

SOME TRANSIENT RESULTS ON THE M/SM/1 SPECIAL SEMI-MARKOV MODEL IN RISK AND QUEUEING THEORIES

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We consider a usual situation in risk theory for which the arrival process is a Poisson process and the claim process a positive $(J - X)$ process inducing a semi-Markov process. The equivalent in queueing theory is the M/SM/1 model introduced for the first time by Neuts (1966).

For both models, we give an explicit expression of the probability of non-ruin on $[0, t]$ starting with u as initial reserve and of the waiting time distribution of the last customer arrived before t . "Explicit expression" means in terms of the matrix of the aggregate claims distributions.

1. THE SPECIAL SEMI-MARKOV MODEL IN RISK THEORY

In a usual situation of the theory of risk, let $(A_n, n \geq 1)$ be the claim inter-arrival times process, $(B_n, n \geq 1)$ the claim amounts process. Moreover, we suppose that m "types" of claims are possible represented by the set:

$$(1.1) \quad I = \{1, 2, \dots, m\} \text{ (with } 1 \leq m < \infty \text{)}.$$

The process starts just after payment of an initial claim of type $J_0 = i$ and after this payment, the fortune of the company is supposed to be u ($u \geq 0$). The process $(J_n, n \geq 0)$ represents the sequence of the successive types of claims. For the simplicity of notations, we also introduce the random variables A_0 and B_0 such that:

$$(1.2) \quad A_0 = B_0 = 0 \quad \text{a.s.}$$

If the claim arrivals process is not explosive, let N_t denote the total number of claims in $(0, t)$ (thus excluded the initial claim) and define:

$$(1.3) \quad X(t) = \sum_{n=0}^{N(t)} B_n \text{ (total amount of claims paid on } (0, t))$$

$$(1.4) \quad Z_t = J_{N(t)} \text{ (type of the last claim occurred before or at } t \text{)}.$$

If we also suppose that the incomes of the company occur at a constant rate c ($c > 0$), then the "fortune" $Z(t)$ of the company at time t is given by

$$(1.5) \quad Z(t) = u + ct - X(t).$$

The matrix $m \times m$ \mathfrak{F} of the "distribution" functions of the aggregate claims at time t will be, by definition:

$$(1.6) \quad \mathfrak{F}(x, t) = (F_{ij}(x, t))$$

where

$$(1.7) \quad F_{ij}(x, t) = P[X(t) \leq x, J_{N(t)} = j \mid J_0 = i] \\ (i, j = 1, \dots, m).$$

Probabilistic assumptions

We assume that the processes introduced satisfy the following assumptions:

1. The claim arrival process is a Poisson process of parameter λ .
2. The process $((J_n, B_n), n \geq 0)$ is a positive $(J-X)$ process (see JANSSEN (1970)); this means that

$$(1.8) \quad P[B_n \leq x, J_n = j \mid (J_k, B_k), k \leq n-1] = Q_{J_{n-1}j}(x) \text{ a.s.}$$

where the matrix \mathbf{Q} , defined by $\mathbf{Q}(x) = (Q_{ij}(x))$ is a matrix of mass functions such that:

$$(1.9) \quad \text{i. } Q_{ij}(x) = 0 \text{ for all } x \leq 0 \text{ for all } i, j \in I$$

$$(1.10) \quad \text{ii. } \sum_{j=1}^m Q_{ij}(+\infty) = 1 \text{ for all } i \in I.$$

From the semi-Markov theory (PYKE (1961)), it is well-known that

- 1° if $p_{ij} = \lim_{x \rightarrow \infty} Q_{ij}(x)$ and $\mathbf{P} = (p_{ij})$, then the process $(J_n, n \geq 0)$ — i.e. the process of claim types—is a homogeneous Markov chain with \mathbf{P} as transition matrix.
- 2° The random variables $B_n, n \geq 0$ are not independent, but only conditionally dependent given the Markov chain $(J_n, n \geq 0)$ — often called the “imbedded Markov chain”.
3. The processes $(A_n, n \geq 0)$ and $((J_n, B_n), n \geq 0)$ are independent.

The main problem

The event “ruin before t ” occurs if the trajectory of $Z(t)$ on $(0, t)$ goes under the time axis before t . More precisely, if $\phi_{ij}(u, t)$ represents the probability of non-ruin on $[0, t]$, starting with $J_0 = i$ and an initial fortune u , and such that $J_{N(t)} = j$, we have, by definition:

$$(1.11) \quad \phi_{ij}(u, t) = P[Z(t') \geq 0, 0 \leq t' \leq t, J_{N(t)} = j \mid J_0 = i]$$

or equivalently by (1.5):

$$(1.12) \quad \phi_{ij}(u, t) = P[\sup_{0 \leq \tau \leq t} (X(\tau) - c\tau) \leq u, J_{N(t)} = j \mid J_0 = i].$$

If we are not interested by the last type observed before t , we have enough with

$$(1.13) \quad \phi_i(u, t) = \sum_{j=1}^m \phi_{ij}(u, t)$$

and if (p_1, \dots, p_m) is an initial distribution on J_0 , we have to compute

$$(1.14) \quad \phi(u, t) = \sum_{j=1}^m p_j \phi_j(u, t).$$

The problem solved in this paper is to find an explicit expression of the matrix ϕ , defined by

$$(1.15) \quad \phi(x, t) = (\phi_{ij}(x, t))$$

in terms of the matrix \mathfrak{F} .

2. THE ANALOGOUS MODEL IN QUEUEING THEORY: THE M/SM/1 MODEL

As quoted by several authors (PRABHU (1961), SEAL (1972), JANSSEN (1977)), a risk model can easily be interpreted as a queueing model and vice versa. It suffices to see the process $(A_n, n \geq 1)$ as the one of the interarrival times between two successive customers (i.e. customers $(n-1)$ and n) in a queueing system with one server and as discipline rule FIFO; then, the process $(B_n, n \geq 1)$ represents the successive service times (i.e. B_n is the service time of the customer number $(n-1)$, $n \geq 1$).

We also suppose that at $t=0$, the customer number 0 just begins his service. Moreover, we have m types of customers and J_n represents the type of customer n . Here N_t gives the "number" of the last customer arrived before or at t . With the same probabilistic assumptions as those of the preceding paragraph, the main problem considered in the queueing optic is to get an explicit expression of the distribution of $W_{N(t)}$ where W_n ($n \geq 0$) represents the waiting of the n th customer. More precisely, we must express the matrix \mathbf{W} in terms of \mathfrak{F} where it is defined by

$$(2.1) \quad \mathbf{W}(x, t) = (\mathbf{W}_{ij}(x, t))$$

with

$$(2.2) \quad W_{ij}(x, t) = P[W_{N(t)} \leq x, J_{N(t)} = j \mid J_0 = i].$$

This model is noted M/SM/1 in the queueing literature (Poisson arrivals and semi-Markov service times) introduced by NEUTS (1966).

3. THE DISTRIBUTION OF AGGREGATE CLAIMS

Introduce the usual notation in semi-Markov theory: for any matrix $m \times m$ of mass functions \mathbf{L} , we note by $\mathbf{L}^{(\bar{n})}$ the n -fold convolution of the matrix \mathbf{L} ,

that is

$$(3.1) \quad \mathbf{L}^{(\bar{0})}(x) = (U_0(x)), \mathbf{L}^{(\bar{1})}(x) = (L_{ij}(x))$$

(where $U_0(x)$ is the distribution function with a unit mass at 0) and for $\mathbf{L}^{(\bar{n})}$ we have:

$$(3.2) \quad L_{ij}^{(\bar{n})}(x) = \sum_k \int_{\mathbb{R}} L_{ik}^{(\bar{n}-1)}(x-y) d L_{kj}(y), n \geq 1.$$

If

$$(3.3) \quad S_n = \sum_{i=0}^n B_i$$

it is clear, from (1.8), that

$$(3.4) \quad Q_{ij}^{(\bar{n})}(x) = \mathbf{R}[S_n \leq x, J_n = j | J_0 = i].$$

From assumption (3), it follows then that:

$$(3.5) \quad \mathbf{F}(x, t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \mathbf{Q}^{(\bar{n})}(x)$$

expression given the matrix of distribution of aggregate claims by means of the semi-Markov kernel \mathbf{Q} .

Let us remark that the assumption (1) gives:

$$(3.6) \quad \mathbf{P}[X(t+s) \leq x, J_{N(t+s)} = j | X(s') = y, J_{N(s')} = i, s' \leq s, X(s) = y, J_{N(s)} = i] = F_{ij}(x-y, t)$$

showing that the process $((X(t), J_{N(t)}), t \geq 0)$ is markovian.

4. LOADINGS OF PREMIUMS

To show how the concept of loading of premiums can be introduced in the special semi-Markov risk model considered here, let us suppose that the quantities—mean cost of a claim of type i —

$$(4.1) \quad \eta_i = \sum_j \int_0^{\infty} x d Q_{ij}(x), \quad i \in I$$

are finite. Moreover, we suppose that the Markov chain $(J_n, n \geq 0)$ is ergodic and that (Π_1, \dots, Π_m) represents the unique stationary probability distribution. Starting with this distribution for J_0 , we get, using (3.5):

$$(4.2) \quad \mathbf{P}[X(t) \leq x] = \sum_i \sum_j \Pi_i F_{ij}(x, t)$$

$$(4.3) \quad = \sum_{n=0}^{\infty} \sum_{i=1}^m \sum_{j=1}^m e^{-\lambda t} \frac{(\lambda t)^n}{n!} \Pi_i Q_{ij}^{(\bar{n})}(x)$$

so that the mean of the aggregate claims at time t is given by

$$(4.4) \quad E[X(t)] = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \left(\sum_{i=1}^m \sum_{j=1}^n \Pi_i \int_0^{\infty} x \, d Q_{ij}^{(n)}(x) \right).$$

The term under brackets is the expectation of S_n or, by (3.3)

$$(4.5) \quad \sum_{k=1}^n E(B_k).$$

As the process $(J_n, n \geq 0)$ is stationary, we have, for all k

$$(4.6) \quad E(B_k) = \sum_{l=1}^m \Pi_l \eta_l.$$

This gives:

$$(4.7) \quad E[X(t)] = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} n \left(\sum_{l=1}^m \Pi_l \eta_l \right)$$

or

$$(4.8) \quad E[X(t)] = \lambda \varphi t$$

with

$$(4.9) \quad \varphi = \sum_{l=1}^m \Pi_l \eta_l.$$

It follows that the mean fortune at time t is given by:

$$(4.10) \quad (c - \lambda \varphi)t$$

which is positive if and only if $c = \lambda \varphi(1 + \eta)$, with $\eta > 0$. The justification of the loading η comes also from the fact that, except some degenerate cases, there exists a reserve u such that for all i, j , $\phi_{ij}(u)$ is positive—where $\phi_{ij}(u) = \lim_t \phi_{ij}(u, t)$ —if and only if $\lambda \varphi < c$ (see JANSSEN (1970)).

5. EXPRESSION OF $\phi_{ij}(u, t)$

The assumptions made—(1), (2), (3)—are such that the method used by PRABHU (1961) and later by SEAL (1974) is valid. For the facility, let us suppose that the mass functions $Q_{ij}(x)$ have densities $q_{ij}(x)$ on $(0, \infty)$; then the PRABHU's integral equation becomes the integral system:

$$(5.1) \quad F_{ij}(u + ct, t) = \phi_{ij}(u, t) + \sum_{k=1}^m \int_0^t \phi_{kj}(0, t - \tau) d_x F_{ik}(u + c\tau, \tau)$$

$i, j = 1, \dots, m$

where

$$(5.2) \quad d_x F_{ik}(u + c\tau, \tau) = c \frac{\partial F_{ik}}{\partial x}(u + c\tau, \tau) d\tau.$$

The system (5.1) gives the $\phi_{ij}(u, t)$ provided we know the values at $u = 0$. These can be computed using (5.1) with $u = 0$:

$$(5.3) \quad F_{ij}(ct, t) = \phi_{ij}(0, t) + \sum_{k=1}^m \int_0^t \phi_{kj}(0, t-\tau) d_x F_{ik}(c\tau, \tau) \\ i, j = 1, \dots, m.$$

To write this system of Volterra integral equations in a more concise way, let us introduce the following matrices:

$$(5.4) \quad \Phi(t) = (\phi_{ij}(0, t)) = (\phi(0, t))$$

$$(5.5) \quad \mathbf{F}(t) = (F_{ij}(ct, t)) = (\mathbf{F}(ct, t))$$

$$(5.6) \quad \mathbf{G}(t) = c \left(\frac{\partial F_{ij}}{\partial x}(ct, t) \right)$$

$$(5.7) \quad (\mathbf{A} * \mathbf{B})(t) = \left(\sum_{k=1}^m \int_0^t A_{ik}(t-v) B_{kj}(v) dv \right)$$

(with A and B $m \times m$ matrices)

$$(5.8) \quad \tilde{\mathbf{A}}(s) = \left(\int_0^\infty e^{-st} A_{ij}(t) dt \right)$$

(Laplace transform for matrices).

The system (5.3) takes the matrix form:

$$(5.9) \quad \mathbf{F}(t) = \Phi(t) + \mathbf{G} * \Phi(t)$$

and using Laplace transforms, we get

$$(5.10) \quad \tilde{\mathbf{F}}(s) = (\tilde{\mathbf{I}} + \tilde{\mathbf{G}}(s)) \tilde{\Phi}(s)$$

and consequently:

$$(5.11) \quad \tilde{\Phi}(s) = (\mathbf{I} + \tilde{\mathbf{G}}(s))^{-1} \tilde{\mathbf{F}}(s)$$

provided the inverse matrix of $\mathbf{I} + \tilde{\mathbf{G}}(s)$ exists.

We can now show the main result and for simplicity, we suppose that the derivatives $q_{ij}(x)$ of $Q_{ij}(x)$ exist for all i and j .

Proposition

If the quantity M defined by

$$(5.12) \quad M = \sup \{q_{ij}(x), i, j \in I, x \geq 0\}$$

is finite, then

$$(5.13) \quad \phi(t) = \sum_{n=0}^{\infty} (-1)^n \mathbf{G}^{(n)} * \mathbf{F}(t)$$

$$(5.14) \quad \phi(u, t) = \mathbf{F}(u + ct, t) - \mathbf{G}_u * \sum_{n=0}^{\infty} (-1)^n \mathbf{G}^{(n)^{1)} * \mathbf{F}(t)$$

where

$$(5.15) \quad \mathbf{G}_u(t) = \left(c \frac{\partial F_{ij}}{\partial x}(u + ct, t) \right).$$

Proof: From (3.5), we deduce that

$$(5.16) \quad \frac{\partial F_{ij}}{\partial x}(x, t) = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} q_{ij}^{(n)}(x)$$

where

$$(5.17) \quad q_{ij}^{(1)}(x) = q_{ij}(x)$$

and

$$(5.18) \quad q_{ij}^{(n)}(x) = \sum_k \int_0^x q_{ik}^{(n-1)}(x-y) q_{kj}(y) dy, n > 1.$$

From (5.12), (5.18), it is clear that, for all $n \geq 1$

$$(5.19) \quad q_{ij}^{(n)}(x) \leq M$$

so that from (5.16):

$$(5.20) \quad \frac{\partial F_{ij}}{\partial x}(x, t) \leq M(1 - e^{-\lambda t}) \leq M.$$

From the definition (5.6), we get

$$\tilde{G}_{ij}(s) \leq c \int_0^{\infty} M e^{-st} dt = \frac{cM}{s}$$

¹ From now, this symbol means the n -fold convolution product for the definition (5.7).

$$\begin{aligned} \tilde{G}_{ij}^2(s) &= \sum \tilde{G}_{ik}(s) \tilde{G}_{kj}(s) \leq m \frac{c^2 M^2}{s^2} \\ &\vdots \\ \tilde{G}_{ij}^n(s) &= \sum \tilde{G}_{ik}^{n-1}(s) \tilde{G}_{kj}(s) \leq m^{n-1} \frac{c^n M^n}{s^n}. \end{aligned}$$

Consequently, the matrix series $\sum \tilde{G}^n(s)$ converges for all $s > m c M$. A well-known consequence of this fact is that the matrix $(\mathbf{I} + \tilde{G}(s))^{-1}$ is invertible and

$$(\mathbf{I} + \tilde{G}(s))^{-1} = \sum_{n=0}^{\infty} (-1)^n \tilde{G}^n(s)$$

of course on $(m c M, \infty)$.

Using the matrix version of a theorem of DOETSCH (1974) and (5.11), we get (5.13).

The result (5.14) follows then from the relations (5.1) written under the matrix form and where $\phi(t)$ is under the form (5.13).

6. RESULTS FOR THE ACTUAL WAITING TIME AT TIME t OF THE M/SM/1 QUEUEING MODEL

The probabilistic assumptions made in the paragraph 1 imply that the process $((J_n, A_n, B_n), n \geq 0)$ is a two-dimensional $(J - X)$ process (JANSSEN, 1979) with kernel $(Q_{ij}(t, x))$ given by:

$$(6.1) \quad Q_{ij}(t, x) = E(t) \cdot Q_{ij}(x)$$

where

$$(6.2) \quad E(t) = \begin{cases} 0 & , t < 0 \\ 1 - e^{-\lambda t} & , t \geq 0. \end{cases}$$

If we suppose that the matrix $\mathbf{P} (= \mathbf{Q}(+\infty))$ is ergodic with a stationary probability distribution (Π_1, \dots, Π_m) , the dual kernel $(\hat{Q}_{ij}(t, x))$ of $(Q_{ij}(t, x))$ is given by (see JANSSEN (1979)):

$$(6.3) \quad \hat{Q}_{ij}(t, x) = \frac{\Pi_j}{\Pi_i} Q_{ij}(t, x)$$

$$(6.4) \quad = \frac{\Pi_j}{\Pi_i} E(t) Q_{ij}(x).$$

Let us now consider the M/SM/1 queueing model whose kernel is given by (6.4). The asymptotical study has been done for the first time by NEUTS (1966). Now the transient behaviour of $\hat{W}_{ij}(x, \tau)$ —defined by (2.2)—can be easily deduced from the last paragraph and our duality results (JANSSEN, 1979). From the proposition 4 of this last reference, we get, for all $x > 0$ and all $t > 0$:

$$(6.5) \quad \Pi_i \int_0^t e^{\lambda t} \hat{W}_{ij}(x, d\tau) = \Pi_j \int_0^t e^{\lambda t} \phi_{ji}(x, d\tau)$$

so that

$$(6.6) \quad \hat{W}_{ij}(x, \tau) = \frac{\Pi_j}{\Pi_i} \phi_{ji}(x, \tau).$$

If Π_a represents the $m \times m$ diagonal matrix whose i th element on the principal diagonal is Π_i , (6.6) takes the form

$$(6.7) \quad \hat{W}(x, \tau) = \Pi_a^{-1} \phi^T(x, \tau) \Pi_a$$

with

$$\hat{W}(x, \tau) = (\hat{W}_{ij}(x, \tau)).$$

(6.7) with the aid of (5.14) gives an explicit expression of the distribution of the actual waiting time in a M/SM/1 model.

7. COMMENTS

a) For $m = 1$, the model considered becomes the classical Cramér's model of risk theory and the M/G/1 queueing model for which it is known (see PRABHU (1961), SEAL (1972)) that:

$$(7.1) \quad \phi(0, t) = \frac{1}{t} \int_0^t F(x, t) dx.$$

Using successive integrations by parts, it is possible to show—in this case—the equivalence of (7.1) and (5.13). It does not seem possible to have an analogous result for $m > 1$, in particular an extension of the analytical proof of DE VYLDER (1977) cannot be used as the variables (B_n) are no more exchangeable.

b) The effect of a suppression of the k^e type of claim is theoretically possible by comparing $\phi(u, t)$ and $\phi_k(u, t)$, representing the non-ruin probability with ($m - 1$) types of claims, k being excluded.

c) The main result can be extended to the non-Poisson case if we suppose that the process (J_n, A_n) is a semi-Markov process of kernel

$$(\phi_{ij} E_i(t))$$

where

$$E_i(t) = \begin{cases} 0 & , t < 0 \\ 1 - e^{-\lambda t} & , t \geq 0 \end{cases}$$

that is a regular continuous Markov process with a finite number of states.

d) The following remarks may be useful for numerical computation.

It is easy to show that

$$(7.2) \quad \mathbf{G}^{(n)} * \mathbf{F}(t) \leq m^n \frac{M^n t^n}{n!}$$

so that approximating $\phi(t)$ by the first $(N - 1)$ terms of (5.13), we have for the absolute value of the error $R_N(t)$, the following upper bound:

$$(7.3) \quad |R_N(t)| \leq \frac{(mMt)^N}{N!} e^{mMt}.$$

For $m = 1$, we can say more. Indeed, let us suppose, without loss of generality, that $c = 1$ and $M \leq 1$. For c , that is well-known in risk theory; if $M > 1$, it suffices to introduce the random variables $(B'_n), (A'_n)$ defined by $B'_n = M^{-1} B_n$ and $A'_n = M^{-1} A_n$ so that the process (A'_n) induces a Poisson one of parameter $\lambda' = M^{-1}\lambda$. Then, if $\phi'(u', t')$ is the probability of non-ruin for this model: $\phi(u, t) = \phi'(Mu, Mt)$. (7.4)

In this case, we have

$$(7.5) \quad \mathbf{G}^{(n)} * \mathbf{F}(t) - \mathbf{G}^{(n+1)} * \mathbf{F}(t) = \mathbf{G}^{(n)} * (U_0 - \mathbf{G}) * \mathbf{F}(t)$$

which is a non-negative quantity as $G(t) \leq 1$ (U_0 is the Heaviside function with a unit mass at 0).

Consequently, the series (5.13) is alternating so that the sign of the error R_N is this of $(-1)^N$ and

$$(7.6) \quad |R_N(t)| \leq \mathbf{G}^{(N)} * \mathbf{F}(t).$$

From (7.2), it follows that:

$$(7.7) \quad |R_N(t)| \leq \frac{t^N}{N!}.$$

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