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PROBLEMS AND SOLUTIONS

PROBLÈMES ET SOLUTIONS

This department welcomes problems believed to be new. Solutions should accompany proposed problems.

Send all communications concerning this department to

Cette section a pour but de présenter des problèmes inédits. Les problèmes proposés doivent être accompagnés de leurs solutions.

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PROBLEMS FOR SOLUTION

P.194. Triads of points ABC, $A'B'C'$ on two lines through O determine the following points of intersection:

$$L = BC'\cdot B'C, \quad M = CA'\cdot C'A, \quad N = AB'\cdot A'B,$$
$$P = OL\cdot AA', \quad Q = OM\cdot BB', \quad R = ON\cdot CC'.$$

The theorem of Pappus tells us that L, M, N are collinear. Prove that P, Q, R are collinear.

K. Leisenring,
University of Michigan

P.195. Let (b_{ij}) be the inverse of the $(n+1)\times(n+1)$ matrix (a_{ij}), where

$$a_{ij} = \frac{(i-n+m-1)^{j-1}}{(j-1)!}$$

(m, n fixed integers with $m > n$). Show that

(i) $$b_{i+1,j} + \sum_{0\le s\le i-1} \binom{i-1}{s} b_{s+1,j} \sum_{\substack{m-n\le p\le m \\ p\ne j+m-n-1}} p^{-1} = 0,$$

(ii) $$\sum_{i=1}^{n+1} b_{ij} \frac{(i+n-m)^n}{n!} = \frac{1}{(n-j+1)!}.$$

J. S. Griffith,
Lakehead University

P.196. Let $g(n)$ be the maximal determinant of ± 1's of order n. Show that if $n=4^a h^b$, where $a, b \geq 0$ and h is the order of a Hadamard matrix, then

$$g(n+1) \geq (1+2^a[3-4h^{-1}]^b)g(n).$$

K. W. SCHMIDT,
UNIVERSITY OF MANITOBA

P.197. Given rings $\mathscr{R}_1$, $\mathscr{R}_2$ of subsets of a nonempty set S, find necessary and sufficient conditions for the union $\mathscr{R}_1 \cup \mathscr{R}_2$ to be a ring of subsets of S.

BERNARD L. D. THORP,
DURHAM, ENGLAND

SOLUTIONS

P.72. Prove that a sufficiently small neighbourhood of the origin on the arc $x=s^2$, $y=s^4-s^7$, $s \geq 0$ is met by no conic more than five times.

N. D. LANE,
MCMASTER UNIVERSITY

Solution by the Proposer. Consider the equations $ax^2+bxy+cy^2+dx+cy+f=0$; $x=s^2$, $y=s^4-s^7$. If $s \leq \frac{1}{2}$, it is readily verified by using Rolle's theorem and Descarte's rule of signs that

$$a(s^2)^2+b(s^2)(s^4-s^7)+c(s^4-s^7)^2+d(s^2)+e(s^4-s^7)+f = 0$$

cannot have *seven* positive real roots. Hence a small neighbourhood of the origin on the arc is met by no conic more than six times. It is well known, however, that every arc of conical order six is a union of a finite number of arcs of conical order five.

REMARK. This is an example of an arc of conical order five which is conically differentiable at an end point and the osculating conic at that point is a double line; cf. N. D. Lane and K. D. Singh, *Arcs of conical order five*, J. Reine Angew. Math. **217** (1965), p. 118.

P.104. It is well known that the idempotent elements of an arbitrary commutative ring with unit form a Boolean ring with respect to the operations $x \oplus y = x+y-2xy$, $x \odot y = xy$. Prove the following theorem or any suitable generalization.

THEOREM. *Let R be a commutative ring with unit and without nonzero nilpotent elements. Suppose further that R contains $3/2$, i.e. an element z such that $z+z$*

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$=e+e+e$. *Let* $S_3=\{x\colon x\in R, x^3=x\}$. *Then* S_3 *is a ring with respect to the operations* $x\oplus y=(x+y)(e-3/2xy)$, $x\odot y=xy$.

R. A. MELTER,
UNIVERSITY OF MASSACHUSETTS

Solution by the Proposer. A ring without nonzero nilpotent elements is a subdirect sum of integral domains. In an integral domain the only elements satisfying $x^3=x$ are $x=0, -1, 1$. Hence it suffices to show that in each of the component domains the equations satisfy the postulates for a ring when restricted to these three elements. But the equations imply the following addition and multiplication tables which are precisely those for $GF(3)$.

$\oplus$	-1	0	1
-1	-1	0	1
0	1	-1	0
1	0	1	-1

$\odot$	-1	0	1
-1	1	0	-1
0	0	0	0
1	-1	0	1

Editor's comment: Even if we allow R to have nilpotent elements S_3 is a ring, as can be seen by direct verification of the ring axioms. For a generalization of this problem see the paper by E. G. Connell in this issue, p. 79.

P.119. Denote Z_n the set of all n-tuples of complex numbers $z\equiv(z_1, z_2, \ldots, z_n)$ with $|z_j|=1$ $(1\leq j\leq n)$ and $\sum_{j=1}^{n} z_j=0$.

For fixed z denote by π_z the set of all n-tuples $\zeta\equiv(\zeta_1, \ldots, \zeta_n)$ where $\zeta_1, \ldots, \zeta_n$ is a permutation of $z_1, \ldots, z_n$. Let $\mu(\zeta)=\max_{0\leq k\leq n}|\zeta_1+\cdots+\zeta_k|$.

Determine

$$M_n = \min_{z\in Z_n} \max_{\zeta\in\pi_z} \mu(\zeta).$$

A. MEIR,
UNIVERSITY OF ALBERTA

This is solved except in the case when n is a power of 2. Dr. J. Schaer, University of Calgary, contributed the following remarks about the problem.

For a given $z\in Z_n$, $\max_{\zeta\in\pi_z}\mu(\zeta)$ is attained for a $\zeta\in\pi_z$ in which the z_j occur in their natural order according to direction, since all z_j that form an acute angle with $\zeta_1+\cdots+\zeta_k$, for which $|\zeta_1+\cdots+\zeta_k|$ is maximum, i.e. $\mu(\zeta)$, must contribute to this maximum sum. Therefore the problem is reduced to the following:

Consider the closed convex polygon formed by taking the z_j in their natural order as sides. Then $\max_{\zeta\in\pi_z}\mu(\zeta)$ is the diameter of that polygon.

PROBLEM. Find the least possible diameter Δ_n of all closed convex polygons u_n with n unit sides.

This problem has been partially solved by S. Vincze [1]. Here are his results:

THEOREM 1. *If n contains an odd prime factor, then* $\Delta_n = (2 \sin \pi/2n)^{-1}$.

So the problem is unsolved only for $n = 2^s > 4$.

THEOREM 2. *If $n \geq 3$ then* $\Delta_n \geq (2 \sin \pi/2n)^{-1}$.

The general upper bound $\Delta_n \leq (\sin \pi/n)^{-1}$ is trivial, but it is not sharp, e.g.

THEOREM 3. $\Delta_8 < (2 \sin \pi/8)^{-1}$.

It is easy to see that $\lim_{n\to\infty} \Delta_n/n = 1/\pi$.

THEOREM 4. *If n contains two odd prime factors (equal or not), then the extremal polygon is not unique.*

If n is not a power of 2, then the regular n-gon is extremal if and only if n is odd. That the regular hexagon is not extremal, had already been observed by P. Erdös; he proved that Δ_6 is attained for six vectors z_j alternately spaced by angles of $\pi/2$ and $\pi/6$.

THEOREM 5. *If U_n is extremal, then each vertex has at least one opposite vertex, i.e. a vertex at a distance equal to the diameter Δ_n.*

The condition $\sum_{j=1}^n z_j = 0$ is superfluous. If it is not satisfied, let $z_0 = -\sum_{j=1}^n z_j$; then $\sum_{j=0}^n z_j = 0$, and the problem is to find the minimal diameter of all closed convex polygons V_n with n unit sides and one arbitrary side. Using the hinge methods with which Vincze proves Theorem 5, one can show that a V_n with $z_0 \neq 0$ cannot be extremal.

Indeed, if A, B denote the endpoints of the side of length $|z_0|$ of the polygon V_n, then according to Theorem 5, which is valid also for V_n, either there exists a vertex C such that $\overline{AC} = \overline{BC} = \Delta$, or two adjacent vertices A', B' such that $\overline{AA'} = \overline{BB'} = \Delta$, and the diameters AA', BB' intersect according to Vincze's Lemma 1. In the first case keep, e.g., the part of V_n between A and C, which does not contain B, fixed, and rotate the part of V_n between B and C, which does not contain A, rigidly about C until B coincides with A. Then, according to Vincze's Lemma 2, all distances from vertices of one part to the other, other than C, will be decreased, so that according to Theorem 5 the resulting U_n is not extremal. In the second case two rotations are required, a first one of the part of V_n between A' and B that does not contain A about A' until $\overline{BB'}$ is decreased to $\overline{B'A}$, and a second one, of the part of V_n between A and B' that does not contain B, about B' until $\overline{AA'}$ is decreased to $\overline{A'B}$, and A and B now coincide. Then again, according to Lemma 2 and Theorem 5 the resulting U_n is not extremal. A fortiori, V_n has not been extremal.

REFERENCE

1. S. Vincze, Acta. Sci. Math. 12 (1950), 136–142.

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P.175. Prove or disprove: Every totally disconnected (no connected subset contains more than one point) topological space is Hausdorff.

D. G. Paulowich,
Dalhousie Universty

Solution by D. Z. Djokovic, University of Waterloo, Waterloo, Ontario. Let X be an infinite set, a, b distinct elements of X. Define a topology on X as follows: $A \subset X$ is open iff $X \backslash A$ is finite or $\{a, b\} \cap A = \varnothing$. Then X is totally disconnected but not Hausdorff since every neighbourhood of a meets every neighbourhood of b.

Also solved by Helen F. Cullen, Arthur S. Finbow, C. J. Knight, Douglas Lind, Frank J. Papp, Robert K. Tamaki and the Proposer.

Editor's comment: Several solvers refer to Example 27 in the recent text by L. A. Steen and J. A. Seebach, *Counterexamples in Topology* (Holt, New York, 1970) which appeared after receipt of this problem.

P.179. Solve the following system of congruences for positive integers x, y, z:

$$\begin{aligned} xy+1 &\equiv 0 \pmod z \\ zx+1 &\equiv 0 \pmod y \\ yz+1 &\equiv 0 \pmod x. \end{aligned}$$

E. J. Barbeau,
University of Toronto

Solution by T. M. K. Davison, McMaster University, Hamilton, Ontario. First we note that x, y, z are relatively prime and the given system of congruences reduces to the single diophantine equation $A+1=kxyz$, where $A=xy+yz+zx$ and k is a positive integer. We can moreover assume that $x \le y \le z$. Hence $A \le 3yz$, with equality occuring iff $x=y=z$ (hence $=1$). In all other cases $yz < A+1 \le 3yz$, and hence $2 \le kx \le 3$. There are just four cases to consider.

Case 1. $k=1$, $x=2$. Substituting in the equation $A+1=kxyz$, we obtain $(y-2)(z-2)=5$ and hence $(x, y, z)=(2, 3, 7)$.

Case 2. $k=2$, $x=1$. In this case $(y-1)(z-1)=2$, i.e. $(x, y, z)=(1, 2, 3)$.

Case 3. $k=1$, $x=3$. Clearly $3y+3z+1=2yz$ is not satisfied with $y=3$. Therefore, $y \ge 4$ and $3y+3z+1 < 8z \le 2yz$. So there are no solutions in this case.

Case 4. $k=3$, $x=1$. In this case $A+1=3yz$ gives $yz+(y-1)(z-1)=2$. Hence $yz=2$, $(y-1)(z-1)=0$, i.e. $(x, y, z)=(1, 1, 2)$.

Thus all solutions of the original system are $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 3)$ and $(2, 3, 7)$.

Also solved by W. J. Blundon, M. R. Pettet, E. Rosenthall and the Proposer.

Editor's comment: L. J. Mordell has considered the more general system of congruences $x_1x_2 \ldots x_n x_i^{-1} + a \equiv 0 \pmod{x_i}$ with $a = \pm 1$ and the x_i positive or negative. A paper on this problem has been submitted to the Bulletin.