Bull. Aust. Math. Soc. (First published online 2025), page 1 of 7*
doi:10.1017/S0004972725100191
*Provisional—final page numbers to be inserted when paper edition is published

RIGHT ENGEL CONDITIONS FOR ORDERABLE GROUPS

MARIA TOTA, PAVEL SHUMYATSKY and CARMELA SICA

(Received 1 April 2025; accepted 19 April 2025)

Abstract

Let g be an element of a group G. For a positive integer n, let $R_n(g)$ be the subgroup generated by all commutators $[\dots[[g,x],x],\dots,x]$ over $x \in G$, where x is repeated n times. Similarly, $L_n(g)$ is defined as the subgroup generated by all commutators $[\dots[[x,g],g],\dots,g]$, where $x \in G$ and g is repeated n times. In the literature, there are several results showing that certain properties of groups with small subgroups $R_n(g)$ or $L_n(g)$ are close to those of Engel groups. The present article deals with orderable groups in which, for some $n \geq 1$, the subgroups $R_n(g)$ are polycyclic. Let $h \geq 0$, $n > 0$ be integers and G be an orderable group in which $R_n(g)$ is polycyclic with Hirsch length at most h for every $g \in G$. It is proved that there are (h,n)-bounded numbers h^* and c^* such that G has a finitely generated normal nilpotent subgroup N with $h(N) \leq h^*$ and G/N nilpotent of class at most c^*. The analogue of this theorem for $L_n(g)$ was established in 2018 by Shumyatsky ['Orderable groups with Engel-like conditions', *J. Algebra* **499** (2018), 313–320].

2020 *Mathematics subject classification*: primary 20F60; secondary 20F45.

Keywords and phrases: orderable groups, polycyclic groups, Engel groups, right Engel elements.

1. Introduction

A group G is called an Engel group if for every $x, y \in G$, the equation $[y, x, x, \dots, x] = 1$ holds, where x is repeated in the commutator sufficiently many times depending on x and y. Throughout the paper, we use the left-normed simple commutator notation $[a_1, a_2, a_3, \dots, a_r] = [\dots[[a_1, a_2], a_3], \dots, a_r]$. The long commutators $[y, x, \dots, x]$, where x occurs $i \geq 0$ times, are denoted by $[y,_i x]$ with $[y,_0 x] = y$. An element $g \in G$ is called a left Engel element if for each $x \in G$, there is a positive integer $n = n(g, x)$ such that $[x,_n g] = 1$. If n can be chosen independently of x, then g is a left n-Engel element of G. If $g \in G$ and for all $x \in G$ there exists a positive integer $n = n(g, x)$ such that $[g,_n x] = 1$, then g is a right Engel element of G. If n can be chosen independently

Funded by the European Union – Next Generation EU, Missione 4 Componente 1 CUP B53D23009410006, PRIN 2022-2022PSTWLB – Group Theory and Applications. The first author acknowledges financial support from FAPDF. The first and the second authors acknowledge financial support from GNSAGA – INdAM.

of x, then g is a right n-Engel element of G. A group G is n-Engel if $[x,_n g] = 1$ for all $x, g \in G$.

Given $g \in G$, write $R_n(g)$ for the subgroup generated by all commutators $[g,_n x]$, where x ranges over G. Similarly, $L_n(g)$ stands for the subgroup generated by all commutators of the form $[x,_n g]$.

There are several recent results showing that certain properties of groups with small subgroups $R_n(g)$ or $L_n(g)$ are close to those of Engel groups (see for instance [3–5, 8, 9]). The present article deals with orderable groups. A group G is called orderable if there exists a full order relation $\leq$ on the set G such that $x \leq y$ implies $axb \leq ayb$ for all $a, b, x, y \in G$, that is, the order on G is compatible with the product of G. Kim and Rhemtulla proved that any orderable n-Engel group is nilpotent ([6], see also [7]). More recently, orderable groups with n-Engel word-values were considered [10]. In the present article, we consider orderable groups G such that the subgroup $R_n(g)$ is polycyclic for each $g \in G$. Recall that a group is polycyclic if and only if it admits a finite subnormal series all of whose factors are cyclic. The Hirsch length $h(K)$ of a polycyclic group K is the number of infinite factors in the subnormal series. It is well known that every finitely generated nilpotent group is polycyclic.

Our aim here is to prove the following theorem.

THEOREM 1.1. *Let h, n be positive integers and G be an orderable group in which $R_n(g)$ is polycyclic with $h(R_n(g)) \leq h$ for every $g \in G$. Then, there exist (h, n)-bounded numbers h^* and c^* such that G has a finitely generated normal nilpotent subgroup N with $h(N) \leq h^*$ and G/N nilpotent of class at most c^*.*

A similar result for $L_n(g)$ was proved in [9]. We remark that while it is well known that the inverse of a right Engel element is left Engel, there is no such straightforward connection between the subgroups $R_n(g)$ and $L_n(g^{-1})$, and our Theorem 1.1 is not a direct consequence of the result in [9].

2. Preliminaries

We write $\langle X\rangle$ for a subgroup generated by a set X and $\gamma_i(G)$ for the ith term of the lower central series of a group G.

The following lemma plays a central role in what follows.

LEMMA 2.1. *Let a group G be a product of a normal abelian subgroup A by a cyclic subgroup $\langle g\rangle$. Then, $L_{n+1}(g) \leq R_n(g^{-1})$ for any $n \geq 1$. In particular, $\gamma_{n+1}(G) \leq R_{n-1}(g^{-1})$ for any $n \geq 2$.*

PROOF. Let $x \in G$ and write $x = yg^i$, where $y \in A$. An easy induction on n shows that $[yg^i,_{n+1} g] = [y,_{n+1} g]^{g^i}[g^i,_{n+1} g]$. We have

$$\begin{aligned}[x,_{n+1} g] &= [yg^i,_{n+1} g] = [y,_{n+1} g]^{g^i} \\ &= [[y, g],_n g]^{g^i} = [g^{-y}g,_n g]^{g^i} = [g^{-y},_n g]^{g^{i+1}} = [g^{-1},_n g^{y^{-1}}]^{yg^{i+1}}.\end{aligned}$$

Since G' is contained in A and A is abelian, it follows that

$$[g^{-1},{}_n\, g^{y^{-1}}]^{yg^{i+1}} = [g^{-1},{}_n\, g^{y^{-1}}]^{g^{i+1}} = [g^{-1},{}_n\, g^{y^{-1}g^{i+1}}].$$

Hence, $L_{n+1}(g) \leq R_n(g^{-1})$.

We obviously have $\gamma_{n+1}(G) \leq L_n(g)$ and so it follows that $\gamma_{n+1}(G) \leq R_{n-1}(g^{-1})$ for any $n \geq 2$. □

Certainly, under the hypotheses of Lemma 2.1, we have $\gamma_{n+1}(G) \leq R_{n-1}(g)$.

LEMMA 2.2 [9, Lemma 2.1]. *Let $G = H\langle g\rangle$, where H is a nilpotent of class c normal subgroup and g is a left n-Engel element. Then, G is nilpotent with class at most cn.*

LEMMA 2.3. *Let $G = H\langle g\rangle$, where H is a nilpotent of class c normal subgroup. For any positive integers c, n, there exists an integer $f = f(c, n)$ such that $\gamma_f(G) \leq R_n(g)$.*

PROOF. We argue by induction on c. If H is abelian, then Lemma 2.1 gives $\gamma_{n+2}(G) \leq R_n(g)$ and so it is enough to choose $f = n + 2$. Assume that $c \geq 2$ and let $Z = Z(H)$. By induction, there exists a bounded number s such that $\gamma_s(G) \leq ZR_n(g)$. Let $R = R_n(g) \cap Z\gamma_s(G)$ and hence $\gamma_s(G) \leq ZR$. Arguing modulo Z, we have $ZR = Z(R_n(g) \cap Z\gamma_s(G)) = Z\gamma_s(G)$. So ZR is normal in G. Set $Z_0 = ZR$ and, for $i = 0, 1, \ldots, s-1$, let Z_i denote the full inverse image of $Z_i(G/Z_0)$. Further, for $i = 0, 1, \ldots, s-1$, we set $G_i = Z_i\langle g\rangle$. It is clear that G/Z_0 is nilpotent and $Z_{s-1} = G_{s-1} = G$.

Since Z is abelian, Lemma 2.1 gives $[Z,{}_{n+1}\, g^{-1}] \leq R_n(g) \cap Z \leq R$. We observe that Z and R are commuting g-invariant subgroups and so $[Z_0,{}_{n+1}\, g^{-1}] = [Z,{}_{n+1}\, g^{-1}][R,{}_{n+1}\, g^{-1}] \leq R$. Let T be the normal closure of $[Z_0,{}_{n+1}\, g^{-1}]$ in G_0. Note that $R \leq G' \leq H$. Hence, $ZR \leq H$ and since R is g-invariant, we have $T \leq R$. As the image of g^{-1} in G_0/T is left $(n+1)$-Engel and ZR/T is nilpotent, Lemma 2.2 implies that there exists a bounded number k such that G_0/T is nilpotent with class at most $k-1$ and so $\gamma_k(G_0) \leq R$.

By induction on i, we will show that there exists a bounded number k_i such that $\gamma_{k_i}(G_i) \leq R$. Once this is done, we will simply set $f = k_{s-1}$. Assume that for some $j \leq s-1$, there exists k_j with the property that $\gamma_{k_j}(G_j) \leq R$. If $j = s-1$, we have nothing to prove, so we suppose that $j \leq s-2$. Since G_{j+1} normalises G_j, it follows that $\gamma_{k_j}(G_j)$ is normal in G_{j+1}. Recall that $\gamma_s(G) \leq G_0$. Then, if $x \in G_{j+1}$, we get $[x,{}_{s-1}\, g] \in \gamma_s(G)$ and hence $[x,{}_{s+k_j-2}\, g] \in \gamma_{k_j}(G_j)$. It follows that the image of g in $G_{j+1}/\gamma_{k_j}(G_j)$ is left $(s+k_j-2)$-Engel. Applying Lemma 2.2 to the factor-group $G_{j+1}/\gamma_{k_j}(G_j) = ((G_{j+1} \cap H)\gamma_{k_j}(G_j)/\gamma_{k_j}(G_j))\langle g\gamma_{k_j}(G_j)\rangle$, we see that it is nilpotent with bounded class, say $k_{j+1} - 1$. We conclude that $\gamma_{k_{j+1}}(G_{j+1}) \leq R$. This completes the proof. □

Given subgroups X and Y of a group G, we denote by X^Y the smallest subgroup of G containing X and normalised by Y. We say that a group G satisfies max if G satisfies the maximal condition on subgroups.

LEMMA 2.4. *Let g and y be elements of a group G, and suppose that for some $n \geq 1$, the subgroup $R_n(g)$ satisfies* max. *Then, $\langle g\rangle^{\langle y\rangle}$ is finitely generated.*

PROOF. Observe that $\langle g\rangle^{\langle y\rangle}$ is generated by all commutators $[g,_i y]$ for $i \geq 0$. Set $Y = \langle g\rangle^{\langle y\rangle} \cap R_n(g)$. We have $\langle g\rangle^{\langle y\rangle} = \langle g, [g,y], \dots, [g,_{n-1} y], Y\rangle$. Since $R_n(g)$ satisfies max, Y is finitely generated and so the lemma follows. □

COROLLARY 2.5. *Let $g_1, \dots, g_m$ be elements of a group G such that for some $n \geq 1$, the subgroups $R_n(g_i)$ satisfy* max *for every $i \in \{1, \dots, m\}$. If $y \in G$, then $\langle g_1, \dots, g_m\rangle^{\langle y\rangle}$ is finitely generated.*

LEMMA 2.6 [9, Lemma 2.8]. *If G is a group generated by two elements x and y, then $G' = \langle [x,y]^{x^r y^s} \mid r,s \in \mathbb{Z}\rangle$.*

Using the previous results, we are able to prove the following lemma.

LEMMA 2.7. *Let $n \geq 1$ and $G = \langle g_1, \dots, g_m\rangle$ such that $R_n([g_i, g_j])$ satisfies* max *for all $i,j \in \{1, \dots, m\}$. Then, G' is finitely generated.*

PROOF. First, assume that $m = 2$. Then, $G' = \langle [g_1, g_2]^{g_1^r g_2^s} \mid r,s \in \mathbb{Z}\rangle$ by Lemma 2.6. However, by repeated applications of Corollary 2.5, $(\langle [g_1, g_2]\rangle^{\langle g_1\rangle})^{\langle g_2\rangle}$ is finitely generated. Now, suppose that $m \geq 3$, and assume that the result is true for subgroups which can be generated by at most $m - 1$ elements from $\{g_1, \dots, g_m\}$. For $i = 1, \dots, m$, set $G_i = \langle g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_m\rangle$. By the induction hypothesis, G_i' is finitely generated and, by Corollary 2.5, the same is true for $(G_i')^{\langle g_i\rangle}$. Moreover, it is easy to see that $K = \langle (G_i')^{\langle g_i\rangle} \mid i = 1, \dots, m\rangle$ is a normal subgroup of G and hence $G' = K$. In particular, G' is finitely generated. □

Now, an easy induction yields the following corollary.

COROLLARY 2.8. *Let G be a finitely generated group such that for each $g \in G$, there exists $n \geq 1$ with the property that $R_n(g)$ satisfies* max. *Then, each term of the derived series of G is finitely generated.*

LEMMA 2.9 [9, Corollary 2.5]. *Let $G = H\langle g\rangle$ be a nilpotent group with a normal torsion-free subgroup H of Hirsch length h. Then, G is nilpotent with h-bounded class.*

3. Proof of Theorem 1.1

It is easy to see that any orderable group is torsion-free. The class of orderable groups is closed under taking subgroups, but a quotient of an orderable group is not necessarily orderable [1, Section 2.1]. A subgroup C of an ordered group $(G, \leq)$ is called convex if $x \in C$ whenever $1 \leq x \leq c$ for some $c \in C$. Obviously, $\{1\}$ and G are convex subgroups of G; and, if C is a convex subgroup, then every conjugate of C is convex. If C is a normal convex subgroup of an ordered group G, then G/C is ordered [1, Section 2.1]. It is also clear that all convex subgroups of an ordered group form, by inclusion, a totally ordered set, which is closed under intersection and union. If C and D are convex subgroups of an ordered group G, with $C < D$, and there is not a convex subgroup H of G such that $C < H < D$, we say that the pair (C, D) is a convex jump in G. Orders on a group G in which $\{1\}$ and G are the only convex subgroups are very well known. By a result of Hölder [1, Theorem 1.3.4], a group G with such an order is order-isomorphic to a subgroup of the additive group of the real numbers under the natural order. This implies that if (C, D) is a convex jump of an ordered group, then C is normal in D and D/C is abelian [1, Lemma 1.3.6].

The following lemma is an application of Lemma 2.4.

LEMMA 3.1. *Let G be an orderable group in which for each g, there exists $n \geq 1$ such that $R_n(g)$ satisfies* max. *Then, each convex subgroup in G is normal.*

PROOF. Suppose that C is convex and not normal in G. Then, there exists $x \in G$ such that $C \neq C^x$. Since convex subgroups form a chain, we have either $C^x < C$ or $C < C^x$. Without loss of generality, assume that $C < C^x$ and let $c^x \in C^x \setminus C$ for a suitable $c \in C$. Then, $C^{x^i} < C^{x^{i+1}}$ for any integer i. Moreover, by Lemma 2.4, the subgroup $\langle c\rangle^{\langle x\rangle}$ is finitely generated so that $\langle c\rangle^{\langle x\rangle} = \langle c^{x^{i_1}}, \dots, c^{x^{i_k}}\rangle$, where $i_1, \dots, i_k$ are integers. We may assume $i_1 < i_2 < \cdots < i_k$. It follows that $\langle c\rangle^{\langle x\rangle} \leq C^{x^{i_k}}$. Hence, $c^{x^{i_k+1}} \in C^{x^{i_k}}$ and therefore $c^x \in C$, which is a contradiction. □

We will need the following result due to Zelmanov [11] (see also [2]).

THEOREM 3.2. *Let $n \geq 1$ and G be a nilpotent torsion-free n-Engel group. Then, G is nilpotent of n-bounded nilpotency class.*

LEMMA 3.3. *Let h, n be positive integers and G be an orderable group in which $R_n(g)$ is polycyclic with $h(R_n(g)) \leq h$ for every $g \in G$. Then, G' is nilpotent with (h, n)-bounded class.*

PROOF. It is sufficient to establish the result under the additional hypothesis that G is finitely generated. Thus, assume that G is finitely generated. Since polycyclic groups satisfy max, we know by Lemma 3.1 that the convex subgroups in G are normal. Let C be a convex subgroup such that G/C is soluble. By Corollary 2.8, all terms of the derived series of G are finitely generated. It follows that G/C has finite rank and therefore, by [1, Theorem 3.3.1], the derived group $(G/C)'$ is nilpotent. Hence, each element of $(G/C)'$ is left Engel. For $x \in G/C$ and $y \in (G/C)'$, let $J_{x,y}$ be the subgroup generated by all commutators $[x,_k y]$, where $k \geq n$. The subgroup $J_{x,y}$ is a y-invariant nilpotent subgroup with $h(J_{x,y}) \leq h$.

In view of Lemma 2.9, the subgroup $J_{x,y}\langle y\rangle$ is nilpotent of h-bounded class. Therefore, there is an (h, n)-bounded number n_0 such that y is n_0-Engel in G/C. Hence, every element of $(G/C)'$ is left n_0-Engel. Now, Theorem 3.2 tells us that $(G/C)'$ is nilpotent of (h, n)-bounded nilpotency class. In particular, we deduce that G/C has (h, n)-bounded derived length, say d.

Let S be the intersection of all convex subgroups N of G such that G/N is soluble. The above argument shows that $G^{(d)} \leq S$. Since all terms of the derived series of G are finitely generated, it follows that $G/G^{(d)}$ satisfies max and we conclude that S is finitely generated, too. Then, if $S \neq 1$, among the convex subgroups properly contained in S, we can choose a maximal one, say D. It follows that (D, S) is a convex jump in G. Hence, S/D is abelian and so G/D is soluble. This is a contradiction since S is the intersection of all convex subgroups N of G such that G/N is soluble. The conclusion is that $S = 1$ and G is soluble with derived length at most d. Again, we observe that G has finite rank whence, by [1, Theorem 3.3.1], G' is nilpotent. Finally, arguing as above, every element of G' is left n_0-Engel. Hence, by Theorem 3.2, the nilpotency class of G' is (h, n)-bounded. □

We are now ready to complete the proof of Theorem 1.1, which we restate here for the reader's convenience.

THEOREM. *Let h, n be positive integers and G be an orderable group in which $R_n(g)$ is polycyclic with $h(R_n(g)) \leq h$ for every $g \in G$. Then, there exist (h, n)-bounded numbers h^* and c^* such that G has a finitely generated normal nilpotent subgroup N with $h(N) \leq h^*$ and G/N nilpotent of class at most c^*.*

PROOF. For any $x \in G$, set $H_x = G'\langle x\rangle$. In view of Lemma 3.3, G' is nilpotent and Lemma 2.3 tells us that there is a bounded number f such that $\gamma_f(H_x) \leq R_n(x)$. It follows that $h(\gamma_f(H_x)) \leq h$ and therefore $h(L_{f-1}(x)) \leq h$. Hence, we can apply the main theorem from [9], which completes the proof. □

Acknowledgement

This work was done during a visit of the first and the second authors at the Department of Mathematics of the University of Salerno. They thank the department for their hospitality.

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MARIA TOTA, Dipartimento di Matematica,
Università di Salerno, 84084 Fisciano (SA), Italy
e-mail: mtota@unisa.it

PAVEL SHUMYATSKY, Department of Mathematics,
University of Brasilia, Brasilia – DF 70910-900, Brazil
e-mail: pavel@mat.unb.br

CARMELA SICA, Departamento de Matemàtica,
Universidade Federal da Bahia,
Ondina 40170-110, Salvador – Bahia, Brasil
e-mail: csica@ufba.br