QUASI-HOMOLOGY AND UNIVERSAL COEFFICIENTS

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In order to study an arbitrary sequence of modules and homomorphisms, we propose a definition of " homology " modules, or what we call quasi-homology modules, for such a sequence. Then we seek partial analogues of the universal coefficient theorems to make some propaganda for the notion.

1. Quasi-homology module. For a sequence

$$C: \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots, \quad n \in \mathbf{Z},$$

of arbitrary modules over a ring with an identity and arbitrary homomorphisms, we define the *n-dimensional quasi-homology module* $\mathscr{H}_n(C)$, the *n-dimensional lower quasi-homology module* $\underline{\mathscr{H}}_n(C)$ and the *n-dimensional upper quasi-homology module* $\overline{\mathscr{H}}_n(C)$ by

$$\mathscr{H}_n(C) = (\operatorname{Ker} d_n + \operatorname{Im} d_{n+1})/(\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}),$$

$$\underline{\mathscr{H}}_n(C) = \operatorname{Ker} d_n/(\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1})$$

and

$$\overline{\mathscr{H}}_n(C) = \operatorname{Im} d_{n+1}/(\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}),$$

respectively.

If C and C' are sequences of modules over the same ring, a *homomorphism* $f: C \to C'$ is a family of homomorphisms $f_n: C_n \to C'_n$, one for each n, such that $d'_n f_n = f_{n-1} d_n$. The mapping

$$\mathscr{H}_n(f): \quad \mathscr{H}_n(C) \to \mathscr{H}_n(C')$$

defined by

$$\mathscr{H}_n(f): \quad c + (\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}) \mapsto f_n(c) + (\operatorname{Ker} d'_n \cap \operatorname{Im} d'_{n+1})$$

is a homomorphism. The same rule defines homomorphisms

$$\underline{\mathscr{H}}_n(f): \quad \underline{\mathscr{H}}_n(C) \to \underline{\mathscr{H}}_n(C')$$

and

$$\overline{\mathscr{H}}_n(f): \quad \overline{\mathscr{H}}_n(C) \to \overline{\mathscr{H}}_n(C').$$

With these definitions, $\mathscr{H}_n$, $\underline{\mathscr{H}}_n$ and $\overline{\mathscr{H}}_n$ are covariant functors on the category of sequences of modules to the category of modules for each n.

A *homotopy* s between two homomorphisms $f, g: C \to C'$ is a family of homomorphisms $s_n: C_n \to C'_{n+1}$, one for each n, such that

$$d'_{n+1} s_n + s_{n-1} d_n = f_n - g_n,$$

$$s_{n-1} d_n d_{n+1} = 0$$

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and

$$d'_n d'_{n+1} s_n = 0$$

for each n. The statements expected of homotopy such as Theorem 2.1, Corollary 2.2 and Proposition 2.3 in [1, p. 40] can readily be ascertained.

A sequence C is said to be *exact*, *demi-exact* or *semi-exact* at C_n according as $\mathscr{H}_n(C) = 0$, $\underline{\mathscr{H}}_n(C) = 0$ or $\overline{\mathscr{H}}_n(C) = 0$. When a sequence C is semi-exact at C_n, we have $\mathscr{H}_n(C) = \underline{\mathscr{H}}_n(C) = H_n(C)$, where $H_n(C)$ is the usual n-dimensional homology module of C. A sequence C is said to be *exact*, *demi-exact* or *semi-exact* according as the sequence C is exact, demi-exact or semi-exact at C_n for every n. Semi-exact sequences are the ones that have been most intensively studied until now. All sequences consisting entirely of epimorphisms, and all sequences consisting entirely of monomorphisms, are demi-exact.

2. Lower and upper quasi-homology modules.

PROPOSITION 2.1. *For any sequence C and for any integer n, we have*

$$\mathscr{H}_n(C) = \underline{\mathscr{H}}_n(C) \oplus \overline{\mathscr{H}}_n(C). \blacksquare$$

We use the sign ▮ to mean that no proof is given or to indicate the end of the proof.

From a sequence

$$C: \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots,$$

we form a new sequence

$$d^{-2}C: \cdots \to \operatorname{Ker} d_n d_{n+1} \xrightarrow{d^{-2}_{n+1}} \operatorname{Ker} d_{n-1} d_n \xrightarrow{d^{-2}_n} \operatorname{Ker} d_{n-2} d_{n-1} \to \cdots,$$

where $d^{-2}_n(c) = d_n(c)$ for all $c \in \operatorname{Ker} d_{n-1} d_n \subset C_n$ for each n.

PROPOSITION 2.2. *For any sequence C, the sequence $d^{-2}C$ is semi-exact and we have*

$$\underline{\mathscr{H}}_n(C) = H_n(d^{-2}C)$$

for each integer n.

Proof. $\operatorname{Im} d^{-2}_{n+1} = \operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1} \subset \operatorname{Ker} d_n = \operatorname{Ker} d^{-2}_n$, and

$$\begin{aligned} \underline{\mathscr{H}}_n(C) &= \operatorname{Ker} d_n / (\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}) \\ &= \operatorname{Ker} d^{-2}_n / \operatorname{Im} d^{-2}_{n+1} = H_n(d^{-2}C). \blacksquare \end{aligned}$$

PROPOSITION 2.3. *For any sequence C and for any integer n, we have isomorphisms*

$$C_{n+1}/\operatorname{Ker} d_n d_{n+1} \rightarrowtail\!\!\!\!\twoheadrightarrow \overline{\mathscr{H}}_n(C) \rightarrowtail\!\!\!\!\twoheadrightarrow \operatorname{Im} d_n d_{n+1}$$

under the mappings

$$c + \operatorname{Ker} d_n d_{n+1} \overset{i_1(C)}{\mapsto} d_{n+1}(c) + (\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}) \overset{i_2(C)}{\mapsto} d_n d_{n+1}(c). \blacksquare$$

3. Universal coefficients for quasi-homology. Throughout the section, let

$$C: \cdots \to C_{n+1} \overset{d_{n+1}}{\to} C_n \overset{d_n}{\to} C_{n-1} \to \cdots$$

be a sequence of modules over a commutative ring with an identity and let G be a module over the same ring. We consider

$$C\otimes G: \cdots \to C_{n+1}\otimes G \overset{d^{\wedge}_{n+1}}{\to} C_n\otimes G \overset{d^{\wedge}_n}{\to} C_{n-1}\otimes G \to \cdots,$$

where $d^{\wedge}_n = d_n \otimes 1_G$. We are interested in the n-dimensional quasi-homology module $\mathscr{H}_n(C\otimes G)$ of $C\otimes G$. By Proposition 2.1, we have the decompositions

$$\mathscr{H}_n(C)\otimes G = (\underline{\mathscr{H}}_n(C)\otimes G) \oplus (\bar{\mathscr{H}}_n(C)\otimes G)$$

and

$$\mathscr{H}_n(C\otimes G) = \underline{\mathscr{H}}_n(C\otimes G) \oplus \bar{\mathscr{H}}_n(C\otimes G).$$

PROPOSITION 3.1. *For any sequence C, for any module G over the same ring and for any integer n, the mapping*

$$\alpha_n: \quad \mathscr{H}_n(C)\otimes G \to \mathscr{H}_n(C\otimes G)$$

defined as follows is a homomorphism:

$$\alpha_n: \quad (c + (\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}))\otimes g \mapsto (c\otimes g) + (\operatorname{Ker} d^{\wedge}_n \cap \operatorname{Im} d^{\wedge}_{n+1}),$$

where

$$c \in \operatorname{Ker} d_n + \operatorname{Im} d_{n+1} \subset C_n, \; g \in G \quad \textit{and} \quad c\otimes g \in \operatorname{Ker} d^{\wedge}_n + \operatorname{Im} d^{\wedge}_{n+1} \subset C_n \otimes G.$$

If

$$\underline{\alpha}_n = \alpha_n \,|\, \underline{\mathscr{H}}_n(C)\otimes G$$

and

$$\bar{\alpha}_n = \alpha_n \,|\, \bar{\mathscr{H}}_n(C)\otimes G,$$

then

$$\underline{\alpha}_n: \quad \underline{\mathscr{H}}_n(C)\otimes G \to \underline{\mathscr{H}}_n(C\otimes G)$$

and

$$\bar{\alpha}_n: \quad \bar{\mathscr{H}}_n(C)\otimes G \to \bar{\mathscr{H}}_n(C\otimes G). \blacksquare$$

PROPOSITION 3.2. *For any sequence C, for any module G over the same ring and for any integer n, if there exist isomorphisms f_{n+1}, f_n* and *f_{n-1} making*

$$\begin{array}{ccccc}
\operatorname{Ker} d^{\wedge}_n d^{\wedge}_{n+1} & \overset{d^{\wedge -2}_{n+1}}{\to} & \operatorname{Ker} d^{\wedge}_{n-1} d^{\wedge}_n & \overset{d^{\wedge -2}_n}{\to} & \operatorname{Ker} d^{\wedge}_{n-2} d^{\wedge}_{n-1} \\
\downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\
\operatorname{Ker} d_n d_{n+1} \otimes G & \overset{d^{-2\wedge}_{n+1}}{\to} & \operatorname{Ker} d_{n-1} d_n \otimes G & \overset{d^{-2\wedge}_n}{\to} & \operatorname{Ker} d_{n-2} d_{n-1} \otimes G
\end{array}$$

commutative, then

$$\underline{\mathscr{H}}_n(C)\otimes G = H_n(d^{-2}C)\otimes G$$

and

$$\underline{\mathscr{H}}_n(C\otimes G) = H_n(d^{-2}C\otimes G).$$

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Proof. Under the hypothesis, the two semi-exact sequences $d^{\wedge -2}(C\otimes G)$ and $d^{-2}C\otimes G$ have the same n-dimensional homology module and Proposition 2.2 yields

$$\underline{\mathscr{H}}_n(C\otimes G) = H_n(d^{\wedge -2}(C\otimes G)) = H_n(d^{-2}C\otimes G). \blacksquare$$

Needless to say, if C is semi-exact, then the hypothesis of Proposition 3.2 is obviously satisfied.

PROPOSITION 3.3. *For any sequence C, for any free module G over the same ring and for any integer n, the hypothesis of Proposition* 3.2 *is satisfied.*

Proof. If G is free, we have a monomorphism

$$j\otimes 1_G\colon \quad \operatorname{Ker} d_{n-1}\, d_n\otimes G \to C_n\otimes G,$$

where j is the injection, i.e. the inclusion, of $\operatorname{Ker} d_{n-1}\, d_n$ into C_n. Since G is free, we have

$$\operatorname{Im}(j\otimes 1_G) = \operatorname{Ker} d^{\wedge}_{n-1}\, d_n^{\wedge},$$

and therefore $j\otimes 1_G$ induces an isomorphism of $\operatorname{Ker} d_{n-1}\, d_n\otimes G$ onto $\operatorname{Ker} d^{\wedge}_{n-1}\, d^{\wedge}_n$. Let f_n be the inverse of this isomorphism. The isomorphisms f_{n+1}, f_n and f_{n-1} clearly satisfy the hypothesis of Proposition 3.2. ∎

PROPOSITION 3.2 means that, under the hypothesis stated, the study of

$$\underline{\alpha}_n\colon \quad \underline{\mathscr{H}}_n(C)\otimes G \to \underline{\mathscr{H}}_n(C\otimes G)$$

is reduced to the study of the homomorphism

$$H_n(d^{-2}C)\otimes G \to H_n(d^{-2}C\otimes G)$$

in the usual universal coefficient theorem for homology [**1**, p. 171]. In particular, we record

PROPOSITION 3.4. *For any sequence C of free modules over a principal ideal domain, for any projective module G over the same domain and for any integer n, if the hypothesis of Proposition* 3.2 *is satisfied, then*

$$\underline{\alpha}_n\colon \quad \underline{\mathscr{H}}_n(C)\otimes G \to \underline{\mathscr{H}}_n(C\otimes G)$$

defined by

$$\underline{\alpha}_n\colon \quad (c+(\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}))\otimes g \mapsto (c\otimes g)+(\operatorname{Ker} d_n^{\wedge} \cap \operatorname{Im} d^{\wedge}_{n+1})$$

is an isomorphism. ∎

COROLLARY 3.5. *For any sequence C of vector spaces over a field, for any vector space G over the same field and for any integer n,*

$$\underline{\alpha}_n\colon \quad \underline{\mathscr{H}}_n(C)\otimes G \to \underline{\mathscr{H}}_n(C\otimes G)$$

defined by

$$\underline{\alpha}_n\colon \quad (c+(\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}))\otimes g \mapsto (c\otimes g)+(\operatorname{Ker} d_n^{\wedge} \cap \operatorname{Im} d^{\wedge}_{n+1})$$

is an isomorphism.

Proof. Combine Proposition 3.3 and Proposition 3.4. ∎

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The usual universal coefficient theorem for homology states among other things that, under certain circumstances, the homomorphism similar to our $\underline{\alpha}_n$ is a monomorphism. It is interesting to note that $\bar{\alpha}_n$ is an epimorphism. This is part of the following theorem.

THEOREM 3.6. *Let C be a sequence of modules and let G be a module over the same ring. For any integer n, let*

$$\bar{\alpha}_n:\quad \mathscr{H}_n(C)\otimes G \to \mathscr{H}_n(C\otimes G)$$

be a mapping defined by

$$\bar{\alpha}_n:\quad (c+(\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}))\otimes g \mapsto (c\otimes g)+(\operatorname{Ker} d_n^\wedge \cap \operatorname{Im} d_{n+1}^\wedge).$$

Then

(1) $\bar{\alpha}_n$ *is an epimorphism.*

(2) *If* C_{n-1} *is projective, then*

$$0 \to \operatorname{Tor}(C_{n-1}/\mathscr{H}_n(C),\, G) \to \mathscr{H}_n(C)\otimes G \xrightarrow{\bar{\alpha}_n} \mathscr{H}_n(C\otimes G) \to 0$$

is exact.

(3) *If G is projective, then (regardless of whether* C_{n-1} *is projective or not)* $\bar{\alpha}_n$ *is an isomorphism.*

Proof. Let j_1 be the injection, i.e. the inclusion, of $\operatorname{Im} d_n d_{n+1}$ into C_{n-1} and recall that $i_2(C)$ and $i_2(C\otimes G)$ are the isomorphisms

$$i_2(C):\quad \mathscr{H}_n(C) \rightarrowtail\!\!\!\!\twoheadrightarrow \operatorname{Im} d_n d_{n+1}$$

and

$$i_2(C\otimes G):\quad \mathscr{H}_n(C\otimes G) \rightarrowtail\!\!\!\!\twoheadrightarrow \operatorname{Im} d_n^\wedge d_{n+1}^\wedge$$

given in Proposition 2.3. From a short exact sequence

$$0 \to \mathscr{H}_n(C) \xrightarrow{j_1 i_2(C)} C_{n-1} \to C_{n-1}/\mathscr{H}_n(C) \to 0,$$

we obtain its fundamental exact sequence

$$\cdots \to \operatorname{Tor}(C_{n-1}, G) \to \operatorname{Tor}(C_{n-1}/\mathscr{H}_n(C), G)$$
$$\to \mathscr{H}_n(C)\otimes G \xrightarrow{j_1^\wedge i_2(C)^\wedge} C_{n-1}\otimes G \to (C_{n-1}/\mathscr{H}_n(C))\otimes G \to 0.$$

A quick computation tells us that

$$\operatorname{Im} j_1^\wedge i_2(C)^\wedge = \operatorname{Im} d_n^\wedge d_{n+1}^\wedge.$$

In view of the commutative diagram

$$\begin{array}{ccc} \mathscr{H}_n(C)\otimes G & \overset{\varepsilon}{\twoheadrightarrow} & \operatorname{Im} j_1^\wedge i_2(C)^\wedge \\ \downarrow \bar{\alpha}_n & & \| . \\ \mathscr{H}_n(C\otimes G) & \overset{i_2(C\otimes G)}{\rightarrowtail\!\!\!\!\twoheadrightarrow} & \operatorname{Im} d_n^\wedge d_{n+1}^\wedge, \end{array}$$

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where ε is the epimorphism induced by $j_1^\wedge i_2(C)^\wedge$ by restricting its codomain, the fundamental exact sequence yields an exact sequence

$$\cdots \to \operatorname{Tor}(C_{n-1}, G) \to \operatorname{Tor}(C_{n-1}/\bar{\mathscr{H}}_n(C), G) \xrightarrow{} \bar{\mathscr{H}}_n(C)\otimes G \xrightarrow{\bar{\alpha}_n} \bar{\mathscr{H}}_n(C\otimes G) \to 0.$$

The conclusions follow from this immediately. ▮

COROLLARY 3.7. *Let C be a sequence of vector spaces over a field and let G be a vector space over the same field. For any integer n, the mapping*

$$\alpha_n\colon\ \mathscr{H}_n(C)\otimes G \to \mathscr{H}_n(C\otimes G)$$

defined by

$$\alpha_n\colon\ (c+(\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}))\otimes g \mapsto (c\otimes g)+(\operatorname{Ker} d_n^\wedge \cap \operatorname{Im} d_{n+1}^\wedge)$$

is an isomorphism.

Proof. Combine Corollary 3.5 and (3) of Theorem 3.6. ▮

4. Universal coefficients for quasi-cohomology. Throughout the section, let

$$C\colon \cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots$$

be a sequence of modules over a commutative ring with an identity and let G be a module over the same ring. We consider

$$\operatorname{Hom}(C, G)\colon \cdots \leftarrow \operatorname{Hom}(C_{n+1}, G) \xleftarrow{d_{n+1}^*} \operatorname{Hom}(C_n, G) \xleftarrow{d_n^*} \operatorname{Hom}(C_{n-1}, G) \leftarrow \cdots,$$

where $d_n^*(f) = fd_n$ for $f \in \operatorname{Hom}(C_{n-1}, G)$. We are interested in the n-dimensional quasi-(co)homology module $\mathscr{H}^n(\operatorname{Hom}(C,G))$ of $\operatorname{Hom}(C,G)$. By Proposition 2.1, we have the decompositions

$$\operatorname{Hom}(\mathscr{H}_n(C), G) = \operatorname{Hom}(\underline{\mathscr{H}}_n(C), G)\oplus \operatorname{Hom}(\bar{\mathscr{H}}_n(C), G)$$

and

$$\mathscr{H}^n(\operatorname{Hom}(C,G)) = \underline{\mathscr{H}}^n(\operatorname{Hom}(C,G))\oplus \bar{\mathscr{H}}^n(\operatorname{Hom}(C,G)).$$

PROPOSITION 4.1. *For any sequence C, for any module G over the same ring and for any integer n, the mapping*

$$\alpha^n\colon\ \mathscr{H}^n(\operatorname{Hom}(C,G)) \to \operatorname{Hom}(\mathscr{H}_n(C), G)$$

defined as follows is a homomorphism: For

$$f+(\operatorname{Im} d_n^* \cap \operatorname{Ker} d_{n+1}^*) \in \mathscr{H}^n(\operatorname{Hom}(C,G)),$$

where

$$f \in \operatorname{Im} d_n^* + \operatorname{Ker} d_{n+1}^* \subset \operatorname{Hom}(C_n, G),$$

let

$$\alpha^n(f+(\operatorname{Im} d_n^* \cap \operatorname{Ker} d_{n+1}^*)) \in \operatorname{Hom}(\mathscr{H}_n(C), G)$$

be such that

$$\alpha^n(f+(\operatorname{Im} d_n^* \cap \operatorname{Ker} d_{n+1}^*)): \quad c+(\operatorname{Ker} d_n \cap \operatorname{Im} d_{n+1}) \mapsto f(c),$$

where

$$c \in \operatorname{Ker} d_n + \operatorname{Im} d_{n+1} \subset C_n.$$

If

$$\underline{\alpha}^n = \alpha^n \,|\, \underline{\mathscr{H}}^n(\operatorname{Hom}(C, G))$$

and

$$\bar{\alpha}^n = \alpha^n \,|\, \bar{\mathscr{H}}^n(\operatorname{Hom}(C, G)),$$

then

$$\underline{\alpha}^n: \quad \underline{\mathscr{H}}^n(\operatorname{Hom}(C, G)) \to \operatorname{Hom}(\underline{\mathscr{H}}_n(C), G)$$

and

$$\bar{\alpha}^n: \quad \bar{\mathscr{H}}^n(\operatorname{Hom}(C, G)) \to \operatorname{Hom}(\bar{\mathscr{H}}_n(C), G). \blacksquare$$

PROPOSITION 4.2. *For any sequence C, for any module G over the same ring and for any integer n, if there exist isomorphisms g_{n+1}, g_n and g_{n-1} making*

$$\begin{array}{ccccc}
\operatorname{Ker} d_{n+3}^* d_{n+2}^* & \xleftarrow{d_{n+1}^{*-2}} & \operatorname{Ker} d_{n+2}^* d_{n+1}^* & \xleftarrow{d_n^{*-2}} & \operatorname{Ker} d_{n+1}^* d_n^* \\
\uparrow g_{n+1} & & \uparrow g_n & & \uparrow g_{n-1} \\
\operatorname{Hom}(\operatorname{Ker} d_n d_{n+1}, G) & \xleftarrow{d_{n+1}^{-2*}} & \operatorname{Hom}(\operatorname{Ker} d_{n-1} d_n, G) & \xleftarrow{d_n^{-2*}} & \operatorname{Hom}(\operatorname{Ker} d_{n-2} d_{n-1}, G)
\end{array}$$

commutative, then

$$\underline{\mathscr{H}}^n(\operatorname{Hom}(C, G)) = H^n(\operatorname{Hom}(d^{-2}C, G))$$

and

$$\operatorname{Hom}(\underline{\mathscr{H}}_n(C), G) = \operatorname{Hom}(H_n(d^{-2}C), G).$$

Proof. Under the hypothesis, two semi-exact sequences $d^{*-2}\operatorname{Hom}(C, G)$ and $\operatorname{Hom}(d^{-2}C, G)$ have the same n-dimensional homology module and Proposition 2.2 yields

$$\underline{\mathscr{H}}^n(\operatorname{Hom}(C, G)) = H^n(d^{*-2}\operatorname{Hom}(C, G)) = H^n(\operatorname{Hom}(d^{-2}C, G)). \blacksquare$$

Needless to say, if C is semi-exact, then the hypothesis of Proposition 4.2 is obviously satisfied. We cannot, however, make a statement which would be labelled as Proposition 4.3 corresponding to Proposition 3.3. Proposition 4.2 means that, under the hypothesis stated, the study of

$$\underline{\alpha}^n: \quad \underline{\mathscr{H}}^n(\operatorname{Hom}(C, G)) \to \operatorname{Hom}(\underline{\mathscr{H}}_n(C), G)$$

is reduced to the study of the homomorphism

$$H^n(\operatorname{Hom}(d^{-2}C, G)) \to \operatorname{Hom}(H_n(d^{-2}C), G)$$

in the usual universal coefficient theorem for cohomology [**1**, p. 77]. In particular, we record

PROPOSITION 4.4. *For any sequence C of free modules over a principal ideal domain, for any*

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injective module G over the same domain and for any integer n, if the hypothesis of Proposition 4.2 is satisfied, then

$$\underline{\alpha}^n: \quad \underline{\mathscr{H}}^n(\mathrm{Hom}(C,G)) \to \mathrm{Hom}(\underline{\mathscr{H}}_n(C),G)$$

defined by

$$\underline{\alpha}^n(f+(\mathrm{Im}\, d_n^* \cap \mathrm{Ker}\, d_{n+1}^*)): \quad c+(\mathrm{Ker}\, d_n \cap \mathrm{Im}\, d_{n+1}) \mapsto f(c)$$

is an isomorphism. ∎

Since we do not have a statement which would be labelled as Proposition 4.3, we cannot make a statement which would be labelled as Corollary 4.5. The usual universal coefficient theorem for cohomology states among other things that, under certain circumstances, the homomorphism similar to our $\underline{\alpha}^n$ is an epimorphism. It is interesting to note that $\bar{\alpha}^n$ is a monomorphism. This is part of the following theorem.

THEOREM 4.6. *Let C be a sequence of modules and let G be a module over the same ring. For any integer n, let*

$$\bar{\alpha}^n: \quad \bar{\mathscr{H}}^n(\mathrm{Hom}(C,G)) \to \mathrm{Hom}(\bar{\mathscr{H}}_n(C),G)$$

be a mapping defined by

$$\bar{\alpha}^n(f+(\mathrm{Im}\, d_n^* \cap \mathrm{Ker}\, d_{n+1}^*)): \quad c+(\mathrm{Ker}\, d_n \cap \mathrm{Im}\, d_{n+1}) \mapsto f(c).$$

Then

(1) *$\bar{\alpha}^n$ is a monomorphism.*
(2) *If C_{n-1} is projective, then*

$$0 \to \bar{\mathscr{H}}^n(\mathrm{Hom}(C, G)) \xrightarrow{\bar{\alpha}^n} \mathrm{Hom}(\bar{\mathscr{H}}_n(C), G) \to \mathrm{Ext}(C_{n-1}/\bar{\mathscr{H}}_n(C), G) \to 0$$

is exact.

(3) *If G is injective, then (regardless of whether C_{n-1} is projective or not) $\bar{\alpha}^n$ is an isomorphism.*

Proof. Let j_1 be the injection, i.e. the inclusion, of $\mathrm{Im}\, d_n d_{n+1}$ into C_{n-1} and recall that $i_2 = i_2(C)$ and $i_1 = i_1(\mathrm{Hom}(C,G))$ are isomorphisms

$$i_2(C): \quad \bar{\mathscr{H}}_n(C) \rightarrowtail\!\!\!\!\twoheadrightarrow \mathrm{Im}\, d_n d_{n+1}$$

and

$$i_1(\mathrm{Hom}(C,G)): \quad \mathrm{Hom}(C_{n-1},G)/\mathrm{Ker}\, d_{n+1}^* d_n^* \rightarrowtail\!\!\!\!\twoheadrightarrow \bar{\mathscr{H}}^n(\mathrm{Hom}(C,G))$$

given in Proposition 2.3. From a short exact sequence

$$0 \to \bar{\mathscr{H}}_n(C) \xrightarrow{j_1 i_2} C_{n-1} \to C_{n-1}/\bar{\mathscr{H}}_n(C) \to 0,$$

we obtain its fundamental exact sequence

$$\cdots \leftarrow \mathrm{Ext}(C_{n-1},G) \leftarrow \mathrm{Ext}(C_{n-1}/\bar{\mathscr{H}}_n(C),G)$$

$$\leftarrow \mathrm{Hom}(\bar{\mathscr{H}}_n(C), G) \xleftarrow{i_2^* j_1^*} \mathrm{Hom}(C_{n-1}, G) \leftarrow \mathrm{Hom}(C_{n-1}/\bar{\mathscr{H}}_n(C), G) \leftarrow 0.$$

A quick computation tells us that

$$\operatorname{Ker} i_2^* j_1^* = \operatorname{Ker} d_{n+1}^* d_n^*.$$

In view of the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}(\bar{\mathscr{H}}_n(C), G) & \overset{\mu}{\leftarrowtail} & \operatorname{Hom}(C_{n-1}, G)/\operatorname{Ker} i_2^* j_1^* \\ \uparrow \bar{\alpha}^n & i_1 & \| \\ \bar{\mathscr{H}}^n(\operatorname{Hom}(C, G)) & \twoheadleftarrow & \operatorname{Hom}(C_{n-1}, G)/\operatorname{Ker} d_{n+1}^* d_n^*, \end{array}$$

where μ is the monomorphism induced by $i_2^* j_1^*$ by factoring its domain, the fundamental exact sequence yields an exact sequence

$$\cdots \leftarrow \operatorname{Ext}(C_{n-1}, G) \leftarrow \operatorname{Ext}(C_{n-1}/\bar{\mathscr{H}}_n(C), G) \leftarrow \operatorname{Hom}(\bar{\mathscr{H}}_n(C), G) \overset{\bar{\alpha}^n}{\leftarrow} \bar{\mathscr{H}}^n(\operatorname{Hom}(C, G)) \leftarrow 0.$$

The conclusions follow from this immediately. ∎

Since we do not have a statement which would be labelled as Corollary 4.5, we cannot make a statement which would be labelled as Corollary 4.7.

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