infty(\mathrm{Y}) < \sup_{t\in\mathbb{R},x\in\Omega} b_0(t,x)$ and $\lambda_\infty(\mathrm{Y})$ is called the approximate principal Lyapunov exponent of the following equation

$$\begin{cases} \partial_t u = \Delta u + \sum_{i=1}^N b_i(t,x)\frac{\partial u}{\partial x_i} + b_0(t,x)u, & x \in \mathbb{R}\times\omega, \\ \partial_\nu u = 0 \quad \text{on } \mathbb{R}\times\partial\omega. \end{cases} \tag{2.4}$$

Let $X$ and $X^+$ be as in Section 1. Denote by $U(t,b)u_0$ the unique classical solution of (2.4) with the initial value $u_0 \in X$ and define $\|U(t,b)\| = \sup_{u\in X,\,\|u\|=1} \|U(t,b)u\|$. It follows from the almost-periodicity of $b(t,x)$ and the subadditive ergodic theorem (see [14]) that $\lim_{t\to\infty}\frac{\ln U(t,b)}{t}$ exists for all $b \in Y$. Then, we have the following definition for the principal Lyapunov exponent of (2.4).

**Definition 2.2** (Principal Lyapunov exponent). *The number*

$$\lambda(\mathrm{Y}) := \lim_{t-s\to\infty} \frac{1}{t-s}\ln\|U(t-s,\sigma_s b)\|, \quad \forall b\in\mathrm{Y}$$

*is called the principal Lyapunov exponent of (2.4).*

In the following, we show that the connections between the approximate principal Lyapunov exponent $\lambda_\infty(\mathrm{Y})$ and the principal Lyapunov exponent $\lambda(\mathrm{Y})$.

**Lemma 2.3.** *$\lambda_\infty(\mathrm{Y})$ and $\lambda(\mathrm{Y})$ have the same sign. In particular, $\lambda_\infty(\mathrm{Y}) = \lambda(\mathrm{Y})$ for $\lambda_\infty(\mathrm{Y}) \geq \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} b_0(t,x_1,y)$.*

**Proof.** In the following proof, to emphasis the dependence of Y on $b$, we write it as $\mathrm{Y}(b)$. We divide the proof into two cases: $\lambda_\infty(\mathrm{Y}(b)) \geq \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} b_0(t,x_1,y)$ and $\lambda_\infty(\mathrm{Y}(b)) < \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} b_0(t,x_1,y)$.

Firstly, we consider the case $\lambda_\infty(\mathrm{Y}(b)) \geq \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} b_0(t,x_1,y)$. By definitions of $\lambda_\infty(Y(b))$ and $\lambda(Y(b))$, we have $\lambda_\infty(Y(b)) \leq \lambda(Y(b))$. Assume that $\lambda_\infty(Y(b)) < \lambda(Y(b))$. Without loss of generality, we may assume that

$$\lambda_\infty(Y(b)) < 0 < \lambda(Y(b)).$$

In fact, we can take $\lambda_0 \in (\lambda_\infty(Y(b)), \lambda(Y(b)))$ and replace $b_0(t,x)$ by $\bar{b}_0(t,x) = b_0(t,x) - \lambda_0$. Then

$$\lambda_\infty(Y(\bar{b})) = \lambda_\infty(Y(b)) - \lambda_0 < 0 < \lambda - \lambda_0 = \lambda(Y(\bar{b}))$$

and

$$\lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} \bar{b}_0(t,x_1,y) < 0.$$

Consider

$$\begin{cases} \partial_t u = \Delta u + \sum_{i=1}^N a_i(t,x)\frac{\partial u}{\partial x_i} + ug(t,x,u), & x\in\mathbb{R}\times\omega, \\ \partial_\nu u = 0 \quad \text{on } \mathbb{R}\times\partial\omega. \end{cases} \tag{2.5}$$

Denote $v(t,x;v_0)$ be the solution of (2.5) with initial value $v(t,x;v_0) = v_0 \in X^+$. Choose $v_0^* \equiv 1$, similar to that in Theorem 3.1 later, we have that $\lim_{t\to\infty}\|v(t,\cdot;v_0^*)\|_\infty = 0$ for $\lambda_\infty(Y(b)) < 0$. Note that $\lambda(Y(b)) > 0$ and $g$ is uniformly continuous on $\mathbb{R}\times\Omega\times[0,\delta]$ for any $\delta > 0$, then there exists large $T > 0$ such that

$$g(t,x,0) \leq g(t,x,v(t,x;v_0^*)) + \frac{\lambda(Y(b))}{2}$$

for any $t \geq T$ and $x \in \mathbb{R}\times\omega$. By comparison principle,

$$U(t,\sigma_T b)v(T,\cdot;v_0^*) \leq e^{\frac{\lambda(Y(b))}{2}t} v(t+T,\cdot;v_0^*), \quad \forall t\geq 0. \tag{2.6}$$

Due to that $v_0^*$ is uniform positive and $g$ is bounded on $\mathbb{R}\times\Omega\times[0,\delta]$ for any $\delta > 0$,

$$\inf_{x\in\mathbb{R}\times\omega} v(T,\cdot;v_0^*) > 0.$$

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Then for any $v_0 \in X^+$ with $\|v_0\| = 1$, there is a constant $M > 0$ such that

$$-Mv(T,\cdot;v_0^*) \le v_0 \le Mv(T,\cdot;v_0^*), \quad x \in \mathbb{R}\times\omega.$$

Then the comparison principle implies that

$$-MU(t,\sigma_T b)v(T,\cdot;v_0^*) \le U(t,\sigma_T v_0) \le MU(t,\sigma_T b)v(T,\cdot;v_0^*), \quad t \ge 0,$$

which implies that

$$\|U(t,\sigma_T b)v_0\|_\infty \le M\|U(t,\cdot)v(T,\cdot;v_0^*)\|_\infty, \quad \forall t \ge 0.$$

Then by the definition of $\lambda(Y(b))$ and together with (2.6), we have that

$$0 < \lambda(Y(b)) \le \frac{\lambda(Y(b))}{2},$$

which is a contradiction. Therefore, $\lambda_\infty(Y(b)) = \lambda(Y(b))$ and they have the same sign in the case $\lambda_\infty(\mathrm{Y}(b)) \ge \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} b_0(t,x_1,y)$.

Secondly, we consider the case $\lambda_\infty(\mathrm{Y}(b)) < \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} b_0(t,x_1,y)$. In this case, we show that if $\lambda_\infty(Y(b)) < 0$, then $\lambda(Y(b)) < 0$. Suppose that $\lambda(Y(b)) \ge 0$. Then there exists $\lambda_0 > 0$ such that

$$\lambda_\infty(Y(b)) + \lambda_0 < \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} b_0(t,x_1,y) + \lambda_0 < 0 < \lambda(Y(b)) + \lambda_0.$$

Let $a(t,x) = g(t,x,0) + \lambda_0$. Then,

$$\lambda_\infty(Y(b')) < \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} a(t,x_1,y) < 0 < \lambda(Y(b')).$$

Similar to above, we have

$$\lambda_\infty(Y(b')) < \frac{\lambda_\infty(Y(b'))}{2},$$

which is a contradiction. Thus $\lambda(Y(b)) < 0$. This completes the proof. □

**Proposition 2.1.** *Suppose that $b_i(t,x)$ $(i = 1,\dots,N)$ and $b_0(t,x)$ are periodic in $t$. If $\lambda_\infty > \lim_{r\to\infty}\sup_{|x_1|>r,t\in\mathbb{R},y\in\omega} b_0(t,x_1,y)$, then $\lambda_\infty$ is the principal eigenvalue (i.e. the eigenvalue with large real part) of the following linear periodic equation*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^N b_i(t,x)\frac{\partial u}{\partial x_i} + b_0(t,x)u, & t>0,\ x\in\mathbb{R}\times\omega,\\ \partial_\nu u = 0, & t>0,\ x\in\mathbb{R}\times\partial\omega. \end{cases}$$

**Proof.** Since $b_i(t,x)$ $(i = 1,\dots,N)$ and $b_0(t,x)$ are periodic in $t$, by the definition of $\lambda_L$, we know that $\lambda_L$ is the principal eigenvalue of (2.1) for any $L > 0$. Thus, there exists a bounded positive time periodic eigenfunction $\phi_L$ of (2.1) corresponding to $\lambda_L$, that is,

$$\begin{cases} \frac{\partial \phi_L}{\partial t} = \Delta\phi_L + \sum_{i=1}^N b_i(t,x)\frac{\partial \phi_L}{\partial x_i} + b_0(t,x)\phi_L - \lambda_L\phi_L, & t>0,\ x\in(-L,L)\times\omega,\\ \partial_\nu\phi_L = 0, & t>0,\ x\in(-L,L)\times\partial\omega,\\ \phi_L(t,-L,y) = \phi_L(t,L,y) = 0, & t>0,\ y\in\omega,\\ \phi_L \text{ is periodic in } t. \end{cases} \tag{2.7}$$

Normialised $\phi_L$ by $\max_{\mathbb{R}\times(-L,L)\times\omega}\phi_L(t,x_1,y) = 1$ and let $(t_L,x_{1L},y_L) \in [0,T]\times(-L,L)\times\omega$ be such that $\phi_L(t_L,x_{1L},y_L) = 1$. Extending by reflection the function $\phi_L$ to large cylinders, as done in [3, Appendix A], then $\phi_L$ is uniformly bounded in $\mathbb{R}\times(-L,L)\times\omega$. Hence, by standard parabolic regularity estimates,

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there exists $\phi_\infty$ such that $\phi_L(t,x) \to \phi_\infty(t,x)$ locally uniformly in $\mathbb{R} \times (-L, L) \times \omega$ as $L \to \infty$, where $\phi_\infty$ is a bounded, nonnegative and time periodic function satisfying

$$\begin{cases} \frac{\partial \phi_\infty}{\partial t} = \Delta \phi_\infty + \sum_{i=1}^{N} b_i(t,x) \frac{\partial \phi_\infty}{\partial x_i} + b_0(t,x)\phi_\infty - \lambda_\infty \phi_\infty, & t > 0,\ x \in \mathbb{R} \times \omega, \\ \partial_\nu \phi_\infty = 0, & t > 0,\ x \in \mathbb{R} \times \partial\omega, \\ \phi_\infty \text{ is periodic in } t. \end{cases} \tag{2.8}$$

Consider (2.7) at point $(t_L, x_{1L}, y_L)$, we have

$$0 = \Delta \phi_L(t_L, x_{1L}, y_L) + b_0(t_L, x_{1L}, y_L)\phi_L - \lambda_L \phi_L \le (b_0(t_L, x_{1L}, y_L) - \lambda_L)\phi_L.$$

Since $\lambda_\infty > \lim_{r\to\infty} \sup_{|x_1|>r, t\in\mathbb{R}, y\in\omega} b_0(t, x_1, y)$, we must have that $x_{1L}$ is bounded as $L \to \infty$. Thus, $(t_L, x_{1L}, y_L) \to (t^*, x_1^*, y^*)$ as $L \to \infty$, and $\phi_\infty(t^*, x_1^*, y^*) = 1$. The strong maximum principal yields that $\phi_\infty > 0$ in $\mathbb{R} \times \Omega$. Then, $\lambda_\infty$ is the principal eigenvalue of (2.8) with principal eigenfunction $\phi_\infty$. □

## 3. Large-time dynamical behaviour in infinite cylinder

In this section, we consider the long-time dynamical behaviour of the solution $u(t, x_1, y; u_0)$ of (1.1) in the infinite cylinder and prove Theorem 1.1.

Let $X$ and $X^+$ be given in Section 1. By the semigroup theory, for any initial value $v_0 \in X^+$, (1.2) admits a unique solution $v(t, x; v_0)$ with $v(0, x; v_0) = v_0$. Set $S = \max\{M_0, \|v_0\|_{L^\infty(\Omega)}\}$, where $M_0$ is the positive constant in (H3). Due to that 0 and $S$ are a sub-and supersolution of (1.2) with initial value $v_0 \in X^+$, then $0 \le v(t, x; v_0) \le S$ in $\overline{\Omega}$. By extending $v(t, x; v_0)$ to a large cylinder $(\mathbb{R} \times \overline{\omega}) \supset\supset \Omega$ through reflection (see [3, Appendix A]) and applying the parabolic maximum principle, we find that

$$0 < v(t, x_1, y; v_0) \le S \quad \text{in } \mathbb{R}^+ \times \Omega.$$

Take $b_1(t,x) = c(t)$, $b_i(t,x) = 0\ (i = 2, \ldots, N)$ and $b_0(t,x) = g(t, x, 0)$ in (2.1). Then for any $L > 0$, we have the following linear problem

$$\begin{cases} \partial_t u = \Delta u + c(t)\partial_1 u + g(t, x, 0)u, & t > 0,\ x \in (-L, L) \times \omega, \\ \partial_\nu u = 0, & t > 0,\ x \in (-L, L) \times \partial\omega, \\ u(t, -L, y) = u(t, L, y) = 0, & t > 0,\ y \in \omega. \end{cases} \tag{3.1}$$

By Definition 2.1, let $\lambda_L$ be the principal Lyapunov exponent of (3.1). Then, all results in Section 2 hold true for $\lambda_L$, particularly, $\lambda_\infty := \lim_{L\to\infty} \lambda_L$.

**Theorem 3.1.** *Assume that (H1)-(H4) hold true. If $\lambda_\infty < 0$, then for any $v_0 \in X^+$,*

$$\lim_{t\to\infty} v(t, x_1, y; v_0) = 0 \tag{3.2}$$

*uniformly for $(x_1, y) \in \Omega$.*

**Proof.** Before proving (3.2), we first show that

$$\lim_{t\to\infty, |x_1|\to\infty} v(t, x_1, y; v_0) = 0 \quad \text{uniformly for } y \in \omega. \tag{3.3}$$

For any fixed $v_0 \in X^+$, we assume that (3.3) does not hold. Then there exist $\epsilon_0 > 0$, $\{t_n\} \subset (0, \infty)$ and $\{x_{1,n}\} \subset \mathbb{R}$ with $t_n \to \infty$ and $|x_{1,n}| \to \infty$, as $n \to \infty$, such that

$$v(t_n, x_{1,n}, y; v_0) \ge \epsilon_0 \quad \text{for some } y \in \omega \text{ and all } n.$$

Let $v^n(t, x_1, y) = v(t + t_n, x_1 + x_{1,n}, y; v_0)$. Then $v^n$ satisfies

$$\begin{cases} \partial_t v^n = \Delta v^n + c(t + t_n)\partial_1 v^n + f(t + t_n, x_1 + x_{1,n}, y, v^n), & t > 0, (x_1, y) \in \Omega, \\ \partial_\nu v^n(t, x_1, y) = 0, & t \in \mathbb{R}, (x_1, y) \in \partial\Omega. \end{cases} \tag{3.4}$$

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Then parabolic estimates and the almost periodicity of $c(t)$ and $f(t,x,v)$ imply that (up to subsequence) there are $v^*$, $c^*$ and $f^*$ such that

$$\lim_{n\to\infty} v^n(t,x_1,y) = v^*(t,x_1,y) \quad \text{locally uniformly in } t\in\mathbb{R}, (x_1,y)\in\Omega,$$
$$\lim_{n\to\infty} f(t+t_n, x_1+x_{1,n}, y, v^n) = f^*(t,x_1,y,v^*) \quad \text{locally uniformly in } t\in\mathbb{R}, (x_1,y)\in\Omega$$

and

$$\lim_{n\to\infty} c(t+t_n) = c^*(t) \quad \text{uniformly in } t\in\mathbb{R},$$

where $v^*(t,x_1,y)$ is a nonnegative bounded solution of

$$\begin{cases} \partial_t v^* = \Delta v^* + c(t)\partial_1 v^* + f(t,x_1,y,v^*), & t\in\mathbb{R}, (x_1,y)\in\Omega, \\ \partial_\nu v^* = 0, & t\in\mathbb{R}, (x_1,y)\in\partial\Omega \end{cases} \tag{3.5}$$

and

$$v^*(0,0,y) = v(t_n, x_{1,n}, y; v_0) \geq \epsilon_0.$$

Note that $f(t,x,u) \leq f_u(t,x,0)u = g(t,x,0)u$ and $\lim_{x_1\to\infty}\sup_{(t,y)\in\mathbb{R}\times\omega} g(t,x_1,y,0) < 0$. Choose $\alpha > 0$ such that

$$\lim_{x_1\to\infty}\sup_{(t,y)\in\mathbb{R}\times\omega} g(t,x_1,y,0) < -\alpha.$$

Recall that $S = \max\{M_0, \|v_0\|_{L^\infty(\Omega)}\}$. For any fixed $t_0\in\mathbb{R}$, $w(t,x) = e^{-\alpha(t-t_0)}S$ is a solution of

$$\begin{cases} \partial_t w = \Delta w + c(t)\partial_1 w - \alpha w, & t\in\mathbb{R}, (x_1,y)\in\Omega, \\ \partial_\nu w = 0, & t\in\mathbb{R}, (x_1,y)\in\partial\Omega. \end{cases} \tag{3.6}$$

We also have that $v^*(t,x_1,y)$ is a subsolution of (3.6). The comparison principle implies that

$$v^*(t,x_1,y) \leq e^{-\alpha(t-t_0)}S \quad \text{for any } t > t_0, (x_1,y)\in\Omega,$$

particularly, $v^*(0,x_1,y) \leq e^{\alpha t_0 S}$ for any $(x_1,y)\in\Omega$ and $t_0 < 0$. Choose $t_0 \ll -1$ such that $e^{\alpha t_0}S \leq \epsilon_0$, then $v^*(0,x_1,y) < \epsilon_0$ for $(x_1,y)\in\Omega$, which is a contradiction. Therefore, for any $\epsilon > 0$, there exist constants $T_\epsilon > 0$ and $L(\epsilon) > 0$ such that

$$v(t,x_1,y;v_0) < \epsilon \quad \text{for } t > T(\epsilon) \text{ and } |x| > L_\epsilon \tag{3.7}$$

and (3.3) hold true.

Given $v_0\in X^+$, $\epsilon > 0$, let

$$v^\epsilon(t,x_1,y) = v(t,x_1,y;v_0) - \epsilon.$$

Then, $v^\epsilon(t,x_1,y)$ satisfies

$$\begin{cases} \partial_t v^\epsilon = \Delta v^\epsilon + c(t)\partial_1 v^\epsilon + f(t,x,v^\epsilon+\epsilon), & t\in\mathbb{R}, x\in\Omega, \\ \partial_\nu v^\epsilon = 0, & t\in\mathbb{R}, x\in\partial\Omega. \end{cases} \tag{3.8}$$

By the assumption on $f$, we have $f(t,x,v^\epsilon+\epsilon) \leq g(t,x,0)(v^\epsilon+\epsilon)$. Then $v^\epsilon(t,x_1,y)$ satisfies

$$\begin{cases} \partial_t v^\epsilon \leq \Delta v^\epsilon + c(t)\partial_1 v^\epsilon + g(t,x,0)(v^\epsilon+\epsilon), & t\in\mathbb{R}, x\in\Omega, \\ \partial_\nu v^\epsilon = 0, & t\in\mathbb{R}, x\in\partial\Omega. \end{cases} \tag{3.9}$$

Since $\lambda_\infty < 0$, there exists $L > 0$ such that $\lambda_L < 0$. Let $\eta:\mathbb{R}\to[0,1]$ be a smooth function with $\eta(x_1) = 1$ for $|x_1| \leq \frac{L}{2}$ and $\eta(x_1) = 0$ for $|x_1| \geq \frac{3L}{4}$. Define

$$v_0^\epsilon(x_1,y) = \eta(x_1)\Big(v_0(x_1,y) - \min_{[-L,L]\times\omega} v_0(x_1,y)\Big), \quad (x_1,y)\in[-L,L]\times\omega.$$

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Clearly, $v_0^\epsilon(x_1,y)\in X^+$. Let $v(t,x_1,y;v_0^\epsilon)$ be the unique solution of the following problem

$$\begin{cases}\partial_t v=\Delta v+c(t)\partial_1 v+g(t,x,0)(v+\epsilon), & t>0, x\in(-L,L)\times\omega,\\ \partial_\nu v=0, & t>0, x\in(-L,L)\times\partial\omega,\\ v(t,-L,y)=v(t,L,y)=0, & t>0, y\in\omega\end{cases}$$

with initial value $v(0,x_1,y;v_0^\epsilon)=v_0^\epsilon(x_1,y)$, $(x_1,y)\in[-L,L]\times\omega$. Let $U_L(t,c)$ be the solution operator of

$$\begin{cases}\partial_t w=\Delta w+c(t)\partial_1 w+f_u(t,x,0)w, & t>0, x\in(-L,L)\times\omega,\\ \partial_\nu w=0, & t>0, x\in(-L,L)\times\partial\omega,\\ w(t,-L,y)=w(t,L,y)=0 & t>0, y\in\omega.\end{cases}$$

Then by the variation of constants formula, we have

$$v(t,\cdot;v_0^\epsilon)=U_L(t,c)v_0^\epsilon+\epsilon\int_0^t U_L(t-s,c\cdot s)g(s,x,0)ds. \tag{3.10}$$

Since $\lambda_L<0$, by Definition 2.1, there exists $M>0$ such that

$$\epsilon\int_0^t U_L(t-s,c\cdot s)g(s,x,0)ds<\epsilon M$$

for $t\geq 0$. Together with (3.10), we have

$$v(t,\cdot;v_0^\epsilon)\leq U_L(t,c)v_0^\epsilon+\epsilon M \quad \text{in } (-L,L)\times\omega, t\geq 0. \tag{3.11}$$

Note that

$$v^\epsilon(t,\pm L,y)=v(t,\pm L,y)-\epsilon<0 \quad \text{for all } t>0, y\in\omega$$

and

$$v^\epsilon(0,x_1,y)=v_0(x_1,y)-\epsilon\leq v_0^\epsilon(x_1,y) \quad \text{for } x_1\in(-L,L), y\in\omega.$$

Then, by (3.9) and the comparison principle, we obtain that

$$v^\epsilon(t,x_1,y)\leq v(t,x_1,y;v_0^\epsilon) \quad \text{for } t>0, x_1\in(-L,L), y\in\omega.$$

By (3.11),

$$v^\epsilon(t,x_1,y)\leq U_L(t,c)v_0^\epsilon+\epsilon M \quad \text{for } t>0, x_1\in(-L,L), y\in\omega.$$

Due to $\lambda_L<0$, Definition 2.1 ensures that there exists $T_\epsilon>0$ such that

$$U_L(t,c)v_0^\epsilon\leq\varepsilon \quad \text{for all } t\geq T_\epsilon.$$

Hence,

$$v^\epsilon(t,x_1,y)\leq(1+M)\epsilon \tag{3.12}$$

for all $t\geq T_\epsilon$, $x_1\in(-L,L)$ and $y\in\omega$. By (3.9), choose large $T>0$ and $L>0$, $v(t,x_1,y;v_0)<\epsilon$ for $t\geq T$, $|x_1|>\frac{L}{2}$ and $y\in\omega$. This, together with (3.12), implies that

$$v(t,x_1,y;v_0)<(2+M)\epsilon \quad \text{for all } t>\max\{T_\epsilon,T\}, (x_1,y)\in\Omega.$$

By the arbitrariness of $\epsilon$, we obtain $\lim_{t\to\infty}v(t,x_1,y;v_0)=0$ uniformly for $(x_1,y)\in\Omega$. This completes the proof. □

Consider the following mixed Dirichlet–Neumann boundary value problem

$$\begin{cases}\partial_t w=\Delta w+c(t)\partial_1 w+f(t,x,w), & t>0, x\in(-L,L)\times\omega,\\ \partial_\nu w=0, & t>0, x\in(-L,L)\times\partial\omega,\\ w(t,-L,y)=w(t,L,y)=0, & t>0, y\in\omega.\end{cases} \tag{3.13}$$

For any $w_0\in X_L^+$, we denote by $w(t,x_1,y;w_0)$ the unique solution of (3.13) with $w(0,x_1,y;w_0)=w_0$. Then we have the following proposition on the dynamics of (3.13).

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**Proposition 3.1.** *Assume that (H1)-(H4) hold true and* $\lambda_L > 0$. *Then (3.13) admits a unique positive almost periodic solution* $w^*(t,x)$ *and for any* $w_0 \in X_L^+ \setminus \{0\}$, *there holds*

$$\lim_{t\to\infty} \|w(t,x_1,y;v_0) - w^*(t,x)\|_{X_L} = 0. \tag{3.14}$$

**Proof.** It can be proved by properly modifying the arguments in [32, Theorem 3.1]. For completeness, we provide the proof in the following.

Let $\mathcal{H}(c,f)$ be the closure of $\{(c\cdot s, f\cdot s) : s\in\mathbb{R}\}$ with the compact open topology. For any $(\widetilde{c},\widetilde{f}) \in \mathcal{H}(c,f)$, assumptions (H1)-(H4) imply that $w = M$, $M \geq M_0$, is an supersolution of (3.13). Hence, by the comparison principle and a priori estimates for parabolic equations, each solution $w(t,x_1,y;w_0,\widetilde{c},\widetilde{f})$ of (3.13) exists globally on $[0,\infty)$, and for any $t>0$, the set $\{w(t,x_1,y;w_0,\widetilde{c},\widetilde{f}) : t>0\}$ is precompact in $X_L^+$. Define the skew-product semiflow $\Pi_t : X_L^+ \times \mathcal{H}(c,f) \to \Pi_t : X_L^+ \times \mathcal{H}(c,f)$ by

$$\Pi_t\big(w_0,\widetilde{c},\widetilde{f}\big) = (w(t,x_1,y;w_0,\widetilde{c},\widetilde{f}),\widetilde{c}_t,\widetilde{f}_t).$$

Then, the omega limit set $\omega(w_0,\widetilde{c},\widetilde{f})$ of the forward orbit $\gamma^+(w_0,\widetilde{c},\widetilde{f}) := \{\Pi_t(w_0,\widetilde{c},\widetilde{f}) : t\geq 0\}$ is well defined, compact and invariant under $\Pi_t, t\geq 0$, for each $(w_0,\widetilde{c},\widetilde{f}) \in X_L^+ \times \mathcal{H}(c,f)$.

We first show that there exists a $\delta > 0$ such that

$$\limsup_{t\to\infty} \|w(t,x_1,y;w_0,\widetilde{c},\widetilde{f})\|_{X_L} \geq \delta, \quad \forall \big(w_0,\widetilde{c},\widetilde{f}\big) \in X_L^+ \times \mathcal{H}(c,f).$$

We set $\lambda_L$ of (3.1) as $\lambda_L(g)$. Since $\lambda_L(g) > 0$, then we can choose a sufficient small $\varepsilon_0 > 0$ such that

$$\lambda_L(f_u - \varepsilon_0) > 0.$$

Since $f$ is uniformly almost periodic in $t$ and $\mathcal{H}(f)$ is compact, there exists $\delta_0 > 0$ such that

$$|\widetilde{f}(t,x_1,y,u) - \widetilde{f}_u(t,x_1,y,0)u| < \varepsilon_0, \quad (x_1,y)\in\Omega,\ t\in\mathbb{R}, u\in(0,\delta_0], \widetilde{f}\in\mathcal{H}(f).$$

Suppose on the contrary that there is $(w_0,\widetilde{c},\widetilde{f}) \in X_L^+ \times \mathcal{H}(c,f)$, such that

$$\limsup_{t\to\infty} \|w(t,x_1,y;w_0,\widetilde{c},\widetilde{f}\|_{X_L} < \delta.$$

Then, there exists $t_0 > 0$ such that

$$\|w(t,x_1,y;w_0,\widetilde{c},\widetilde{f})\|_{X_L} < \delta, \quad \forall t\geq t_0, (x_1,y)\in\Omega.$$

Since $w(t_0,x_1,y;w_0,\widetilde{c},\widetilde{f}) > 0$, then

$$\|w\big(t,x_1,y;w_*,\widetilde{c}_{t_0},\widetilde{f}_{t_0}\big)\|_{X_L} < \delta, \quad \forall t\geq 0, (x_1,y)\in\Omega,$$

where $w_* = w(t_0,x_1,y;w_0,\widetilde{c},\widetilde{f}), \widetilde{c}_{t_0} = c(t+t_0)$ and $\widetilde{f}_{t_0} = f(t+t_0,x,w)$. Then $w\big(t,x_1,y;w_*,\widetilde{c}_{t_0},\widetilde{f}_{t_0}\big)$ satisfies the following differential inequality

$$\begin{cases} \partial_t w \geq \Delta w + \widetilde{c}(t)\partial_1 w + (f_u(t,x_1,y,0) - \varepsilon_0)w, & t>0, x\in(-L,L)\times\omega, \\ \partial_\nu w = 0, \quad t>0, x\in(-L,L)\times\partial\omega, \\ w(t,-L,y) = w(t,L,y) = 0, \quad t>0, y\in\omega. \end{cases}$$

Similar to (2.2), $\lambda_L(f_u - \varepsilon_0) = \lim_{t\to\infty} \frac{1}{t}\int_0^t k_L(\sigma_\tau f_u)d\tau$ and $e^{\int_0^t k_L(\sigma_\tau f_u)d\tau}\phi_L(\sigma_t f_u)$ is a solution of

$$\begin{cases} \partial_t w = \Delta w + \widetilde{c}(t)\partial_1 w + (f_u(t,x_1,y,0) - \varepsilon_0)w, & t>0, x\in(-L,L)\times\omega, \\ \partial_\nu w = 0, \quad t>0, x\in(-L,L)\times\partial\omega, \\ w(t,-L,y) = w(t,L,y) = 0, \quad t>0, y\in\omega. \end{cases}$$

Choose a sufficiently small $\varepsilon > 0$ such that $w_* \geq \varepsilon\phi_L$. By the standard comparison theorem, we have

$$w\big(t,x_1,y;w_*,\widetilde{c}_{t_0},\widetilde{f}_{t_0}\big) \geq \varepsilon e^{\int_0^t k_L(\sigma_\tau f_u)d\tau}\phi_L(\sigma_t f_u), \quad t\geq 0, (x_1,y)\in\Omega.$$

Letting $t\to\infty$, we have

$$\lim_{t\to\infty} w\big(t,x_1,y;w_*,\widetilde{c}_{t_0},\widetilde{f}_{t_0}\big) = \infty, \quad \forall (x_1,y)\in\Omega,$$

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which is a contradiction.

Following from the invariance of $\omega(w_0,\widetilde{c},\widetilde{f})$ and $\Pi_t((X_L^+\setminus\{0\})\times\mathcal{H}(c,f))\subset intX_L^+\times\mathcal{H}(c,f)$, $\forall t>0$, we have

$$\omega\big(w_0,\widetilde{c},\widetilde{f}\big)\subset intX_L^+\times\mathcal{H}(c,f),\quad \forall\big(w_0,\widetilde{c},\widetilde{f}\big)\in(X_L^+\setminus\{0\})\times\mathcal{H}(c,f). \tag{3.15}$$

For each $(\widetilde{c},\widetilde{f},t)\in\mathcal{H}(c,f)\times(0,\infty)$, for any $w_0'(\,\cdot\,)$, $w_0''(\,\cdot\,)\in X_L^+$, when $w_0'(\,\cdot\,)>w_0''(\,\cdot\,)$, by comparison principal, we have

$$w\big(t,\cdot;w_0',\widetilde{c},\widetilde{f}\big)>w\big(t,\cdot;w_0'',\widetilde{c},\widetilde{f}\big)\quad\text{for all } t>0.$$

Then, $w(t,\cdot;\cdot,\widetilde{c},\widetilde{f})$ is strongly monotone on $X_L^+$ for each $(\widetilde{c},\widetilde{f},t)\in\mathcal{H}(c,f)\times(0,\infty)$. Moreover, by assumptions (H1)–(H4), $w(t,\cdot;\cdot,\widetilde{c},\widetilde{f})$ is also subhomongenous on $X_L^+$ for each $(\widetilde{c},\widetilde{f},t)\in\mathcal{H}(c,f)\times(0,\infty)$. Let $w_0'\in X_L^+\setminus\{0\}$ be fixed and $K_0=\omega(w_0',\widetilde{c},\widetilde{f})$. By [32, Theorem 2.1] and (3.15), for every $w_0\in X_L^+\setminus\{0\}$,

$$\lim_{t\to\infty}\|w\big(t,\cdot;w_0,\widetilde{c},\widetilde{f}\big)-w\big(t,\cdot;w_0^*,\widetilde{c},\widetilde{f}\big)\|_{X_L}=0,$$

where $(w_0^*,\widetilde{c},\widetilde{f})\in K_0$. Therefore, $w(t,\cdot;w_0^*,\widetilde{c},\widetilde{f})$ is a unique positive almost periodic solution of (3.13). This completes the proof. □

**Theorem 3.2.** *Assume that (H1)-(H4) hold true. If $\lambda_\infty>0$, then for any $L>0$ and $v_0\in X^+\setminus\{0\}$,*

$$\liminf_{t\to\infty}\ \inf_{x_1\in[-L,L],y\in\omega} v(t,x_1,y;v_0)>0.$$

**Proof.** By the definition of $\lambda_\infty$, due to $\lambda_\infty>0$, there is $L^*>0$ such that $\lambda_L>0$ for any $L>L^*$. Then, by Proposition 3.1, there is a unique positive almost periodic solution $w^*(t,x)$ of (3.13). Note that any nonnegative solution of (1.2) is a supersolution of (3.13). Then,

$$v(t,x_1,y;v_0)\geq w(t,x_1,y;v_0)\quad\text{in }(-L,L)\times\omega.$$

Then by (3.14), we have

$$\liminf_{t\to\infty}\ \inf_{x_1\in[-L,L],y\in\omega} v(t,x_1,y;v_0)>0.$$

This completes the proof. □

## 4. Forced wave solutions

In this section, we study the existence, uniqueness, stability, and exponential decay of forced wave solutions of (1.1). By (1.5), in the following, we need to prove that there is an entire solution $V(t,x_1,y)$ of (1.2), which is bounded, positive and almost periodic in $t$ for any $(x_1,y)\in\Omega=\mathbb{R}\times\omega$.

**Lemma 4.1.** *Assume that (H1)-(H4) holds true. When $\lambda_\infty>0$, equation (1.2), that is,*

$$\begin{cases}\partial_t v=\Delta v+c(t)\partial_1 v+f(t,x_1,y,v), & (x_1,y)\in\mathbb{R}\times\Omega,\\ \partial_\nu v=0 \quad \text{on } \partial\Omega,\end{cases} \tag{4.1}$$

*admits a unique bounded positive solution $V(t,x_1,y)$ with*

$$\lim_{|x_1|\to\infty}\ \sup_{t\in\mathbb{R},y\in\omega} V(t,x_1,y)=0. \tag{4.2}$$

**Proof.** Let $M_0$ be given in (H3). Then $\overline{v}(t,x)=M_0$ is a supersolution of (4.1). Let $v(t,x_1,y;\overline{v},c,f)$ be a solution of (1.2) with the initial condition $v(0,x_1,y;\overline{v},c,f)=\overline{v}$. By comparison principle, $v(t,x_1,y;\overline{v},c,f)\leq\overline{v}(t,x)$ for $x\in\Omega$ and $t>0$. Since $c$ and $f$ are almost periodic function, there exists $t_n\to\infty$ such that $c\cdot t_n\to c$ and $f\cdot t_n\to f$ uniformly as $n\to\infty$. By a priori estimates for parabolic equations, there exists a function $V(t,x)$ such that $v(t+t_n,x_1,y;\overline{v},c,f)$ converges to $V(t,x_1,y)$ locally uniformly in $(t,x_1,y)\in\mathbb{R}\times\Omega$ as $n\to\infty$. Then, $V(t,x_1,y)$ is a bounded entire solution satisfying (4.1) for $t\in\mathbb{R}$ and $(x_1,y)\in\Omega$. By Theorem 3.2, we have $\inf_{\mathbb{R}\times[-L,L]\omega} V(t,x_1,y)>0$ for all $L>0$.

Following from the arguments for (3.3), we also have $\lim_{|x_1|\to\infty}\sup_{t\in\mathbb{R},y\in\omega} V(t,x_1,y)=0$, that is, (4.2) holds true. In the following, we prove the uniqueness of $V(t,x_1,y)$. Suppose that there are two bounded positive solutions $V(t,x_1,y)$ and $V'(t,x_1,y)$ of (4.1). We first show that $V(t,x_1,y)\le V'(t,x_1,y)$. For any $\epsilon>0$, define

$$K_\epsilon := \{k>0 : kV'\ge V-\epsilon \quad \text{in } \mathbb{R}\times\overline{\Omega}\}.$$

By (4.2), there exists $R(\epsilon)>0$ such that

$$V(t,x_1,y)-\epsilon<0, \qquad \forall\,|x_1|>R(\epsilon), y\in\omega, t\in\mathbb{R}.$$

Together with $V'>0$ in $\mathbb{R}\times\overline{\Omega}$, we find that the set $K_\epsilon$ is nonempty. Set $k(\epsilon):=\inf\{k:k\in K_\epsilon\}$. Then, $k:\mathbb{R}^+\to\mathbb{R}$ is a nonincreasing function and thus $\lim_{\epsilon\to 0^+}k(\epsilon)$ exists. Set $k^*=\lim_{\epsilon\to 0^+}k(\epsilon)$. Assume that

$$k^*>1. \tag{4.3}$$

For any $0<\epsilon<\sup_{\mathbb{R}\times\Omega} V$ small enough, we have $k(\epsilon)>0$,

$$k(\epsilon)V'-V+\epsilon\ge 0 \quad \text{in } \mathbb{R}\times\overline{\Omega}$$

and there exists a sequence $\{t_n^\epsilon\}\subset\mathbb{R}$ such that

$$\lim_{n\to\infty}\inf_{(x_1,y)\in\mathbb{R}\times\omega}[k_\epsilon V'(t_n^\epsilon,x_1,y)-V(t_n^\epsilon,x_1,y)+\epsilon]=0.$$

Without loss of generality, we assume that there are $V'_\epsilon(t,x_1,y)$, $V_\epsilon(t,x_1,y)$ and $(c^\epsilon,f^\epsilon)\in\mathcal{H}(c,f)$ such that

$$V'_\epsilon(t,x_1,y)=\lim_{n\to\infty}V'\big(t+t_n^\epsilon,x_1,y\big),\ V_\epsilon(t,x_1,y)=\lim_{n\to\infty}V\big(t+t_n^\epsilon,x_1,y\big)$$

and

$$f^\epsilon(t,x,u)=\lim_{n\to\infty}f(t+t_n^\epsilon,x_1,y,u)$$

locally uniformly in $(t,x_1,y)\in\mathbb{R}\times\mathbb{R}\times\omega$, $u>0$ and

$$c^\epsilon(t)=\lim_{n\to\infty}c(t+t_n^\epsilon) \quad \text{uniformly in } t\in\mathbb{R}.$$

Then, $V_\epsilon$ and $V'_\epsilon$ are bounded positive solution of (1.2) with $c$ and $f$ replaced by $c^\epsilon$ and $f^\epsilon$, respectively. Moreover, there hold $k_\epsilon V'_\epsilon\ge V_\epsilon-\epsilon$ and

$$\inf_{(x_1,y)\in\mathbb{R}\times\omega}[k_\epsilon V'_\epsilon(0,x)-V(0,x)+\epsilon]=0.$$

By (4.2) again, there is $(x_1^\epsilon,y^\epsilon)\in\mathbb{R}\times\omega$ such that

$$k_\epsilon V'_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)-V\left(0,x_1^\epsilon,y^\epsilon\right)+\epsilon=0.$$

Then, the function $t\to k_\epsilon V'_\epsilon(t,x_1^\epsilon,y^\epsilon)-V(t,x_1^\epsilon,y^\epsilon)+\epsilon$ attains its minimum value at 0 and $k_\epsilon V'_\epsilon(0,x_1,y)-V(0,x_1,y)+\epsilon$ attains its minimum value at $(x_1^\epsilon,y^\epsilon)$.

Then, at point $\left(0,x_1^\epsilon,y^\epsilon\right)$, by (H1)-(H4), we have

$$\begin{aligned}
0&=k_\epsilon\frac{\partial V'_\epsilon}{\partial t}\left(0,x_1^\epsilon,y^\epsilon\right)-\frac{\partial V_\epsilon}{\partial t}\left(0,x_1^\epsilon,y^\epsilon\right)\\
&=\left[k_\epsilon\Delta V'_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)+c^\epsilon(0)\partial_1\left(k_\epsilon V'_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)\right)+f^\epsilon\left(0,x_1^\epsilon,y^\epsilon,k_\epsilon V'_\epsilon\right)\right]\\
&\quad-\left[\Delta V_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)+c^\epsilon(0)\partial_1 V_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)+f^\epsilon\left(0,x_1^\epsilon,y^\epsilon,V_\epsilon\right)\right]\\
&=\left[k_\epsilon\Delta V'_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)-\Delta V_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)\right]+c^\epsilon(0)\left[\partial_1\left(k_\epsilon V'_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)\right)-\partial_1 V_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)\right]\\
&\quad+f^\epsilon(0,x_1^\epsilon,y^\epsilon,k_\epsilon V'_\epsilon)-f^\epsilon(0,x_1^\epsilon,y^\epsilon,V_\epsilon)\\
&\ge f^\epsilon(0,x_1^\epsilon,y^\epsilon,k_\epsilon V'_\epsilon)-f^\epsilon(0,x_1^\epsilon,y^\epsilon,V_\epsilon)\\
&\ge\left[k_\epsilon V'_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)-V_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)\right]\left[g^\epsilon(0,x_1^\epsilon,y^\epsilon,k_\epsilon V'_\epsilon)+V_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)g_u^\epsilon\left(0,x_1^\epsilon,y^\epsilon,\eta_\epsilon\right)\right],
\end{aligned}$$

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where $\eta_\epsilon$ is between $k_\epsilon V'_\epsilon(0,x_1^\epsilon,y^\epsilon)$ and $V_\epsilon(0,x_1^\epsilon,y^\epsilon)$. Since $k_\epsilon V'_\epsilon(0,x_1^\epsilon,y^\epsilon)-V_\epsilon(0,x_1^\epsilon,y^\epsilon)<0$, we obtain that

$$g^\epsilon(0,x_1^\epsilon,y^\epsilon,k_\epsilon V'_\epsilon)+V_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)g_u^\epsilon(0,x_1^\epsilon,y^\epsilon,\eta_\epsilon)\geq 0.$$

From (H2), $g_u^\epsilon\leq 0$. Then

$$0\leq g^\epsilon(0,x_1^\epsilon,y^\epsilon,k_\epsilon V'_\epsilon)+V_\epsilon\left(0,x_1^\epsilon,y^\epsilon\right)g_u^\epsilon(0,x_1^\epsilon,y^\epsilon,\eta_\epsilon)\leq g^\epsilon(0,x_1^\epsilon,y^\epsilon,k_\epsilon V'_\epsilon)\leq 0.$$

By (H4), $\{x_1^\epsilon\}$ must be bounded. From $k_\epsilon V'_\epsilon(0,x_1^\epsilon,y^\epsilon)-V(0,x_1^\epsilon,y^\epsilon)+\epsilon=0$, we get that $\{k_\epsilon\}$ is bounded and then $k^*\in[1,\infty)$. Choose a sequence $\{\epsilon_n\}$ in $\mathbb{R}^+$ such that

$$\lim_{n\to\infty}\epsilon_n=0,\quad \xi:=\lim_{n\to\infty}x_1^{\epsilon_n}\in\mathbb{R},\quad \eta:=\lim_{n\to\infty}y^{\epsilon_n}\in\overline{\omega},$$

and

$$V'^*(t,x_1,y)=\lim_{n\to\infty}V'_{\epsilon_n}(t,x_1,y),\qquad V^*(t,x_1,y)=\lim_{n\to\infty}V_{\epsilon_n}(t,x_1,y)$$

locally uniformly in $(t,x)\in\mathbb{R}\times\Omega$,

$$c^*(t)=\lim_{n\to\infty}c^{\epsilon_n}(t)\quad\text{uniformly in } t\in\mathbb{R},$$

and

$$f^*(t,x_1,y,u)=\lim_{n\to\infty}f^{\epsilon_n}(t,x,u)\quad\text{uniformly in } t\in\mathbb{R}, x\in\Omega, u>0.$$

Then $V^*$ and $V'^*$ are bounded positive entire solutions of (4.1) with $c$ and $f$ are replaced by $c^*$ and $f^*$. Moreover, we have $k^*V'^*\geq V^*$ and $k^*V'^*(0,\xi,\eta)-V^*(0,\xi,\eta)=0$. Set $W:=k^*V'^*-V^*$ satisfies $W(0,\xi,\eta)=0$. Then,

$$\begin{aligned}W_t=&\Delta W+c^*(t)\partial_1 W+k^*V'^*g(t,x,V'^*)-V^*g(t,x,V^*)\\ \geq&\Delta W+c^*(t)\partial_1 W+[g(t,x,k^*V'^*)+V^*g_u(t,x,\eta)]W,\quad \forall t\in\mathbb{R},x\in\Omega\end{aligned}$$

and

$$\partial_\nu W=0,\quad t\in\mathbb{R},x\in\partial\Omega.$$

Since $W$ is nonnegative in $\mathbb{R}\times\Omega$, vanishes at $(\xi,\eta)$, the strong maximum principle and the Hopf lemma yield $W=0$. Then

$$g(t,x,k^*V'^*)=g(t,x,V'^*)=g(t,x,V^*),$$

which implies that $k^*=1$ and then $V(t,x_1,y)\leq V'(t,x_1,y)$. Similarly, we can also prove $V(t,x_1,y)\geq V'(t,x_1,y)$. Therefore, $V(t,x_1,y)=V'(t,x_1,y)$ and $V(t,x_1,y)$ is a unique bounded positive solution of (4.1). This completes the proof. □

**Lemma 4.2.** *A function $f(t,x)$ is almost periodic in $t$ uniformly with respect to $x\in E\subset\mathbb{R}^K$ if and only if it is uniformly continuous on $\mathbb{R}\times E$ and for every pair of sequences $\{s_n\}_{n=1}^\infty$, $\{r_m\}_{m=1}^\infty$, there are subsequences $\{s'_n\}_{n=1}^\infty\subset\{s_n\}_{n=1}^\infty$, $\{r'_n\}_{n=1}^\infty\subset\{r_n\}_{n=1}^\infty$ such that for each $(t,x)\in\mathbb{R}\times\mathbb{R}^K$,*

$$\lim_{m\to\infty}\lim_{n\to\infty}f(t+s'_n+r'_m,x)=\lim_{n\to\infty}f(t+s'_n+r'_n,x).$$

**Proof.** See [10, Theorems 1.17 and 2.10]. □

**Theorem 4.1.** *Assume that (H1)–(H4) hold true. If $\lambda_\infty>0$, $V(t,x_1,y)$ is a bounded, positive, unique and almost periodic entire solution of (4.1). For any $v_0\in X^+\setminus\{0\}$, there holds*

$$\lim_{t\to\infty}\|v(t,x_1,y;v_0)-V(t,x_1,y)\|_\infty=0. \tag{4.4}$$

**Proof.** From Lemma 4.1, we need to show that $V(t,x_1,y)$ is almost periodic in $t$ for $(x_1,y)$ in bounded sets. We write $V(t,x_1,y;c,f)$ for $V(t,x_1,y)$ to indicate the dependence on $c$ and $f$. The Schauder estimates ensure that $V(t,x_1,y;c,f)$ is uniformly continuous in $(t,x_1,y)\in\mathbb{R}\times\Omega$. It then suffices to show that for

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each $x$ in bounded sets, $V(t, x_1, y; c, f)$ is almost periodic in $t$. To this end, let $\{\alpha_n\}$ and $\{\beta_n\}$ be any two sequence of $\mathbb{R}$. Without loss of generality, we may assume that

$$c' = \lim_{m\to\infty} c(t+\beta_m), \quad c'' = \lim_{n\to\infty} c(t+\alpha_n+\beta_n),$$

$$f'(t,x,u) = \lim_{m\to\infty} f(t+\beta_m, x, u), \quad f''(t,x,u) = \lim_{n\to\infty} f(t+\alpha_n+\beta_n, x, u)$$

locally uniformly in $(t, x, u) \in \mathbb{R} \times \Omega \times \mathbb{R}$ and

$$V'(x) = \lim_{m\to\infty} V(\beta_m, x; c, f) \quad V''(x) = \lim_{n\to\infty} V(\alpha_n+\beta_n, x; c, f)$$

locally uniformly in $x \in \Omega$. Therefore,

$$V(t, x; v', c', f') = \lim_{m\to\infty} V(t+\beta_m, x; c, f), \quad V(t, x; v'', c'', f'') = \lim_{n\to\infty} V(t+\alpha_n+\beta_n, x; c, f) \tag{4.5}$$

locally uniformly in $(t, x) \in \mathbb{R} \times \Omega$, where $V(t, x; v', c', f')$ and $V(t, x; v'', c'', f'')$ are bounded positive entire solution of (4.1) with $c$ and $f$ replaced by $c', f'$ and $c'', f''$, respectively. By Lemma 4.1,

$$\inf_{\mathbb{R}\times[-L,L]\times\omega} V(t, x; v', c', f') > 0 \quad \text{and} \quad \inf_{\mathbb{R}\times[-L,L]\times\omega} V(t, x; v'', c'', f'') > 0$$

for all $L > 0$.

Let

$$V'(t, x) := \lim_{n\to\infty} V(t+\alpha_n, x; v', c', f')$$

locally uniformly in $(t, x) \in \mathbb{R} \times \Omega$. Since $\lim_{n\to\infty} c' \cdot \alpha_n = c''$, then $V'$ is a bounded positive entire solutions of (1.2) with $c$ and $f$ are replaced by $c'$ and $f'$, and $\inf_{\mathbb{R}\times[-L,L]\times\omega} V' > 0$. By the uniqueness of strictly positive solution, see Lemma 4.1, $V'(t, x) = V(t+\alpha_n, x; v'', c'', f'')$. It then follows from (4.5) that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty} V(t+\alpha_n+\beta_m, x; c, f)\\
&= \lim_{n\to\infty} V(t+\alpha_n, x; v', c', f') = V(t+\alpha_n, x; v'', c'', f'')\\
&= \lim_{n\to\infty} V(t+\alpha_n+\beta_n, x; c, f)
\end{aligned}$$

holds for all $t \in \mathbb{R}$ and uniformly for $x$ in bounded subsets. Hence, by Lemma 4.3, $V(t, x; c, f)$ is almost periodic in $t$ uniformly for $x$ in bounded sets.

Next, we show that (4.4) holds true. Let $v_0 \in X^+ \setminus \{0\}$. Assume that the conclusion fails. Then there exist $\varepsilon_0 > 0$ and $\{t_n\} \subset (0, \infty)$ satisfying $\lim_{n\to\infty} t_n = \infty$ and $\{(x_1^n, y^n)\} \subset \Omega$ such that

$$\left| v\left(t_n, x_1^n, y^n; v_0, c, f\right) - V\left(t_n, x_1^n, y^n; c, f\right)\right| \geq \varepsilon_0, \quad \forall n \in \mathbb{N}. \tag{4.6}$$

It is no loss of generality to assume that $y^n$ converges to some $\eta \in \overline{\omega}$. We may assume without loss of generality that $c \cdot t_n \to c'$ as $n \to \infty$. When $\lim_{n\to\infty} |x_1^n| = \infty$, by (3.3), we must have $\lim_{n\to\infty} V(t_n, x_1^n, y^n; c, f) = 0$ and $\lim_{n\to\infty} v(t_n, x_1^n, y^n; v_0, c, f) = 0$, which is a contradiction with (4.6). When $\{x_1^n\}$ is bounded, set $x_1^n \to x_1'$ as $n \to \infty$. By standard parabolic estimates, we may assume that

$$v\left(t+t_n, x_1, y; v_0, c, f\right) \to v_1\left(t, x_1, y; c', f'\right) \quad \text{and} \quad V(t+t_n, x_1, y; c, f) \to v_2(t, x_1, y; c', f')$$

locally uniformly in $t \in \mathbb{R}$ and $(x_1, y) \in \Omega$, respectively. Then, $v_i(t, x_1, y; c', f')$ $(i = 1, 2)$ are bounded positive entire solution of (4.1) with $c$ and $f$ replaced by $c'$ and $f'$. Moreover, $\inf_{\mathbb{R}\times[-L,L]\times\omega} v_i(\cdot,\cdot,\cdot; c', f') > 0$ for all $L > 0$. The uniqueness in Lemma 4.1 implies that $v_1(t, x_1, y; c', f') = v_2(t, x_1, y; c', f')$ for all $t \in \mathbb{R}$ and $(x_1, y) \in \Omega$. By (4.6) again, when $n \to \infty$, there is $|v_1(0, x_1', \eta; c') - v_2(0, x_1', \eta; c')| \geq \varepsilon_0$. This is a contradiction. Hence, (4.4) holds true. This completes the proof. □

In the following, we will mainly study the effects of fluctuations on the shifting speed and the exponential decay of the forced wave solution $V(t, x_1, y)$ of (4.1). Assume the shifting speed function $c(t) = c + \sigma(t)$ for some fluctuations $\sigma(t)$. We would like to see if the fluctuation $\sigma(t)$ can drive the

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species to extinction and how large it needs to be in order to make this happen. Consider the following specified model

$$\begin{cases} \partial_t u = \Delta u + f\left(t, x_1 - ct - A\int_0^t \sigma(s)ds, y, u\right), \ t > 0, x \in \Omega, \\ \partial_\nu u(t,x) = 0, \quad t > 0, x \in \partial\Omega, \end{cases} \tag{4.7}$$

where $c > 0$, $A \geq 0$ and $\sigma : \mathbb{R} \to \mathbb{R}$ is Holder continuous. The linearisation of (4.7) in the moving frame around the extinction state 0 is

$$\begin{cases} \partial_t w = \Delta w + [c + A\sigma(t)]\partial_1 w + g(t,x,0)w, \quad x \in \Omega, \\ \partial_\nu w = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

Here, $A\sigma(t)$ is the fluctuation on the shifting speed $c$ and the parameter $A$ characterises the amplitude if fluctuations.

Define $\lambda^A$ the principal Lyapunov exponent of (4.8) and $\lambda_L^A$ restricted on $(-L, L)$. Then, $\lambda_\infty^A = \lim_{L\to\infty} \lambda_L^A$ and $\lambda_L^A \leq \lambda^A$. Let $\lambda(\Omega)$ is the generalised Neumann principal eigenvalue of the following problem

$$\begin{cases} -(\Delta\phi + c\partial_1\phi) = \lambda(\Omega)\phi, \quad x \in \mathbb{R} \times \omega, \\ \partial_\nu \phi = 0, \quad x \in \mathbb{R} \times \partial\omega \end{cases}$$

and we refer to see Remark 2.1(2) for the definition of generalised Neumann principal eigenvalue.

**Theorem 4.2.** *(1) If $A = 0$, then there is a critical speed $c^* > 0$ such that $\lambda_\infty^0(c) > 0$ for $c \in [0, c^*)$ and $\lambda_\infty^0(c) < 0$ for $c > c^*$.*

*(2) If $A \neq 0$, assume that $\sigma(t)$ and $f(t,x,u)$ are periodic in $t$. Then*

$$\lambda_\infty^A \geq -\lambda(\Omega) + \inf_{t\in\mathbb{R}, x\in\Omega} g(t,x,0).$$

**Proof.** (1) When $A = 0$, by the Liouville transformation $v(t, x_1, y) = u(t, x_1, y)e^{\frac{c}{2}x_1}$, problem (4.1) reduces to

$$\begin{cases} \partial_t v = \Delta v + f\left(t, x_1, y, v(t, x_1, y)e^{-\frac{c}{2}x_1}\right) e^{\frac{c}{2}x_1} - \frac{c^2}{4}v, \ t > 0, x \in \Omega, \\ \partial_\nu v(t,x) = 0, \quad t > 0, x \in \partial\Omega. \end{cases} \tag{4.9}$$

The associated linearised problem associated with (4.9) is

$$\begin{cases} \partial_t w = \Delta w + g(t,x,0)w - \frac{c^2}{4}w, \ t > 0, x \in \Omega, \\ \partial_\nu w(t,x) = 0, \quad t > 0, x \in \partial\Omega. \end{cases} \tag{4.10}$$

Let $\lambda_\infty^0(0)$ is the principal Lyapunov exponent of

$$\begin{cases} \partial_t w = \Delta w + g(t,x,0)w, \ t > 0, x \in \Omega, \\ \partial_\nu w(t,x) = 0, \quad t > 0, x \in \partial\Omega. \end{cases} \tag{4.11}$$

Then

$$\lambda_\infty^0(c) = \lambda_\infty^0(0) - \frac{c^2}{4}.$$

Define

$$c^* = \begin{cases} 2\sqrt{\lambda_\infty^0} & \text{if } \lambda_\infty^0 > 0, \\ 0 & \text{otherwise}. \end{cases}$$

It then follows that $\lambda_\infty^0(c) > 0$ for $c \in [0, c^*)$ and $\lambda_\infty^0(c) < 0$ for $c > c^*$.

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(2) Fix $A > 0$. Note that $\sigma(t)$ and $f(t, x, u)$ are periodic in $t$. Then $\lambda_L^A$ is the principal eigenvalue of (4.10) on $(-L, L) \times \Omega$ and $t \in \mathbb{R}$. Then

$$\lambda_L^A \geq \Lambda_L + \inf_{t\in\mathbb{R}, x\in\Omega} g(t, x, 0), \tag{4.12}$$

where $\Lambda_L$ is the principal eigenvalue of

$$\begin{cases} \partial_t w = \Delta w + [c + A\sigma(t)]\partial_1 w, & x \in (-L, L) \times \omega, \\ \partial_\nu w = 0, & x \in (-L, L) \times \partial\omega, \\ w(t, L, y) = w(t, -L, y) = 0, & y \in \omega \end{cases}$$

and $\phi_L(t, x_1, y)$ is the periodic eigenfunction corresponding to $\Lambda_L$. Set

$$z = x_1 + A \int_0^t \sigma(s)ds \quad \text{and} \quad \Phi_L(t, z, y) = \phi_L(t, x_1, y).$$

Then $\Phi_L(t, z, x)$ is a periodic function defined on periodic domain

$$D_T = \cup_{t\in\mathbb{R}} \left( \{t\} \times [-L + A \int_0^t \sigma(s)ds, L + A \int_0^t \sigma(s)ds] \times \omega \right)$$

and $\Phi_L(t, z, x)$ also satisfies

$$\begin{cases} \partial_t \Phi_L = \Delta \Phi_L + c\partial_1 \Phi_L - \Lambda_L \Phi_L, & x \in (-L, L) \times \omega, \\ \partial_\nu \Phi_L = 0, & x \in (-L, L) \times \partial\omega, \\ \Phi_L(t, L, y) = \Phi_L(t, -L, y) = 0, & y \in \omega. \end{cases}$$

Let $\widetilde{L} > 0$ large such that $\mathbb{R} \times [-\widetilde{L} - 1, \widetilde{L} + 1] \times \omega \subset D_T$. Moreover, $\widetilde{L} \to \infty$ as $L \to \infty$. Then on $\mathbb{R} \times [-\widetilde{L} - 1, \widetilde{L} + 1] \times \omega$, we find that $\Phi_L$ is periodic in $t$ and $\inf \Phi_L > 0$ and

$$\begin{cases} \partial_t \Phi_L = \Delta \Phi_L + c\partial_1 \Phi_L - \Lambda_L \Phi_L, & [-\widetilde{L} - 1, \widetilde{L} + 1] \times \omega, \\ \partial_\nu \Phi_L = 0, & x \in (-L, L) \times \partial\omega. \end{cases}$$

Set $\widetilde{\Lambda}_L$ is the principal eigenvalue of

$$\begin{cases} \Delta\phi + c\partial_1\phi = \widetilde{\Lambda}\phi, & x \in (-\widetilde{L}, \widetilde{L}) \times \omega, \\ \partial_\nu \phi_L = 0, & x \in (-\widetilde{L}, \widetilde{L}) \times \partial\omega, \\ \phi(\widetilde{L}, y) = \phi(-\widetilde{L}, y) = 0, & y \in \omega. \end{cases}$$

Then the comparison principal implies that $\Lambda_L \geq \widetilde{\Lambda}_L$. By (4.12), we get

$$\lambda_L^A \geq \widetilde{\Lambda}_L + \inf_{t\in\mathbb{R}, x\in\Omega} g(t, x, 0).$$

Following from [3, Proposition 1], we have that $\lim_{L\to\infty} \widetilde{\lambda}_L$ exists and $\lim_{L\to\infty} \widetilde{\lambda}_L = -\lambda(\Omega)$. Thus, we obtain that $\lambda_\infty^A \geq -\lambda(\Omega) + \inf_{t\in\mathbb{R}, x\in\Omega} g(t, x, 0)$. This completes the proof. □

In order to use Theorem 4.2(1), let $V(t, x_1, y)$ be the forced wave solution of (4.1) with $c(t) \equiv c$, that is $A = 0$.

**Lemma 4.3.** *Let $v(t, x_1, y) \in W^{1,2}_{N+1,loc}(\mathbb{R} \times \Omega_L)$ with $\partial_\nu v \leq 0$ on $\mathbb{R} \times \partial\Omega$ for every $L > 0$ and let $C, \gamma > 0$ be such that*

$$\forall (t, x_1, y) \in \mathbb{R} \times \Omega, \quad 0 < v(t, x_1, y) \leq Ce^{\sqrt{\gamma}|x_1|},$$

$$\limsup_{|x_1|\to\infty} \frac{\partial_t v - \Delta v(t, x_1, y)}{v(t, x_1, y)} < -\gamma$$

*uniformly in $t \in \mathbb{R}$ and $y \in \omega$. Then there exists a constant $\kappa > 0$ such that*

$$\forall (t, x_1, y) \in \mathbb{R} \times \Omega, \quad v(t, x_1, y) \leq \kappa e^{-\sqrt{\gamma}|x_1|}. \tag{4.13}$$

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**Proof.** The proof is similar to that in [3]. For sake of completeness, we include the proof here. By the hypotheses on $v$, there exist $\varepsilon > 0$ and $L > 0$ such that

$$\partial_t v - \Delta v < (-\gamma - 2\varepsilon)v$$

for a.e. $t \in \mathbb{R}$ and $(x_1, y) \in \Omega \setminus \overline{\Omega}_L$. We list the following functions as that in [3, Lemma 3.2]. For $a > 0$, let $\theta_a : [L, L+a] \to \mathbb{R}$ be the solution to

$$\begin{cases} \theta'' = (\gamma + \varepsilon)\theta, \quad \text{in } (L, L+a), \\ \theta(L) = ke^{\sqrt{\gamma}L}, \\ \theta(L+a) = ke^{\sqrt{\gamma}(L+a)}. \end{cases} \tag{4.14}$$

Hence, $\theta_a(\rho) = A_a e^{-\sqrt{\gamma+\varepsilon}\rho} + B_a e^{\sqrt{\gamma+\varepsilon}\rho}$, $\rho \in [L, L+a]$ with

$$A_a = ke^{(\sqrt{\gamma}+\sqrt{\gamma+\varepsilon})L}\left(1 - \frac{e^{\sqrt{\gamma}a} - e^{-\sqrt{\gamma+\varepsilon}a}}{e^{\sqrt{\gamma+\varepsilon}a} - e^{-\sqrt{\gamma+\varepsilon}a}}\right)$$

and

$$B_a = ke^{(\sqrt{\gamma}-\sqrt{\gamma+\varepsilon})L}\left(1 - \frac{e^{\sqrt{\gamma}a} - e^{-\sqrt{\gamma+\varepsilon}a}}{e^{\sqrt{\gamma+\varepsilon}a} - e^{-\sqrt{\gamma+\varepsilon}a}}\right).$$

Fix $\tau \in \mathbb{R}$ and define

$$\theta_a(t, x_1, y) := \theta_a(|x_1|) + (\tau - t)\delta_a,$$

where

$$\delta_a = \frac{\varepsilon}{2} \min_{[L,L+a]\times\omega} \theta_a > 0.$$

By the similar computation in [2, Lemma 3.2],

$$\begin{cases} \partial_t\theta_a \geq \Delta\theta_a - (\gamma + 2\varepsilon)\theta_a, \quad t \in (-\infty, \tau), (x_1, y) \in (L, L+a) \times \omega, \\ \partial_\nu\theta_a = 0, \quad t \in (-\infty, \tau), (x_1, y) \in (L, L+a) \times \partial\omega. \end{cases} \tag{4.15}$$

Note that $v$ is a subsolution of (4.15) and $v \leq \theta_a$ on $\{\pm L, \pm(L+a)\} \times \omega$, $t \in (-\infty, \tau)$. Then the comparison principle implies that

$$v \leq \theta_a \quad \text{in } (x_1, y) \in \Omega_{L+a} \setminus \widetilde{\Omega}_L, t < \tau, \quad \text{for any } a > 0.$$

For $|x_1| > L$ and $y \in \omega$, we have

$$v(\tau, x_1, y) \leq \lim_{a\to\infty} \theta_a(\tau, x_1, y) = ke^{\sqrt{\gamma}+\sqrt{\gamma+\varepsilon}L}e^{-\sqrt{\gamma+\varepsilon}|x_1|}.$$

Due to the arbitrarily of $\tau$, we obtain that (4.13) holds true. This completes the proof. □

**Theorem 4.3.** *Assume that (H1)-(H4) hold true. Let $V(t, x_1, y)$ be a solution of (4.1) with $c(t) \equiv c$. Then there exist two constants $h$ and $\beta > 0$ such that*

$$\forall (t, x_1, y) \in \mathbb{R} \times \Omega. \quad V(t, x_1, y) \leq he^{-\beta|x_1|}.$$

**Proof.** The function $v(t, x_1, y) = V(t, x_1, y)e^{\frac{c}{2}x_1}$ is a solution of (4.9). Hence,

$$\frac{\partial_t v - \Delta v(t, x_1, y)}{v(t, x_1, y)} = \zeta(t, x_1, y) - \frac{c^2}{4}$$

for $t \in \mathbb{R}$ and $(x_1, y) \in \Omega$, where $\zeta(t, x_1, y) \leq g(t, x_1, y, 0)$.

Set $\zeta = -\lim_{r\to\infty} \sup_{|x_1|>r, t\in\mathbb{R}, y\in\omega} g(t, x_1, y, 0)$ and (H4) implies $\zeta > 0$. Then for $\gamma \in (\frac{c^2}{4}, \frac{c^2}{4} + \zeta)$,

$$\lim_{L\to\infty} \sup_{|x_1|>L, y\in\Omega} \frac{\partial_t v - \Delta v(t, x_1, y)}{v(t, x_1, y)} \leq \lim_{L\to\infty} \sup_{|x_1|>L, y\in\Omega} g(t, x_1, y, 0) - \frac{c^2}{4} \leq -\zeta - \frac{c^2}{4} < -\gamma.$$

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On the other hand, $v(t,x_1,y)e^{-\sqrt{\gamma}|x_1|} \le v(t,x_1,y)e^{-\frac{c}{2}|x_1|}$, which is bounded. Then by Lemma 4.3, there exist two constants $h$ and $\beta > 0$ such that

$$V(t,x_1,y) = v(t,x_1,y)e^{-\frac{c}{2}x_1} \le he^{-\sqrt{\gamma}|x_1| - \frac{c}{2}|x_1|} \le he^{-\beta|x_1|}, \forall (t,x_1,y) \in \mathbb{R} \times \Omega.$$

This completes the proof. □

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